DERIVATIVES OF $p$-ADIC SIEGEL EISENSTEIN SERIES
AND $p$-ADIC DEGREES OF ARITHMETIC CYCLES

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Abstract

Kudla has given for each $n \geq 1$ a genus $n$ weight $\frac{n+1}{2}$ Siegel Eisenstein series with odd functional equation whose central derivative he speculates to have arithmetic content. Specifically, these incoherent Eisenstein series vanish at $s = 0$ and their derivatives are nonholomorphic modular forms whose Fourier coefficients seem be degrees of 0-cycles on certain Shimura varieties. When $n$ is odd, we search for evidence of a $p$-adic analogue which relates the derivative of a $p$-adic Siegel Eisenstein series to $p$-adic degrees of 0-cycles. Indeed, when $n = 1$ or 3, we construct an analogous $p$-adic Siegel Eisenstein series, compute the Fourier expansion of its derivative, and relate the resulting Fourier coefficients to $p$-adic degrees of the same 0-cycles studied by Kudla.

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Declaration

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Introduction

There are several well-known connections between Fourier coefficients of Siegel Eisenstein series and the arithmetic of certain varieties. One example studied by Gross and Keating [4] concerns the classical Siegel Eisenstein series of genus $g$ and weight 2

$$
E(\tau, s) = \sum \frac{\left( \text{det}(c\tau + d) \right)^s}{\left| \text{det}(c\tau + d) \right|^{2s}},
$$

where $\tau = u + iv$ is in the genus $g$ Siegel upper half-space and the sum is over representatives $(^*_* c d)$ of the Siegel parabolic in $\text{Sp}_g(\mathbb{Z})$. This series has a Fourier expansion

$$
E(\tau, s) = \sum_T a_T(\tau, s)q^T.
$$

When $g = 2$, the Fourier coefficients at $s = 0$ are related to the intersection theory of the surface $S_C = \text{Spec} \mathbb{C}[j, j']$. Specifically, letting $D_m$ be the divisor on $S_C$ defined by the vanishing of the modular polynomial $\phi_m(j, j') \in \mathbb{Z}[j, j']$, there is a constant $C$ so that

$$
(D_{m_1} \cdot D_{m_2})_{S_C} = C \sum_{T=(^*_* m_1 \cdot *_{m_2}) > 0} a_T
$$

where $a_T = a_T(\tau, 0)$ is independent of $\tau$. When $g = 3$, the Eisenstein series vanishes at $s = 0$, but the derivative $\frac{\partial}{\partial s} E(\tau, s)|_{s=0}$ is a non-holomorphic modular form with Fourier coefficients $a'_T = \frac{\partial}{\partial s} a_T(\tau, s)|_{s=0}$ also independent of $\tau$. Computations by Kudla and Zagier suggested that these Fourier coefficients are related to the intersection theory of the arithmetic threefold $S = \text{Spec} \mathbb{Z}[j, j']$ by

$$
(D_{m_1} \cdot D_{m_2} \cdot D_{m_3})_S = C' \sum_{T > 0} a'_T
$$

where the sum is over positive definite $T$ with diagonal entries $m_1, m_2, m_3$. Thus we see an interesting phenomenon. The Fourier coefficients at a special value of $s$ are related to the geometry of a complex surface. But when the Eisenstein series vanishes, the Fourier coefficients of the derivative are related to the geometry of an arithmetic threefold.

Certain Siegel Eisenstein series $E(\tau, s)$ with even functional equations relate to degrees of 0-cycles on complex surfaces in a particularly simple way: there is a class of 0-cycles $Z(T)$ parametrized by
symmetric $T > 0$ so that the Fourier expansion of $E$ is given by

$$E(\tau, s_0) = \sum_T \deg(Z(T))q^T$$

at the center $s_0$ of the functional equation [11]. Motivated by the previous example, as well as other examples arising from his work on triple product $L$-functions [5], Kudla gave for each $n \geq 1$ a genus $n$ weight $\frac{n+1}{2}$ Siegel Eisenstein series with odd functional equation whose central derivative he speculated would have a Fourier expansion of the same simple form [9]. These Eisenstein series are called incoherent Eisenstein series, and the goal of the so-called Kudla program is to formulate and prove connections between their central derivatives and degrees of 0-cycles on certain Shimura varieties. These connections have been proved for most nonsingular Fourier coefficients in some of the simplest cases [12, 13, 14].

The goal of this thesis is to seek evidence of a $p$-adic analogue to the Kudla program in the case of odd genus. Concretely, this means two things. First, it requires constructing $p$-adic incoherent Eisenstein series analogous to the ones described by Kudla; these $p$-adic Eisenstein series should vanish at some point $k_0$ analogous to the central point $s_0$. Second, it means relating the $p$-adic derivative at $k_0$ to $p$-adic degrees of arithmetic cycles, for an appropriate definition of $p$-adic degree. In two of the simplest cases, we carry out these tasks here. For example, in the genus 1 case we prove the following theorem.

**Theorem.** Let $F = \mathbb{Q}(\sqrt{-\ell})$ be an imaginary quadratic field for a prime $\ell > 3$ with $\ell \equiv 3 \pmod{4}$. If $p$ is prime with $\left(\frac{p}{\ell}\right) = 1$, there is an incoherent $p$-adic Siegel Eisenstein series $E(k) = \sum_{t \geq 0} a_t(k)q^t$ of genus 1 associated to $F$ such that when $p \nmid t$

$$\frac{d}{dk}a_t(k)|_{k=1} = \frac{1}{2} \deg(Z(t))$$

for a family of 0-cycles $Z(t)$ on a certain Shimura variety isomorphic to $\text{Spec} \, \mathcal{O}_H$, $H$ the Hilbert class field of $F$.

We prove a similar result in the genus 3 case, but for a more restricted class of Fourier coefficients.

The structure of this thesis is as follows. Section 1 concerns the $p$-adic Eisenstein series side of the picture. In it, we will review constructions of the relevant Siegel Eisenstein series, give basic facts about them, construct analogous $p$-adic Siegel Eisenstein series in the genus 1 and 3 cases, and compute their derivatives. Section 2 then turns to the geometric side. We start by defining $p$-adic Arakelov divisors, which come equipped with a degree map. We will then show how to extend
this degree map to 0-cycles. Finally, we will compute $p$-adic degrees of the 0-cycles given by Kudla in the genus 1 and 3 cases and relate them to the derivatives computed earlier. In Kudla’s work, comparisons between Fourier coefficients of Eisenstein series and degrees of 0-cycles are made by simply computing both quantities separately, though a proof that directly relates them would be highly desirable. The same is quite true of the results presented here.

1 $p$-adic Siegel Eisenstein series

In [9], Kudla shows for $n \geq 1$ how to construct certain incoherent Siegel Eisenstein series $E(g, s, \Phi)$ of degree $n$ and weight $\frac{n+1}{2}$. These Eisenstein series have an odd functional equation in $s$, so that $E(g, 0, \Phi) = 0$. They also have a Fourier expansion

$$E(g, s, \Phi) = \sum_T E_T(g, s, \Phi)$$

over symmetric matrices $T$. For $\det T \neq 0$, these Fourier coefficients factor into local Whittaker functions

$$E_T(g, s, \Phi) = \prod_p W_{T,p}(g_p, s, \Phi_p)$$

over primes $p$ of the number field. Kudla is interested in the arithmetic content of the central derivative $\frac{\partial}{\partial s} E(g, s, \Phi)|_{s=0}$, which he speculates is related to degrees of 0-cycles on a certain Shimura variety. Kudla and his collaborators have shown such a result holds when $n = 1$ [14], and they have also proved weaker results for most nonsingular Fourier coefficients when $n = 2, 3, 4$ [9, 12, 13]. When $n$ is odd, these Siegel Eisenstein series are integer-weight modular forms on the genus $n$ symplectic group $Sp_n$, and we will restrict attention to this case; however, note that in general these Eisenstein series may be half-integer-weight modular forms on the metaplectic cover of $Sp_n$.

In this section we will review a slightly more general version of this construction and use it to study analogous $p$-adic Eisenstein series in the cases $n = 1, 3$. Section 1.1 begins by defining and stating key properties of the local factors $W_{T,p}$. Then section 1.2 defines the relevant Eisenstein series. In sections 1.3 and 1.4 we take up the cases $n = 1$ and 3, respectively.

Let $F$ be a totally real number field (later we will take $F = \mathbb{Q}$ for simplicity). Define the following notation. Let

$$G = Sp_n(F) = \{ x \in SL_{2n}(F) : x^Twx = w \}$$
be the group of $2n \times 2n$ symplectic matrices with respect to the symplectic form given by

$$w = \begin{pmatrix} 1_n & \ast \\ -\ast & 1_n \end{pmatrix}.$$  

Then $G$ has a Siegel parabolic subgroup $P = NM$ with 

$$N = \left\{ n(b) = \begin{pmatrix} 1_n & b \\ b^T & 1_n \end{pmatrix} : b \in \text{Sym}_n(F) \right\}$$

$$M = \left\{ m(a) = \begin{pmatrix} a & (a^T)^{-1} \\ (a^T)^{-1} & a \end{pmatrix} : a \in \text{GL}_n(F) \right\}$$

for $\text{Sym}_n(R) = \{ a \in M_n(R) : a = a^T \}$ denoting the symmetric matrices with coefficients in a ring $R$. For a prime $p$ of $F$, we similarly define $G_p$, $P_p$, $N_p$ and $M_p$ by replacing $F$ by $F_p$. When $p$ is nonarchimedean, let $O_p$ be the ring of integers of $F_p$ and let $K_p = \text{Sp}_n(O_p)$ be the maximal compact subgroup of $G_p$. When $F_p = \mathbb{R}$ let

$$K_p = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a + ib \in U(n) \right\}.$$ 

Thus in all cases, $G_p = P_pK_p$. 

Finally, let $\mathbb{A}$ be the adeles of $F$, and define $G_\mathbb{A}$, $P_\mathbb{A}$, etc. in the obvious way. We set $K_f = \prod_{p \mid \infty} K_p$ and $K_\infty = \prod_{p \mid \infty} K_p$ so that $G_\mathbb{A} = P_\mathbb{A}K_fK_\infty$. 

