Math 136: Differential Geometry– Homework 1

Due Wednesday, September 13

(1) [Kühnel, Chapter 2, Problem 1 (page 49)]. Prove that the curvature and torsion of a Frenet curve \( c(t) \) in \( \mathbb{R}^3 \) are given by the formulas

\[
\kappa(t) = \frac{\| \dot{c} \times \ddot{c} \|}{\| \dot{c} \|^3}, \quad \text{and} \quad \tau(t) = \frac{\det (\dot{c}, \ddot{c}, \dddot{c})}{\| \dot{c} \times \ddot{c} \|^2}.
\]

(2) [Kühnel, Chapter 2, Problem 4 (page 49)]. Show that a regular curve between points \( p, q \in \mathbb{R}^n \) with minimal length is necessarily the line segment from \( p \) to \( q \). **Hint:** Consider the Schwartz inequality \( \langle X, Y \rangle \leq \|X\| \cdot \|Y\| \) for the tangent vector and the difference vector \( p - q \).

(3) [Kühnel, Chapter 2, Problem 8 (page 50)]. Show that the Frenet two-frame of a plane curve with given curvature functions \( \kappa(s) \) can be described by the exponential series for the matrix

\[
\begin{bmatrix}
0 & \int_0^s \kappa(t) dt \\
-\int_0^s \kappa(t) dt & 0
\end{bmatrix}.
\]

Show that this implies that the Frenet frame is

\[
\begin{bmatrix}
e_1(s) \\
e_2(s)
\end{bmatrix} = \sum_{i=0}^{\infty} \frac{1}{i!} \begin{bmatrix}
0 & \int_0^s \kappa(t) dt \\
-\int_0^s \kappa(t) dt & 0
\end{bmatrix}^i \begin{bmatrix}e_1(0) \\
e_2(0)
\end{bmatrix}.
\]

Here we mean that the components of the first row of the matrix on the right hand side describe the vector \( e_1(s) \) and similarly for the components of the second row of the right hand side.

(4) Let \( c(s) : I \to \mathbb{R}^3 \) be a Frenet curve, parametrized by arc-length. We call the vector \( e_2(s) := c''/\|c''\| \) the principal normal vector. We say that \( c(s) \) is a Bertrand curve if there is a function \( r(s) : I \to \mathbb{R} \) such that the curve

\[
\tilde{c}(s) := c(s) + r(s)e_2(s) \tag{1}
\]

has \( e_2(s) \) as its principal normal vector for every \( s \in I \). In this case we say that \( c, \tilde{c} \) are a Bertrand pair of curves. Suppose that \( c(s) \) is NOT planar. Prove the following statements:
(i) Suppose that $c, \tilde{c}$ are a Bertrand pair of curves. Show that $r(s)$ is constant, and in particular, the distance between the curves $\|c - \tilde{c}\|$ is constant.

(ii) Show that the angle between the tangent vectors $c', \tilde{c}'$ is constant. That is, show that
\[
\frac{\langle e_1(s), \tilde{c}'(s) \rangle}{\|\tilde{c}'(s)\|} = const.
\]

Deduce that there are constants $a, b \in \mathbb{R}$ with $a > 0$ such that $aK + b \tau \equiv 1$.

(iii) Give an explicit example of a non-planar pair of Bertrand curves.