### 1.1 Local Theory

In this section, we define the local factors $W_{T,p}(g_p, s, \Phi_p)$ mentioned above. These factors depend on a choice of section $\Phi_p(s)$ of a degenerate principal series representation. We explain how to obtain such a section from a Schwartz function $\varphi_p$ on $V^n$ for a quadratic space $V$ over $F_p$. We also explain a decomposition of the degenerate principal series representation into irreducible components and relate this to properties of $W_{T,p}$. The exposition in this section largely follows [9, §1] and [10, §2].

For ease of notation, we will drop the subscript $p$ in this section, so that $F$ denotes $F_p$, $G$ denotes $G_p$, and so on. In particular, $F$ is a local field. We fix, once and for all, a nontrivial additive character $\psi : F \to \mathbb{C}^\times$ and an odd integer $n \geq 1$.

Let $\chi$ be a quadratic character of $F^\times$. For $s \in \mathbb{C}$ we have the degenerate principal series
representation

$$I_n(s, \chi) = \text{Ind}_P^G(\chi| \cdot |^s)$$

normalized so that a section $$\Phi(s) \in I_n(s, \chi)$$ is a smooth function $$G \to \mathbb{C}$$ satisfying

$$\Phi(n(b)m(a)g, s) = \chi(a)|a|^{s+\frac{n+1}{2}} \Phi(g, s)$$

for $$g \in G$$. If $$F = \mathbb{R}$$ we additionally require that $$\Phi(s)$$ be $$K$$-finite. Here and throughout, $$\chi(a)$$ denotes $$\chi(\det a)$$ and $$|a|$$ denotes $$|\det a|$$.

We call a section $$\Phi(s)$$ \textit{standard} if $$\Phi(s)|_K$$ is independent of $$s$$. Given any $$\Phi \in I_n(0, \chi)$$, we can extend $$\Phi$$ to a standard section via

$$\Phi(g, s) = \Phi(g)|a|^s$$

where $$g = n(b)m(a)k$$. An especially important element of $$I_n(s, \chi)$$ is the \textit{standard spherical section} $$\Phi^0(s)$$, which is defined to be the $$K$$-invariant standard section, normalized so that $$\Phi(1, s) = 1$$. Thus

$$\Phi^0(n(b)m(a)k, s) = \chi(a)|a|^{s+\frac{n+1}{2}}.$$ 

For $$s \in \mathbb{C}$$ with $$\text{Re}(s) > \frac{n+1}{2}$$, $$g \in G$$, and $$\Phi(s) \in I_n(s, \chi)$$, we define the Whittaker function

$$W_T(g, s, \Phi) = \int_N \Phi(w^{-1}ng, s)\psi_T(n)^{-1}dn$$

where $$T \in \text{Sym}_n(F)$$, $$w$$ is given in (1), $$\psi_T(n(b)) = \psi(\text{tr}(Tb))$$ for $$b \in \text{Sym}_n(F)$$, and $$dn$$ is the self-dual measure on $$N$$ with respect to the pairing $$\langle n(b_1), n(b_2) \rangle = \psi(\text{tr}(b_1b_2))$$. There is an analytic continuation of $$W(g, s, \Phi)$$ to an entire function of $$s$$ [6, 22].

The Whittaker function $$W_T$$ gives rise to a functional $$W_T(s) : I_n(s, \chi) \to \mathbb{C}$$ via $$W_T(s)(\Phi) = W_T(1, s, \Phi)$$. We will be especially interested in the functional $$W_T(0) : I_n(0, \chi) \to \mathbb{C}$$. It will be important to know on which irreducible components of $$I_n(0, \chi)$$ this functional vanishes. It ends up that the irreducible components of $$I_n(0, \chi)$$ are parametrized by quadratic spaces of dimension $$n+1$$ over $$F$$ in the manner described below. More generally, given a quadratic space $$V$$ of dimension $$m$$ and a Schwartz-Bruhat function $$\varphi$$ on $$V^n$$, we will construct below a corresponding $$\Phi \in I_n(s_0, \chi_V)$$ for $$s_0 = \frac{n+1}{2} - \frac{n+1}{2}$$ and $$\chi_V$$ a quadratic character depending on $$V$$.

Indeed, let $$V$$ be a quadratic space over $$F$$ of dimension $$m$$, and let $$Q$$ be the matrix of the nondegenerate quadratic form on $$V$$ with respect to some basis. Denote by $$\det V$$ the image of $$\det Q$$
in $F^\times/(F^\times)^2$. We then have a quadratic character $\chi_V$ on $F^\times$ given by
\[
\chi_V(x) = (x, (-1)^{(\frac{m-1}{2})}\det V)_F
\]
where $(-,\cdot)_F$ is the quadratic Hilbert symbol.

Associated to $\psi$, there is a Weil representation $\omega$ of $G$ on $S(V^n)$, the space of Schwartz-Bruhat functions on $V^n$. This gives rise to a $G$-intertwining map
\[
\lambda: S(V^n) \to I_n(s_0, \chi_V)
\]
with $s_0 = \frac{m}{2} - \frac{n+1}{2}$. We can extend any $\Phi = \lambda(\varphi)$ to a standard section $\Phi(s)$ as above. If $F$ is nonarchimedean, this construction has the following important special case. Let $L \subset V$ be a self-dual $\mathcal{O}$-lattice, i.e.
\[
L^* = \{x \in V : (x, y) \in \mathcal{O} \text{ for all } y \in L\} = L
\]
for the inner product $(\cdot, \cdot)$ on $V$. We denote by $\varphi^0 = \varphi^0_L$ the characteristic function of $L^n \subset V^n$. Then the standard section corresponding to $\varphi^0$ is the normalized spherical section $\Phi^0(s)$ defined above [21, §4.4].

When $\dim V = m = n + 1$, this construction gives rise to a subrepresentation
\[
R_n(V) = \lambda(S(V^n)) \subseteq I_n(0, \chi_V)
\]
(for an alternate description of $R_n(V)$, see [9, §1]). Remarkably, these are all the irreducible components of $I_n(0, \chi)$.

**Proposition 1.1.1** ([9, Prop. 1.1]).  
(a) $R_n(V)$ is irreducible.

(b) Let $\chi$ be a quadratic character of $F^\times$. Then we have a decomposition
\[
I_n(0, \chi) = \bigoplus_{\dim V = n + 1} R_n(V).
\]

(c) If $F$ is nonarchimedean, the quadratic spaces $V$ of dimension $n + 1$ are classified by their
character \( \chi_V \) and Hasse invariant \( \varepsilon(V) \in \{ \pm 1 \} \). Thus

\[
I_n(0, \chi) = R_n(V^+) \oplus R_n(V^-)
\]

where \( V^\pm \) is the quadratic space of dimension \( n + 1 \), character \( \chi \), and Hasse invariant \( \pm 1 \).

(d) If \( F = \mathbb{R} \), the quadratic spaces \( V \) of dimension \( n + 1 \) are classified by their signature \( (p,q) \).

Thus

\[
I_n(0, \chi) = \bigoplus_{\substack{p+q=n+1 \\ \chi(-1)=-(-1)^{\frac{n+1}{2}+q}}} R_n(V_{p,q})
\]

where \( V_{p,q} \) is the quadratic space of signature \( (p,q) \).

Moreover, whether the functional \( W_T(0) : I_n(0, \chi) \to \mathbb{C}^\times \) vanishes on \( R_n(V) \) is determined by the arithmetic of \( V \). Recall that we say an \( m \)-dimensional quadratic space \( V \) represents \( T \in \text{Sym}_n(F) \) if there exists \( x \in V^m \) (viewed as an \( n \times m \) matrix) such that \( Q[x] = T \), where as above \( Q \) is a matrix for the quadratic form on \( V \) and we write \( Q[x] = x^TQx \).

**Proposition 1.1.2** ([9, Prop. 1.4]). If \( T \in \text{Sym}_n(F) \) with \( \det T \neq 0 \) then \( W_T(0) \) is nonzero on \( R_n(V) \) if and only if \( V \) represents \( T \).

It is also convenient to have the following description of quadratic spaces of dimension \( n + 1 \) that represent \( T \).

**Proposition 1.1.3** ([9, Prop. 1.3]). Let \( T \in \text{Sym}_n(F) \). Then there is a unique quadratic space \( V_T \) such that \( \dim V_T = n + 1 \) and \( V_T \) represents \( T \). The Hasse invariant of \( V_T \) satisfies

\[
\varepsilon(V_T) = \varepsilon(T)\chi(T)(\det T, -(-1)^{\frac{n(n+1)}{2}})_{F}.
\]

When \( F \) is nonarchimedean, quadratic spaces \( V \) of a given dimension are classified by their Hasse invariants; hence (3) is a necessary and sufficient condition for \( V \) to represent \( T \). These conditions on when \( W_T(0) \) vanishes will be important for understanding the order to which Fourier coefficients vanish in the next section.

### 1.2 Siegel Eisenstein series

We return to the notation fixed at the start of section 1, so that \( F \) again denotes a totally real number field with adele ring \( \mathbb{A} \). Fix once and for all a nontrivial additive character \( \psi : \mathbb{A}/F \to \mathbb{C}^\times \).
and odd $n \geq 1$. Given a quadratic character $\chi$ of $\mathbb{A}^\times / F^\times$, we again have the induced principal series representation $I_n(s, \chi) = \text{Ind}_{P}^{G}(\chi | \cdot |^{s})$ so that sections of $I_n(s, \chi)$ are smooth $K_\infty$-finite functions satisfying (2). As before, a section $\Phi(s)$ of $I_n(s, \chi)$ is standard if the restriction $\Phi(s)|_{K}$ is independent of $s$. As explained in [21, §4.4], $I_n(s, \chi)$ is the restricted tensor product $\bigotimes'_p I_{n,p}(s, \chi)$ with respect to the standard spherical sections $\{\Phi^0_p(s)\}$.

Given a standard section $\Phi(s) \in I_n(s, \chi)$, there is a Siegel Eisenstein series defined on $\text{Re}(s) > \frac{n+1}{2}$ via

$$E(g, s, \Phi) = \sum_{\gamma \in P \setminus G} \Phi(\gamma g, s).$$

This Eisenstein series has a meromorphic continuation to all of $\mathbb{C}$ with no poles on the line $\text{Re}(s) = 0$ and a functional equation relating $s$ and $-s$ [1]. It also has a Fourier expansion

$$E(g, s, \Phi) = \sum_{T \in \text{Sym}_n(F)} E_T(g, s, \Phi)$$

for

$$E_T(g, s, \Phi) = \int_{N \setminus N_a} E(n g, s, \Phi) \psi_T(n)^{-1} \, dn$$

where $\psi_T(n)$ and $dn$ are defined similarly to above. When $\text{det} T \neq 0$, we can write this as

$$E_T(g, s, \Phi) = \int_{\text{Sym}_n(\mathbb{A})} \Phi(w^{-1} n(b) g, s) \psi(- \text{tr}(Tb)) \, db$$

where $db = \prod_p db_p$ is the product of the local self-dual measures on $\text{Sym}_n(F_p)$ with respect to the pairings $\langle b_1, b_2 \rangle = \psi_p(\text{tr}(b_1 b_2))$. When $\Phi(s)$ is factorizable, meaning $\Phi(s) = \otimes_p \Phi_p(s)$ for $\Phi_p(s) \in I_{n,p}(s, \chi)$, we further have

$$E_T(g, s, \Phi) = \prod_p W_{T,p}(g_p, s, \Phi_p)$$

where $W_{T,p}$ are the Whittaker functions defined in section 1.1.

One way to obtain sections of $I_n(s, \chi)$ is as follows. Let $V$ be an $m$-dimensional quadratic space over $F$ so that the quadratic character $\chi_V = \prod_p \chi_{V_p}$ of $\mathbb{A}^\times / F^\times$ is $\chi$. Suppose we have chosen a self-dual lattice $L_p \subset V_p$ for all but finitely many finite primes $p$. We then get the characteristic functions $\varphi^0_p = \varphi^0_{L_p} \in S(V^*_p)$. We define the Schwartz space

$$S(V(\mathbb{A})^n) = \bigotimes'_p S(V^*_p).$$
to be the restricted tensor product with respect to $\{\varphi_p^0\}$. By taking the tensor product of the maps $\lambda_p : S(V^n) \to I_{n,p}(s_0, \chi_V)$ defined in section 1.1 we obtain a map $\lambda : S(V(\mathbb{A})^n) \to I_n(s_0, \chi)$ which is well-defined since $\lambda_p(\varphi_p^0) = \Phi_p^0(s)$. Moreover, if $\varphi = \otimes_p \varphi_p$ is a factorizable Schwartz function then $\lambda(\varphi) = \otimes_p \lambda_p(\varphi_p)$ is a factorizable section.

**Example 1.2.1.** Take $n = 1$, $F = \mathbb{Q}$, and $V = \mathbb{Q}^{2k}$ with quadratic form $Q = 1_{2k}$; note that $\text{Sp}_1(\mathbb{Q}) = \text{SL}_2(\mathbb{Q})$. We suppose also that $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$ is the standard additive character trivial described at the start of section 1.3. We take our factorizable Schwartz function $\varphi = \otimes_p \varphi_p$ so that when $p < \infty$, $\varphi_p = \varphi_p^0$ is the characteristic function of the lattice $\mathbb{Z}_p^n \subset \mathbb{Q}_p^n$, and for $p = \infty$ we take $\varphi_\infty(x) = e^{-\pi \text{tr} Q(x)}$

to be the Gaussian. Then $\lambda_p(\varphi_p) = \Phi_p^0$ for $p < \infty$ and $\lambda_\infty(\varphi_\infty) = \Phi_\infty^k$, the weight $k$ eigenfunction defined by

$$\Phi_\infty^k(gk(\theta), s) = e^{ik\theta} \Phi_\infty(g, s)$$

and $\Phi_\infty^k(1, s) = 1$, where

$$k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in \text{SO}(2).$$

Using the formulas in section 1.3, it is easy to see that the Eisenstein series

$$E_k(\tau, s) = \text{Im}(v)^{-\frac{1}{2}(s+1)} E(g_{\tau, s}, \Phi_\infty^k, (\otimes_p \Phi_p)) = \sum_{\gamma = (a\ b\ c\ d) \in \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z})} \frac{1}{(c\tau + d)^k} \text{Im}(\gamma \tau)^{\frac{1}{2}(s+k-1)}.$$

is the classical weight $k$ Eisenstein series. Here $\Gamma_\infty = P \cap \text{SL}_2(\mathbb{Z})$ and $g_{\tau} \in G_\mathbb{A}$ is such that $g_{\tau, \infty} = n(u)n(v\frac{1}{2})$ and $g_{\tau, p} = 1$ for $p < \infty$ where $\tau = u + iv \in \mathbb{H}$. In particular, when $k > 2$

$$E_k(\tau, k - 1) = \sum_{\gamma = (a\ b\ c\ d) \in \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z})} \frac{1}{(c\tau + d)^k}$$

is a holomorphic modular form of weight $k$.

Proposition 1.1.1 gives rise to a decomposition of $I_n(0, \chi)$ into irreducible components

$$I_n(0, \chi) = \bigoplus_V \Pi_n(V) \oplus \bigoplus_C \Pi_n(C)$$

where $V$ runs over quadratic spaces of dimension $n + 1$ with $\chi_V = \chi$, we set $\Pi_n(V) = \otimes_p R_{n,p}(V_p)$,
and \( \mathcal{C} \) runs over incoherent collections of dimension \( n+1 \) and \( \chi_{\mathcal{C}} = \chi \), defined as follows.

**Definition 1.2.2.** An incoherent collection of dimension \( m \) and character \( \chi \) over \( F \) is a collection \( \mathcal{C} = \{ \mathcal{C}_p \} \) of \( m \)-dimensional quadratic spaces \( \mathcal{C}_p \) over \( F_p \) with character \( \chi_{\mathcal{C}_p} = \chi_p \) such that

(i) \( \mathcal{C} \) is almost everywhere unramified, meaning that for some \( m \)-dimensional quadratic space \( V \) over \( F \) with character \( \chi_V = V \), we have \( \mathcal{C}_p \cong V_p \) for almost all \( p \), and

(ii) the product formula for Hasse invariants fails, meaning

\[
\prod_p \varepsilon_p(\mathcal{C}_p) = -1.
\]

We define \( \Pi_n(\mathcal{C}) = \otimes_p R_{n,p}(\mathcal{C}_p) \) just as for \( \Pi_n(V) \). We call a standard section \( \Phi(s) \in I_n(s,\chi) \) incoherent if \( \Phi(0) \in \Pi_n(\mathcal{C}) \) for some incoherent collection \( \mathcal{C} \) of dimension \( n+1 \). The critical fact is as follows.

**Proposition 1.2.3.** If \( \Phi(s) \in I_n(s,\chi) \) is an incoherent standard section then \( E(g,0,\Phi) = 0 \).

In fact, for \( T \in \text{Sym}_n(F) \) with \( \det T \neq 0 \) we can bound the order of vanishing of \( E_T(g,s,\Phi) \) from below by examining local Whittaker functions.

**Definition 1.2.4.** Let \( \mathcal{C} \) be an incoherent collection of dimension \( n+1 \). For nonsingular \( T \in \text{Sym}_n(F) \), we set

\[
\text{Diff}(T,\mathcal{C}) = \{ p : \varepsilon_p(\mathcal{C}_p) \neq \varepsilon_p(T)\chi_{\mathcal{C}_p}(T)(\det T, -(-1)^{\frac{n(n+1)}{2}})F_p \}
\]

to be the set of places at which (3) fails.

By Proposition 1.1.3, \( \text{Diff}(T,\mathcal{C}) \subseteq \{ p : T \text{ is not represented by } \mathcal{C}_p \} \); in fact, these sets can only differ at infinite places. This gives the following.

**Proposition 1.2.5.** Let \( T \in \text{Sym}_n(F) \) be nonsingular, and let \( \Phi(s) \) be a standard section with \( \Phi(0) \in \Pi_n(\mathcal{C}) \).

(a) \( |\text{Diff}(T,\mathcal{C})| \) is odd.

(b) \( W_{T,p}(g_p,0,\Phi_p) = 0 \) for all \( p \in \text{Diff}(T,\mathcal{C}) \).

**Proof.** Part (b) follows immediately from Proposition 1.1.2. For part (a) first note that \( \text{Diff}(T,\mathcal{C}) \) is finite. Indeed, pick some \( V \) so that \( \mathcal{C}_p = V_p \) for all almost all \( p \). We must have that \( \varepsilon_p(V_p) = \)
χ_{V_p}(T) = 1 for almost all p, so ε_p(C_p) = χ_{C_p}(T) = 1 for almost all p as well. Similarly ε_p(T) = (\det T, -(-1)^{\frac{n(n+1)}{2}})_{F_p} = 1 for almost all p, so that both sides of (3) equal 1 for almost all p, as desired.

To see that |Diff(T, C)| is odd take the product of both sides of (3) over all p.

$$\prod_p \varepsilon_p(C_p) = -1$$

by definition, whereas product formulas give

$$\prod_p \varepsilon_p(T)\chi_{V_p}(T)(\det T, -(-1)^{\frac{n(n+1)}{2}})_{F_p} = 1$$

since det T ∈ F. Thus (3) must be off by a sign for an odd number of primes.

**Corollary 1.2.6.** If \( \frac{\partial}{\partial s} E_T(g, s, \Phi)|_{s=0} \neq 0 \) then Diff(T, C) = \{p\} for some p.

We conclude this section by describing the main type of Eisenstein series studied by Kudla and his collaborators.

**Example 1.2.7.** Let \( V \) be a quadratic space of dimension \( n + 1 \) over \( F = \mathbb{Q} \). We want to form an incoherent collection \( C \) that differs from \( \{V_p\} \) at exactly one prime. In order to obtain an Eisenstein series with good analytic properties, we also want the section at \( \infty \) to be \( \Phi_{\infty} = \Phi_{\infty}^{\frac{n+1}{2}} \), the weight \( \frac{n+1}{2} \) eigenfunction, defined by

$$\Phi_{\infty}^{\frac{n+1}{2}} \left( g \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, s \right) = \det(a + ib)^{\frac{n+1}{2}} \Phi_{\infty}^{\frac{n+1}{2}} (g, s)$$

and \( \Phi_{\infty}^{\frac{n+1}{2}} (1, s) = 1 \), where \( a + ib \in U(n) \) so that \( \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in K_{\infty} \). As in example 1.2.1, \( \Phi_{\infty}^{\frac{n+1}{2}} = \lambda_\infty (\varphi_\infty) \) where

$$\varphi_\infty(x) = e^{-\pi \text{tr} Q[x]}$$

is the Gaussian on \( V_{\infty}^n \) with respect to the quadratic form \( Q \) on \( V \). In order for \( \varphi_\infty \) to be a Schwartz function, \( V_{\infty} \) must have signature \( (n + 1, 0) \).

Given these constraints, there are two possible types of incoherent collections.

1. Starting with a quadratic space of signature \( (p, q) \) with \( p \equiv n - 1 \) (mod 4), we replace \( V_{\infty} \) with \( V'_{\infty} \) of signature \( (n + 1, 0) \) to form the collection \( C = \{V_p\}_{p<\infty} \cup \{V'_{\infty}\} \).
Starting with a quadratic space of signature \((n + 1, 0)\), we pick a prime \(q < \infty\) and replace \(V_q\) with \(V'_q\) of opposite Hasse invariant and \(\chi_{V_q} = \chi_{V'_q}\) to form the collection \(C = \{V'_p\}_{p \neq q} \cup \{V'_q\}\).

In either case, we fix the same \(\varphi_f = \otimes_{p < \infty} \varphi_p \in S(V(A_f)^n)\), and we consider the Eisenstein series \(E(g, s, \Phi_{\infty}^{n+1} \otimes \Phi_f)\) associated to \(\Phi_{\infty}^{n+1} \otimes \Phi_f = \lambda(\varphi_{\infty} \otimes \varphi_f)\).

It ends up that these cases give rise to the same Eisenstein series; in fact, we even have that \(\Phi_f = \lambda_f(\varphi_f)\) is the same in either case. To see this, we use the Bruhat decomposition

\[
G = \bigcup_{j=0}^{n} P w_j P
\]

where

\[
w_j = \begin{pmatrix} 1_{n-j} & 0 & 1_j \\
 & 1_{n-j} & 0 \end{pmatrix}
\]

and formulas for the local Weil representation at \(p\) (see [21])

\[
\omega_p(m(a)) \varphi_p(x) = \chi_p(a)|a|_{p}^{\frac{n+1}{2}} \varphi_p(xa)
\]

\[
\omega_p(n(b)) \varphi_p(x) = \psi_p(bQ[x]) \varphi_p(x)
\]

\[
\omega_p(w_j) \varphi_p(x) = \gamma_p(V_p, \psi_p)^{-j} \int_{V'_p} \psi_p(-2 \text{tr}(z^T Q x_j)) \varphi_p(x_j + z) dz
\]

where \(\gamma_p\) is the local Weil index, an eighth root of unity, and \(x_j\) denotes the last \(j\) coordinates of \(x\).

It suffices to see that \(\Phi_p(g) = \omega(g) \varphi_p(0)\) does not depend on the choice of \(Q\) for \(p < \infty\), which is clear from applying the above formulas with \(x = 0\).

Hence, we may choose to consider the incoherent collection of type (1) above. Kudla has speculated that when we start with \(V\) of signature \((n - 1, 2)\), the central derivative \(\frac{\partial}{\partial s} E(g, s, \Phi_{\infty}^{n+1} \otimes \Phi_f)|_{s=0}\) should be related to a generating function for degrees of 0-cycles on the Shimura variety associated to \(G\text{Spin}(V)\). The Eisenstein series considered in sections 1.3 and 1.4 are the cases \(n = 1, 3\) of this example.

### 1.3 The case \(n = 1\)

In this section, we will take \(F = \mathbb{Q}\) and \(n = 1\), so that \(G = \text{SL}_2(\mathbb{Q})\). Let \(\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times\) be the standard additive character defined so that \(\psi_\infty(x) = e^{2\pi ix}\) and \(\psi\) is trivial on \(\hat{\mathbb{Z}} = \prod_{p < \infty} \mathbb{Z}_p\). Also
let \( \chi : \mathbb{A}^\times / \mathbb{Q}^\times \to \mathbb{C}^\times \) be a quadratic character, and fix squarefree \( \kappa \in \mathbb{Z} \) so that

\[
\chi(x) = (x, \kappa)_\mathbb{A}
\]

where \((\cdot, \cdot)_\mathbb{A} = \prod_p (\cdot, \cdot)_{\mathbb{Q}_p}\) is the global Hilbert symbol. Let \( k \in \mathbb{N} \) with \((-1)^k = \text{sign}(\kappa)\), and let \( \Phi^k_{\infty} \in I_{1, \infty}(s, \chi) \) be the weight \( k \) eigenfunction defined by (5). Given a standard factorizable section \( \Phi = \Phi^k_{\infty} \otimes \Phi_f \) where \( \Phi_f = \otimes_{p < \infty} \Phi_p \), we have the usual factorization (4) of the Fourier coefficient \( E_t(g, s, \Phi) \) for \( t \neq 0 \) into local Whittaker functions. Similarly, we have an expression for the constant term

\[
E_0(g, s, \Phi) = \Phi(g, s) + \prod_p W_{0,p}(g_p, s, \Phi_p).
\]

In this context, [15] gives formulas for the Fourier expansion of

\[
E(\tau, s, \Phi^k_{\infty} \otimes \Phi_f) = v^{-\frac{1}{2}(s+1)}E(g_\tau, s, \Phi^k_{\infty} \otimes \Phi_f),
\]

which we review here. Afterwards, we will apply these results to compute the \( p \)-adic incoherent Eisenstein series associated to an imaginary quadratic field.

To state the relevant results, we set the following notation. Write

\[
W_{t,p}(s, \Phi_p) = W_{t,p}(1, s, \Phi_p) \quad \text{for} \quad p < \infty, \quad \text{and}
\]

\[
W_{t,\infty}(\tau, s, \Phi_\infty) = v^{-\frac{1}{2}(s+1)}W_{t,\infty}(g_\tau, s, \Phi_\infty).
\]

For a finite prime \( p \) set

\[
\sigma_{s,p}(t, \chi_p) = \sum_{r=0}^{\text{ord}_p t} (\chi_p(p)p^r)^r
\]

and for a finite set of finite primes \( S \) set

\[
\sigma^S_s(t, \chi) = \prod_{p \not\in S} \sigma_{s,p}(t, \chi_p) = \prod_{p | t \not\in S} \sigma_{s,p}(t, \chi_p).
\]

Similarly, let

\[
L^S(s, \chi) = \prod_{p \not\in S} L_p(s, \chi_p) = \prod_{p \not\in S} (1 - \chi_p(p)p^{-s})^{-1}
\]

be the \( L \)-function for \( \chi \) with the local factors at all \( p \in S \) removed. In particular note that
\[ \sigma_{s,p}(0, \chi) = L_p(-s, \chi_p). \] We also write

\[ W_{m,S}(s, \Phi_f) = \prod_{p \in S} W_{m,p}(s, \Phi_p). \]

Finally, let \( q^t = e^{2\pi it}. \)

**Proposition 1.3.1** ([15, Props. 2.1, 2.3, 2.6]). Let \( S \) be a finite set of primes so that \( \Phi_p(s) = \Phi_p^0(s) \) for \( p \notin S. \)

1. **For a finite prime** \( p, \)

\[ W_{t,p}(s, \Phi_p^0) = \frac{\sigma_{-s,p}(t, \chi)}{L_p(s+1, \chi_p)}. \]

In particular when \( t = 0 \) we have

\[ W_{0,p}(s, \Phi_p^0) = \frac{L_p(s, \chi_p)}{L_p(s+1, \chi_p)}. \]

2. **When** \( k > 1, \)

\[ W_{t,\infty}(\tau, k-1, \Phi_\infty^k) = \begin{cases} 0, & t \leq 0 \\ \frac{(-2\pi i)^k}{\Gamma(k)} t^{k-1} q^t, & t > 0. \end{cases} \]

3. **For** \( k > 2, E(\tau, k-1, \Phi_\infty^k \otimes \Phi_f) \) is a weight \( k \) holomorphic modular form with Fourier expansion

\[ E(\tau, k-1, \Phi_\infty^k \otimes \Phi_f) = \Phi_f(1, k-1) + \frac{(-2\pi i)^k}{\Gamma(k)L^S(k, \chi)} \sum_{t > 0} W_{t,S}(k-1, \Phi_f) \sigma_{t-k}^S(t, \chi) t^{k-1} q^t. \]

We now apply these formulas to the \( n = 1 \) case described in example 1.2.7, and also considered in [14]. Let \( V \) be a quadratic space of signature \((0, 2)\). Thus \( V \) can be realized as an imaginary quadratic field \( F \) with quadratic form \( Q = -N_{F/\mathbb{Q}} \) given by the norm form. For simplicity, we will take \( F = \mathbb{Q}(\sqrt{-\ell}) \) for a prime \( \ell > 3 \) with \( \ell \equiv 3 \pmod{4} \). Thus

\[ \chi(x) = \chi_V(x) = (x, -\ell)_\mathbb{A}. \]

The section \( \Phi \) is determined by fixing the following “natural” choice of \( \varphi = \otimes_p \varphi_p \in S(V(\mathbb{A})). \) Take \( \varphi_\infty(x) = e^{-2\pi N_{F/\mathbb{Q}}(x)} \) to be the Gaussian. And for \( p < \infty, \) take \( \varphi_p = \varphi_{L_p}^0 \) to be the characteristic function of the lattice \( L_p = \mathcal{O}_F \otimes \mathbb{Z}_p \subset F_p. \) For \( p \neq \ell, \) \( L_p \) is self-dual so that \( \lambda_p(\varphi_{L_p}^0) = \Phi_p^0. \) Proposition 1.3.1 gives us formulas for all Whittaker functions except for \( W_{t,\ell}(s, \Phi_\ell). \) This is remedied here.
Proposition 1.3.2 ([14, Lemma 2.2]). We have

\[ W_{t,\ell}(s, \Phi_\ell) = i\ell^{-1/2} \frac{1 - \chi(t)\ell^{-s(\text{ord}_\ell(t) + 1)}}{L_\ell(s+1, \chi_\ell)}. \]

Corollary 1.3.3. The Fourier expansion of \( \tau(k-1, \Phi_{\infty}^\ell \otimes \Phi_f) \) is 0 for \( k = 1 \), and for odd \( k \geq 3 \)

\[ E(\tau, k, 2^{k-1}L(s, \chi)) = \sum_{t>0} \left(1 - \chi(t)\ell^{(1-k)(\text{ord}_\ell(t) + 1)}\right) \sigma_{-k}^1(t, \chi)t^{k-1}q^t. \]

Proof. After applying Propositions 1.3.1 and 1.3.2, all that remains is to show

\[ \Phi_f(1, k-1) = 1 \quad \text{and} \quad \frac{(-2\pi i)^k i\ell^{-1/2}}{\Gamma(k)L(k, \chi)} = \frac{2^{k-1}}{L(1-k, \chi)}. \]

Since \( \Phi_p = \Phi_0^\ell \) for \( p \neq \ell \), this first claim reduces to showing \( \Phi_\ell(1, k-1) = 1 \). This follows immediately from [14, Eq. 2.10 and Lemma 2.2].

For the second claim we must determine the functional equation of \( L(s, \chi) \). We do this by observing

\[ L(s, \chi) = L(s, \chi) \]

for the mod \( \ell \) Dirichlet character \( \chi \). Indeed, recalling that \( \chi_p(x) = (x, -\ell)_p \) and applying [17, Ch. III, Thm. 1] to compute this local Hilbert symbol, we see that \( \chi \) is ramified only at \( \ell \) and that

\[ \chi_p(p) = \begin{cases} \left(\frac{-\ell}{p}\right), & p \neq 2, \ell \\ (-1)^{\frac{p-1}{2}}, & p = 2 \end{cases} = \left(\frac{p}{\ell}\right) \]

for \( p \neq \ell \). Thus

\[ L(s, \chi) = \prod_{p \neq \ell} \left(1 - \chi_p(p)p^{-s}\right)^{-1} = \prod_{p \neq \ell} \left(1 - \left(\frac{p}{\ell}\right)p^{-s}\right)^{-1} = L(s, \chi). \]

Now, using the functional equation for \( L(s, \chi) \) and special values of \( \Gamma \) at negative half-integers we get

\[ L(k, \chi) = \ell^{s-k} \pi^{k-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} - \frac{k-1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)} L(1-s, \chi) \]

\[ = \ell^{s-k} \pi^k \frac{2^{k-1}(-1)^{\frac{k-1}{2}}}{(k-1)!} L(1-s, \chi). \]
This proves the second claim.

1.3.1 \( p \)-adic interpolation

Just as the classical \( p \)-adic Eisenstein series are formed by interpolating Fourier expansions of the classical Eisenstein series \( E_k(\tau) \) as \( k \) ranges over a \( p \)-adically dense set of weights \( k \in \mathbb{N} \), we shall do the same with the family of Eisenstein series

\[
E_k(\tau) = \frac{1}{2}L(1 - k, \chi)E(\tau, k - 1, \Phi_{\infty}^k \otimes \Phi_f)
\]

\[
= \frac{1}{2}L(1 - k, \chi) + \ell^{k-1} \sum_{t \geq 0} \left(1 - \chi_\ell(t)\ell^{(1-k)(\text{ord}_\ell t + 1)}\right) \sigma_{1-k}(t, \chi) t^{k-1} q^t
\]

for odd \( k \geq 3 \).

Define the \( p \)-stabilization of \( E_k \) by

\[
E^*_k(\tau) = E_k(\tau) - p^{k-1}E_k(p\tau).
\]

Then when \( p \neq \ell \) and

\[
\chi_p(p) = \chi_\ell(p) = \left( \frac{p}{\ell} \right) = 1
\]

we have

\[
E^*_k(\tau) = \frac{1}{2}L(1 - k, \chi)(1-p^{1-k}) + \ell^{k-1} \sum_{t \geq 0} \left(1 - \chi_\ell(t)\ell^{(1-k)(\text{ord}_\ell t + 1)}\right) \sigma_{1-k}(t, \chi) t^{k-1} q^t,
\]

which can be thought of as “removing the \( p \) part” from the Fourier expansion of \( E_k \). Note that \( E^*_k \) is fixed by the operator \( U_p \), defined on Fourier expansions by

\[
\left( \sum a_t q^t | U_p \right) = \sum a_{pt} q^t.
\]

Moreover, the Fourier coefficients of \( E^*_k \) vary \( p \)-adically continuously with \( k \). We can therefore define \( E^*_k \) for

\[
k \in \lim_{\leftarrow}(\mathbb{Z}/p\mathbb{Z})^\times = \mathbb{Z}_p^\times \times \mathbb{Z}/(p - 1)\mathbb{Z} \cong \text{hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)
\]

via (6).

Crucially we find that \( E^*_1 = 0 \). In particular, the non-vanishing of \( L(1, \chi) \), propositions 1.3.1 and
1.3.2, and equation (6) show that for \( t > 0 \)

\[
E^*_{t,k} = \left( \prod_{r \neq p, \infty} W_{t,r}(k-1, \Phi_r) \right) g_t(k)q^t
\]

for some function \( g_t(k) \), where \( E^*_{t,k} \) is the \( t \)th Fourier coefficient. This will allow us to analyze \( \frac{d}{dk} E^*_{t,k} |_{k=1} \).

**Proposition 1.3.4.** Recall that \( V = F = \mathbb{Q}(\sqrt{-\ell}) \) as a quadratic space over \( \mathbb{Q} \) with \( Q = -N_{F/\mathbb{Q}} \). Let \( C \) be the incoherent collection obtained by changing \( V_\infty \) to the space of signature \((2,0)\).

- (a) For \( r \neq \ell, \infty \), \( V_r \) represents \( t \) if and only if \( \chi_r(t) = 1 \). In particular \( V_p \) represents \( t \).
- (b) \( V_{\ell} \) represents \( t \) if and only if \( \chi_{\ell}(t) = -1 \).
- (c) \( \frac{d}{dk} E^*_{t,k} |_{k=1} \neq 0 \) only when \( \text{Diff}(V,C) = \{r\} \) for some \( r \neq p, \infty \).

**Proof.** Using [17, Ch. IV, Th. 6, Corollary] we have that \( V_r \) represents \( t \) if and only if

\[
(t, -\ell)_r = \epsilon_r(Q).
\]

The left-hand side is simply \( \chi_r(t) \). And

\[
\epsilon_r(Q) = (-1, -\ell)_r = \begin{cases} 
1, & r \neq \ell \\
-1, & r = \ell
\end{cases}
\]

giving parts (a) and (b). Although part (c) follows directly from (7) and proposition 1.1.2, it’s worth noticing that one can also see it by using the explicit formulas in propositions 1.3.1 and 1.3.2 along with (7). \( \square \)

We are now able to compute the derivative of \( E^*_{t,k} \) at \( k = 1 \). Let \( \log : \mathbb{Z}_p^* \to \mathbb{Z}_p \) be the \( p \)-adic logarithm.

**Theorem 1.3.5.** Let \( t > 0 \).

- (a) If \( \chi_r(t) = 1 \) for all primes \( r < \infty \) then

\[
\frac{d}{dk} E^*_{t,k} |_{k=1} = \log \ell (\text{ord}_t t + 1) \sigma_0^{p, \ell}(t, \chi).
\]
(b) If $\chi_\ell(t) = -1$, $\chi_r(t) = -1$, and $\chi_u(t) = 1$ for all primes $u \neq \ell, r$ then

$$a_t = \log r(\text{ord}_r t + 1)\sigma_0^{p,\ell,r}(t, \chi).$$

(c) In all other cases, $a_t = 0$.

Proof. By the above discussion, or by computing directly with (6), we see that under the assumptions of part (a), $E_{t,k}^* \frac{1}{1-\chi_\ell(t)e^{(1-k)\text{ord}_\ell t + 1}}$ does not vanish at $k = 1$ so that

$$\frac{d}{dk}E_{t,k}^*|_{k=1} = \frac{d}{dk} \left(1 - \chi_\ell(t)e^{(1-k)\text{ord}_\ell t + 1}\right)|_{k=1} \sigma_0^{p,\ell}(t, \chi) = \log \ell(\text{ord}_\ell t + 1)\sigma_0^{p,\ell}(t, \chi).$$

Similarly in part (b), we see that $\sigma_{1-k,r}(t, \chi)$ is the only vanishing part of $E_{t,k}^*$ at $k = 1$ and

$$\frac{d}{dk}(1-r+\ldots-\chi^{\text{ord}_r t(1-k)})|_{k=1} = \frac{1}{2} \log r(\text{ord}_r t + 1),$$

giving the result. Part (c) follows from 1.3.4. \qed

1.4 The case $n = 3$

We now turn to the $n = 3$ case, considered by Kudla and Rapoport in [12]. Now $V$ is a quadratic space over $\mathbb{Q}$ of signature $(2, 2)$. Fix a $\mathbb{Z}$-lattice $\Lambda$ in $V$; then there is an integer $N$ so that for all $p \nmid N$, $\Lambda_p$ is a self-dual $\mathbb{Z}_p$-lattice in $V_p$ [12, §12]. For all $p$, let $\varphi_p$ be the characteristic function of $\Lambda_p$. Let $\Phi_f = \otimes_p \Phi_p$ be the corresponding section of $I_{3,f}(s, \chi)$ for $\chi = \chi_V$ as in section 1.2. For $k \geq 2$ let also $\Phi_{\infty}(s)$ be as defined in example 1.2.7.

We would like to study the Eisenstein series $E(\tau, s, \Phi_{\infty} \otimes \Phi_f) = (\det v)^{-\frac{1}{2}(s+2)}E(g_{\tau, s}, \Phi_{\infty} \otimes \Phi_f)$ for $\tau = u + iv \in \text{Sym}_3(\mathbb{C})$ with $u$ and $v$ real, where as usual $g_{\tau, \infty} = n(u)m(v^{1/2})$ and $g_{\tau, p} = 1$ for all $p < \infty$. In particular, we would like to say that some appropriate $p$-stabilization of $E(\tau, k - 2, \Phi_{\infty} \otimes \Phi_f)$ varies $p$-adically continuously with $k$. However, it will no longer be feasible to write down explicit formulas for the Fourier coefficients as in section 1.3. A particular challenge is presented by a lack of information about the Whittaker function $W_{T,p}(1, s, \Phi_p)$, since this could provide an obstruction to $p$-adic continuity in some cases. To circumvent these issues we pick $p$ so that $\chi$ does not ramify at $p$, and we will study only the Fourier coefficients $E_T(\tau, s, \Phi_{\infty} \otimes \Phi_f)$ where $T \in \text{Sym}_3(\mathbb{Z})_{>0}$ is positive definite with $p \nmid \det T$. Due to the increased complexity of the computations involved in these genus 3 Eisenstein series, this section will be less self-contained than previous sections. In particular, we will rely more heavily on results in [12], and proposition 1.4.1

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Let $T \in \text{Sym}_n(\mathbb{Z})_{>0}$ with $p \nmid \det T$ and consider the Fourier coefficient $E_T(\tau, s, \Phi^k_{\infty} \otimes \Phi_f)$. We would like to normalize $E_T$ so that $E_T(\tau, k-2, \Phi^k_{\infty} \otimes \Phi_f)$ varies $p$-adically continuously with $k$. To do so, we use results from [9, Appendix], where it is shown that

$$W_{T, \ell}(1, s, \Phi_\ell) = A_{T, \ell}(\ell^{-s})$$

for some polynomial $A_\ell(X) \in \mathbb{Z}[\frac{1}{\ell}][X]$. When $\ell \nmid \det T$ and $\chi$ does not ramify at $\ell$ (note that $p$ satisfies these hypotheses by assumption), we moreover have that

$$W_{T, \ell}(1, s, \Phi_\ell) = \frac{1}{L_\ell(s + 2, \chi)_\ell(2s + 2)} = \frac{1}{(1 - \chi_\ell(\ell)\ell^{-s-2})(1 - \ell^{-2s-2})}.$$  

Additionally, by results of Shimura ([19, §3] along with [18]) when $k \geq 4$ is even, $W_{T, \infty}(g_\tau, k-2, \Phi^k_{\infty})$ is nonzero only when $T \in \text{Sym}_4(\mathbb{Z})_{>0}$, and in this case one has

$$W_{T, \infty}(g_\tau, k-2, \Phi^k_{\infty}) = \frac{(-2\pi i)^k}{\Gamma_n(k)}(\det T)^{k-2}(\det v)^{k/2}q^T$$

where $\Gamma_n$ is a hypergeometric functions nonvanishing on the positive integers and $q^T = e^{2\pi i \text{tr}(Tg_\tau)}$. Since $p \nmid \det T$, we have that $(\det T)^k$ varies $p$-adically continuously with $k$. So in summary, for $k \geq 4$ even we can define

(8) $$E^*_{T, k}(\tau) = L(k, \chi)\zeta(2k-2) \cdot \frac{\Gamma_n(k)}{(-2\pi i)^k} E_{T, k}(\tau, k-2, \Phi^k_{\infty} \otimes \Phi_f) = \prod_{\ell \mid \det T} A_{T, \ell}(\ell^{2-k}) \cdot (\det T)^{k-2} q^T.$$

Also, by the nonvanishing of $L(2, \chi)$ and $\zeta(2)$, we know that $A_{T, \ell}(1) = 0$ exactly when $W_{T, \ell}(1, 0, \Phi_\ell) = 0$. Since $W_{T, \ell}(1, 0, \Phi_\ell)$ vanishes for some $\ell$, we therefore have that $E^*_{T, k}(\tau)$ varies $p$-adically continuously over $k \geq 2$ even with $E^*_{T, 2} = 0$. Thus if $p \neq 2$ we can extend $E^*_{T, k}$ to a function of $k \in \mathbb{Z}_p^\times \times \mathbb{Z}/(p-1)\mathbb{Z}$ via (8).

Despite having only a non-explicit equation (8) giving $E^*_{T, k}$ as a function of $k \in \mathbb{Z}_p^\times \times \mathbb{Z}/(p-1)\mathbb{Z}$, we can nevertheless use the $p$-adic continuity of $E^*_{T, k}$ and the method in [12, §7] to compute $\frac{d}{dk} E^*_{T, k}|_{k=2}$. Let $\mathcal{C}$ be the incoherent collection defined from $V$ by replacing $V_{\infty}$ with a space of signature $(4, 0)$. We know when $|\text{Diff}(T, \mathcal{C})| \geq 2$ that $\frac{d}{dk} E^*_{T, k}|_{k=2} = 0$. Thus suppose that $\text{Diff}(T, \mathcal{C}) = \{\ell\}$ for a prime $\ell \neq p$ with $\ell \nmid \det T$; that is suppose that $\ell$ is the only prime with $T$ not represented by $V_\ell$. For convenience we make the additional technical assumption that $\ell$ is inert in the center of $C^+(V)$, where $C^+(V)$ is the even part of the Clifford algebra of $V$ and its
center is a real quadratic field; however, similar formulas can be obtained without this additional assumption. Let \( V'_\ell \) be the unique 4-dimensional quadratic space over \( \mathbb{Q}_\ell \) with \( \chi_{V'_\ell} = \chi_\ell \) and \( \varepsilon_\ell(V'_\ell) = -\varepsilon_\ell(V_\ell) \). Pick some standard section \( \Phi_\ell \in R_{n,\ell}(V'_\ell) \). Then by propositions 1.1.2 and 1.1.3, we have \( W_{T,\ell}(1, 0, \Phi'_\ell) \neq 0 \). Take \( \Phi'_f = \Phi'_\ell \otimes (\otimes_{r \neq \ell} \Phi_r) \). We have

\[
E_T(g_\tau, k - 2, \Phi^k \otimes \Phi_f) = \frac{W_{T,\ell}(1, k - 2, \Phi_\ell)}{W_{T,\ell}(1, k - 2, \Phi'_\ell)} E_T(g_\tau, k - 2, \Phi^k \otimes \Phi'_f).
\]

Since \( \ell \neq p \) we know from the above that \( W_{T,\ell}(1, k - 2, \Phi_\ell) \) and \( W_{T,\ell}(1, k - 2, \Phi'_\ell) \) are rational functions in \( \ell^{-k} \). Hence, normalizing by the function

\[
G(k) = L(k, \chi) \zeta(2k - 2) \frac{\Gamma_n(k)}{(-2\pi i)^k}
\]

from above, we see that

\[
\frac{d}{dk} E_{T,k}^*|_{k=2} = \left( \frac{d}{dk} W_{T,\ell}(1, k - 2, \Phi_\ell) \right)|_{k=2} W_{T,\ell}(1, 0, \Phi'_\ell)^{-1} G(2) E_T(g_\tau, 0, \Phi^2_\infty \otimes \Phi'_f).
\]

Note that \( \Phi^2_\infty \otimes \Phi'_f \) is not incoherent since we have changed the factor at \( \ell \).

**Proposition 1.4.1.** There is a natural number \( \Delta_T \), a constant \( C \) independent of \( T \) with \( C \cdot G(2) \in \mathbb{Q} \), and a polynomial \( e_T \in \mathbb{Z}[X] \) so that

\[
\frac{d}{dk} E_{T,k}^*|_{k=2} = -C \cdot G(2) \cdot \Delta_T \cdot e_T(\ell) \log \ell
\]

where \( \log \) denotes the \( p \)-adic logarithm.

**Remark 1.4.2.** (a) The quantities \( C \) and \( \Delta_T \) can be defined explicitly by certain volumes and orbital integrals given in [12]. Similarly, there is a simple formula for \( e_T \). However, defining these quantities would take considerable extra work. Instead we will indicate in the proof of proposition 1.4.1 where in [12] these definitions may be found.

(b) The formula for when \( \ell \) is split in the center of \( C^+(V) \) is very similar, but easier; see [12, §11].

**Proof.** We will compute using (9). By the discussion preceding the proposition, \( W_{T,\ell}(1, k - 2, \Phi_\ell) \) is a polynomial in \( \ell^{2-k} \). Using results of Kitaoka on local representation densities [7], Kudla and Rapoport compute this polynomial [12, Proposition 11.5] and take its derivative. Their result gives

\[
\frac{d}{dk} W_{T,\ell}(1, k - 2, \Phi_\ell)|_{k=2} = \gamma(V_\ell)(1 - \ell^{-2})(1 + \ell^{-2}) e_T(\ell) \log \ell
\]
where $\gamma(V_\ell)$ is a Weil index and $e_T(\ell)$ is the quantity denoted by $e_\ell(T)$ in [12, Proposition 6.2]; we also see that when $T \in \text{Sym}_n(\mathbb{Z})$, $e_T(\ell)$ is a polynomial in $\ell$. They also compute $W_{T,\ell}(1, 0, \Phi')$ to be $2\gamma(V'_\ell)\ell^{-4}(\ell^2 - 1)$ using the same method, where $\gamma(V'_\ell)$ is another Weil index and we have $\gamma(V_\ell)/\gamma(V'_\ell) = -1$. Thus

$$\left( \frac{d}{dk} W_{T,\ell}(1, k - 2, \Phi') \right)_{k=2} W_{T,\ell}(1, k - 2, \Phi')^{-1} = -\frac{1}{2}(\ell^2 + 1)e_T(\ell) \log \ell.$$ 

It remains to compute $E_T(g, 0, \Phi^2 \otimes \Phi'_f)$. This is done using the extended Siegel-Weil formula, which relates this value to a certain theta integral (see [9, Corollary 3.3]). One finds

$$E_T(g, 0, \Phi^2 \otimes \Phi'_f) = \frac{C \cdot \Delta_T \cdot q_T}{\ell^2 + 1}$$

where, in the notation of [12, Eq. 7.13] we have

$$\frac{C q_T}{\ell^2 + 1} = \frac{1}{2} \text{vol}(SO(V')(\mathbb{R})) \frac{\text{vol}(\text{pr}(K))}{\ell^2 + 1} W^2_T(g) = \frac{1}{2} \text{vol}(SO(V')(\mathbb{R})) \text{vol}(\text{pr}(K')) W^2_T(g)$$

$$\Delta_T = \text{vol}(K')^{-1} \text{vol}(Z(\mathbb{Q}) \backslash Z(\mathbb{A}_f)) \text{O}_T(\varphi'_f)$$

and $\Delta_T \in \mathbb{N}$ by [12, Eq. 7.9]. Since we know by the above discussion that $C \cdot G(k) \in \mathbb{Q}$, we are done.

\section{$p$-adic degrees of 0-cycles}

Let $V$ be a signature $(n - 1, 2)$ quadratic space over a totally real number field. Given a Schwartz function $\varphi_f \in S(V(\mathbb{A}_f)^{n+1})$, we obtain a corresponding incoherent Siegel Eisenstein series of weight $\frac{n+1}{2}$ and genus $n$ as described in example 1.2.7. Kudla has speculated that the central derivative of this Eisenstein series is related to Arakelov degrees of 0-cycles on (an integral model of) the Shimura variety associated to $GSpin(V)$. In the previous section, we computed central derivatives of the corresponding $p$-adic Siegel Eisenstein series. In this section we will take up the task of computing $p$-adic Arakelov degrees of the 0-cycles.

In section 2.1 we begin by giving a theory of $p$-adic Arakelov divisors on $\text{Spec}(\mathcal{O}_F)$ for a number field $F$. We also explain how to extend the degree map to compute the $p$-adic degrees of 0-cycles on other schemes. Then in sections 2.2 and 2.3 we describe the 0-cycles given by Kudla in the cases $n = 1$ and 3, respectively, and then compute their $p$-adic Arakelov degrees.
2.1 Degrees of \(p\)-adic Arakelov divisors

In this section we give a theory of \(p\)-adic Arakelov divisors which will allow us to compute their degrees, which will be \(p\)-adic numbers. We start by describing the classical theory of Arakelov divisors and their degrees in language amenable to a \(p\)-adic analogue; the exposition here will largely follow Lang [16]. Then, building off of ideas of Besser [2], we will develop the details of this \(p\)-adic analogue. Finally, we define the pushforward of a cycle and use it to more generally define \(p\)-adic Arakelov degrees for 0-cycles on schemes over a number field.

2.1.1 Classical theory

Let \(F\) be a number field with ring of integers \(\mathcal{O}_F\), and let \(Z = \text{Spec}(\mathcal{O}_F)\). We begin by defining the degree of a metrized line bundle on \(Z\). Given a line bundle \(\mathcal{L}\) on \(Z\), we have an associated \(\mathcal{O}_F\)-module \(L\), i.e. the module of sections of \(\mathcal{L}\). We will tacitly identify \(L\) and \(\mathcal{L}\) in what follows, as well as identifying \(L_z = L \otimes_{\mathcal{O}_F} F_z\) for a closed point \(z\) of \(Z\), also viewed as a prime of \(F\).

**Definition 2.1.1.** A metrized line bundle on \(Z\) is a pair \((\mathcal{L}, \{\log_v\})\) of a line bundle \(\mathcal{L}\) with a choice of \(\log\) function \(\log_v : \mathcal{L}_v - 0 \to \mathbb{R}\) for each place \(v \mid \infty\) of \(F\), meaning that

\[
\log_v(ax) = \log |a|_v + \log_v(x)
\]

for \(a \in F_v\) and \(x \in \mathcal{L}_v\).

We want to define the degree of a metrized line bundle. For this, first pick \(0 \neq s \in L_F = L \otimes_{\mathcal{O}_F} F\), so that \(s\) is a global rational section of \(\mathcal{L}\). This corresponds to picking an isomorphism \(\theta : F \cong L_F\) with \(\theta(1) = s\). This realizes \(L\) as the fractional ideal \(\theta^{-1}(L) \subset F\). For a closed point \(z\) of \(Z\), we have \(g_z \in F\) so that

\[
L_z = g_z^{-1} s \mathcal{O}_z,
\]

or equivalently \(\theta^{-1}_z(L_z) = g_z^{-1} \mathcal{O}_z\) where \(\theta_z = \theta \otimes_{\mathcal{O}_F} F_z\) and \(\mathcal{O}_z\) is the ring of integers of \(F_z\). Note that \(g_z\) is unique up to multiplication by an element of \(\mathcal{O}_z^\times\). Thus we have a well-defined value

\[
\deg_{s,z}(\mathcal{L}) = \text{ord}_z(g_z) \log |\mathcal{O}_z/z \mathcal{O}_z| = \text{ord}_z(g_z) \log N_z
\]
where \( N_z = N_{F/Q}(z) \) is the norm of the prime \( z \). For a place \( v|\infty \) of \( F \), we define

\[
\deg_{s,v}(\mathcal{L}, \{\log_v\}) = -[F_v : \mathbb{R}] \log_v(s).
\]

Summing all of these contributions up, we define the degree of \((\mathcal{L}, \{\log_v\})\) to be

\[
\deg_s(\mathcal{L}, \{\log_v\}) = \sum_z \deg_{s,z}(\mathcal{L}) + \sum_v \deg_{s,v}(\mathcal{L}, \{\log_v\}).
\]

This degree is independent of the choice of \( s \). To see this, note that any other choice must be of the form \( as \) for some \( a \in F^\times \). This has the effect of multiplying each \( g_z \) by \( a \). Thus

\[
\deg_{as}(\mathcal{L}, \{\log_v\}) = \sum_z (\text{ord}_z a + \text{ord}_z g_z) N_z - \sum_v [F_v : \mathbb{R}] (\log |a|_v + \log_v(s))
\]

\[
= - \sum_z \log |a|_z - \sum_v [F_v : \mathbb{R}] \log |a|_v + \deg_s(\mathcal{L}, \{\log_v\})
\]

\[
= \deg_s(\mathcal{L}, \{\log_v\})
\]

by the product formula.

Suppose

\[
D = D_f + \sum_{v|\infty} r_v \cdot v
\]

is an Arakelov divisor, meaning that \( D_f = \sum_z r_z \cdot z \), \( r_z \in \mathbb{Z} \), is a Cartier divisor and \( r_v \in \mathbb{R} \) for \( v|\infty \). Then taking \( L = \prod_z z^{-r_z} \) to be a fractional ideal of \( \mathcal{O}_F \) and \( \log_v \) so that \( \log_v(1) = -r_v \), we obtain a metrized line bundle \((\mathcal{L}, \{\log_v\})\). Moreover, we can compute its degree with respect to the section \( 1 \in L_F = F \) to be

\[
\deg_1(\mathcal{L}, \{\log_v\}) = \sum_z r_z \log N_z + \sum_v r_v.
\]

It thus make sense to define

\[
\deg D = \sum_z r_z \log N_z + \sum_v r_v,
\]

which is the usual definition of the degree of an Arakelov divisor.

Before moving on to the \( p \)-adic setting, we reformulate this notion of degree slightly. Let

\[
\ell : K^\times / F^\times \to \mathbb{R}
\]
be a continuous homomorphism. We think of $\ell$ as being a “global log.” Since $\ell_z(O_z^\times) = 0$ for any finite prime $z$, we have that a choice of $\ell$ is equivalent to the data:

- a value of $\ell_z(\pi_z)$ where $\pi_z$ is a uniformizer for each finite prime $z$, and
- homomorphisms $t_v: \mathbb{R} \to \mathbb{R}$ for $v|\infty$ such that the following diagram commutes.

$$
\begin{array}{c}
F_v^\times \\
\downarrow \log|v|_v \\
\mathbb{R} \\
\downarrow t_v \\
\mathbb{R} \\
\end{array}
$$

If $(\mathcal{L}, \{\log_v\})$ is a metrized line bundle, then we pick an isomorphism $\theta: F \xrightarrow{\sim} L_F$ as before. We have that $\theta^{-1}_z(L_z)$ is a fractional ideal of $O_z$, and therefore generated by $\pi_z^r$ for some $r \in \mathbb{Z}$. We abuse notation and write $\ell_z(\theta^{-1}_z(L_z)) = \ell_z(\pi_z^r)$. We can then define the degree of $(\mathcal{L}, \{\log_v\})$ to be

$$
\deg_{\ell, \theta}(\mathcal{L}, \{\log_v\}) = \sum_{v|\infty} t_v(\log_v(\theta_v(1))) - \sum_{z|\infty} \ell_z(\theta^{-1}_z(L_z)).
$$

As before, it is easy to see that this value does not depend on the isomorphism $\theta$. However, it does depend on the choice of $\ell$. To obtain the canonical degree function defined above, we use the following canonical choice of $\ell$:

- $\ell_z(\pi_z) = \log N_z$ for $z < \infty$, and
- $t_v(x) = -x$ for $v|\infty$.

### 2.1.2 $p$-adic analogue

We are now ready to generalize the notion of the degree of an Arakelov divisor to the $p$-adic setting. Strictly speaking, we could do this only for $\text{Spec } \mathbb{Z}$ and then use the pushforward map defined below to lift the definition to an arbitrary number field. But the fun is worth the fine, so we will instead give a definition by generalizing the theory of metrized line bundles from above.

As before, let $F$ be a number field and $Z = \text{Spec}(\mathcal{O}_F)$. Fix some continuous homomorphism $\ell: \mathbb{A}^\times/F^\times \to \mathbb{Q}_p$, and note that $\ell_z(O_z^\times) = 0$ for $z \nmid p$. This determines the data:

- a value of $\ell_z(\pi_z)$ for each finite prime $z \nmid p$,
- a homomorphism $t_v: F_v \to \mathbb{Q}_p$ for each $v|p$ so that the diagram

$$
\begin{array}{c}
O_v^\times \\
\downarrow \log_v \\
F_v \\
\downarrow t_v \\
\mathbb{Q}_p \\
\end{array}
$$

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commutes where \( \log_v : \mathcal{O}_v^\times \to F_v \) is the \( v \)-adic logarithm, and

- a branch of \( \log_v \) so that the above diagram commutes with \( \mathcal{O}_v^\times \) replaced with \( F_v^\times \).

**Definition 2.1.2.** A \( p \)-adic metrized line bundle on \( Z \) is a pair \( (\mathcal{L}, \{ \lg_v \}_v|p) \) consisting of a line bundle \( \mathcal{L} \) on \( Z \) and for each prime \( v|p \) of \( F \) a log function \( \lg_v : L_v - 0 \to \mathbb{Q}_p \), meaning that

\[
\lg_v(ax) = \log_v(a) + \lg_v(x)
\]

for \( a \in F_v \).

Note that the above definition depends on the branch of \( \log_v \) chosen, and therefore on the choice of \( \ell \).

We now define the degree of a \( p \)-adic metrized line bundle \( (\mathcal{L}, \{ \lg_v \}) \). Let \( \theta : F \xrightarrow{} L_F \) be an isomorphism as before. We then define

\[
\deg_{\ell, \theta}(\mathcal{L}, \{ \lg_v \}) = \sum_{v|p} t_v(\lg_v(\theta_v(1))) - \sum_{z|p} \ell_z(\theta_z^{-1}(L_z)).
\]

As before, this value does not depend on \( \theta \), but it may depend on \( \ell \). The canonical choice of \( \ell \) is given by

- \( \ell_z(\pi_z) = \log_p N z \),
- the Iwasawa branch of \( \log_v \) for each \( v|p \) so that \( \log_v(\pi_v) = 0 \), and
- \( t_v(x) = -x \) for \( v|p \).

As before, we define a \( p \)-adic Arakelov divisor \( D \) to be a formal sum

\[
D = D_f + D_\infty
\]

where \( D_f = \sum z \cdot r_z \cdot z \) is a Cartier divisor on \( Z \) and \( D_\infty = \sum_{v|p} s_v \cdot X_v \). Using the canonical \( \ell \) given above and repeating the similar procedure as before for converting \( D \) into a \( p \)-adic metrized line bundle \( (\mathcal{L}, \{ \lg_v \}) \), we get

\[
\deg D = \sum_{z|p} r_z \log N z + \sum_{v|p} s_v.
\]

Note that this only depends on \( L \) as an \( \mathcal{O}_F[\frac{1}{p}] \)-module, or equivalently it does not depend on the \( r_z \) for \( z|p \).
2.1.3 The pushforward map

Let $X$ and $Y$ be schemes over $\mathcal{O}_F$ locally of finite type, and let $f : X \to Y$ be a morphism. Given an irreducible $k$-cycle $Z \subset X$, i.e. a dimension $k$ closed subscheme, we define the pushforward (see [20, Tag 02R3]) of $Z$ by

$$f_*(Z) = \deg(Z/f(Z)) \cdot f(Z)$$

if $\deg(Z/f(Z))$ is finite, and we define the pushforward to be 0 otherwise. Here $\deg(Z/f(Z)) = [F(Z) : F(f(Z))]$ is the degree of extension of the function fields. It is clear that $f_*$ extends to a map from the group of $k$-cycles on $X$ to the group of $k$-cycles on $Y$.

Example 2.1.3. Let $E/F$ be an extension of number fields, giving a morphism

$$f : X = \text{Spec } \mathcal{O}_E \to Y = \text{Spec } \mathcal{O}_F.$$

Let $D = \sum_v r_v \cdot v$ be a divisor on $X$, so that each $v$ is a prime of $E$. We see that $f(v) = w$ is the prime of $F$ lying under $v$ and that

$$\deg(v/w) = [\mathcal{O}_E/v : \mathcal{O}_F/w] = f_v|_w$$

is the inertia degree. Thus

$$f_*(D) = \sum_w r_v f_v|_w \cdot w.$$

Taking the degrees of Arakelov divisors, we find

$$\deg f_*(D) = \sum r_v f_v|_w \log Nw = \sum r_v \log Nv = \deg D.$$

Hence, at least for finite divisors, the degree map commutes with pushforward. Clearly the same holds when taking $p$-adic Arakelov degrees as well.

As hinted at by the above example, for an archimedean place $v$, one can define $f_*(v) = [E_v : F(f(v))]$ so that degree commutes with pushforward on all Arakelov divisors. With these definitions, one could define the ($p$-adic) Arakelov degrees above entirely in terms Arakelov degrees on $\text{Spec } \mathcal{O}_F$ by $\deg(D) = \deg(\pi_*(D))$ where $\pi : \text{Spec } \mathcal{O}_F \to \text{Spec } \mathcal{Z}$ is the natural projection and $D$ is any ($p$-adic) Arakelov divisor on $\text{Spec } \mathcal{O}_F$.

In general, the pushforward is defined for elements of the arithmetic Chow group $\overline{CH}_k(X)$ of a scheme, which consist of $k$-cycles on $X$ with an attached Green current (see [3, §3.6]). However,
the cycles we encounter in the following sections will only have finite components, so we will not
describe this theory here.

If \( D = \sum_{x \in X} r_x \cdot x \) is a 0-cycle on a scheme \( \pi : X \to \text{Spec } \mathcal{O}_F \) then we have a pushforward

\[
\pi_*(D) = r_x \deg(x/\pi(x)) \cdot \pi(x)
\]
to a divisor on \( \text{Spec } \mathcal{O}_F \). Since \( \pi(x) = v \) is a prime of \( \mathcal{O}_F \), we see that \( \deg(x/\pi(x)) \) is simply
the dimension of the field \( \mathcal{O}_x \) over \( \mathcal{O}_F/v \). We define the \((p\text{-adic})\) Arakelov degree of \( D \) to be

\[
\deg(D) = \begin{cases} 
\sum_x \log |\mathcal{O}_{D,x}|, & \text{Arakelov degree} \\
\sum_{x \not\in \pi^{-1}(p)} \log |\mathcal{O}_{D,x}|, & \text{p-adiic Arakelov degree}
\end{cases}
\]

where \( \mathcal{O}_{D,x} \) the local ring at \( x \) of the scheme defined by \( D \) (taking into account point multiplicity),
and \( \log \) denotes the \( p \)-adic logarithm in the second case.

### 2.2 The case \( n = 1 \)

As in section 1.3, let \( F = \mathbb{Q}(\sqrt{-\ell}) \) be an imaginary quadratic field with \( \ell > 3 \) a prime with
\( \ell \equiv 3 \pmod{4} \). In [14, §5], the authors define a moduli space \( M \) of elliptic curves with complex
multiplication by \( \mathcal{O}_F \). Hence for a ring \( R \), the points of \( M(R) \) parametrize pairs \( (E, \iota) \) of an elliptic
curve \( E \) over \( R \) and a complex multiplication \( \iota : \mathcal{O}_F \to \text{End}(E) \) (see [14, Eq. 5.1] for a precise
definition of complex multiplication in this context). The main theorem of complex multiplication
then gives

\[
M \cong \text{Spec}(\mathcal{O}_H)
\]

where \( H \) is the Hilbert class field of \( F \) [14, Cor. 5.4].

Cycles on \( M \) are picked out by elliptic curves with \textit{special endomorphisms}, defined as follows.
Given an elliptic curve \( (E, \iota) \) with complex multiplication by \( \mathcal{O}_F \), \( y \in \text{End}(E) \) is a special endomorphism
if

\[
y \circ \iota(a) = \iota(\bar{a}) \circ y
\]

for all \( a \in \mathcal{O}_F \). We see that the set \( V(E, \iota) \) of special endomorphisms on \( (E, \iota) \) is an \( \mathcal{O}_F \)-module via
the action \( a \cdot y = \iota(a) \circ y \) for \( a \in \mathcal{O}_F \) and \( y \in V(E, \iota) \).

If \( \kappa \) is a field of characteristic 0 and \( (E, \iota) \in M(\kappa) \), we must have \( \iota : \mathcal{O}_R \sim \to \text{End}(E) \) so that
\( V(E, \iota) = 0 \). A more complicated argument shows that \( \iota \) is also an isomorphism when \( \kappa \) is characteristic \( p \) for a prime split in \( F \). Finally, suppose that \( \kappa \) is a field of characteristic \( p \) for a prime nonsplit in \( F \). In this case \( V(E, \iota) \) is an \( \mathcal{O}_F \)-module of rank 1 in a quaternion algebra, and for any \( y \in V(E, \iota) \) we have that \( y^2 \in \mathbb{Z} \). We can thus define a quadratic form \( Q(y) = -y^2 \) on \( V(E, \iota) \), viewed as a \( \mathbb{Z} \)-module of rank 2.

We are ready to define cycles \( Z(t) \) on \( M \) for each \( t \in \mathbb{Z} \). Indeed, one can define \( Z(t) \) as the moduli space of \( (E, \iota, y) \) where \( y \in V(E, \iota) \) with \( Q(y) = t \). Then the \( Z(t) \) are considered as 0-cycles on \( M \) via the forgetful map \((E, \iota, y) \mapsto (E, \iota)\).

We have already seen that the fibre of \( Z(t) \) in characteristic 0 or \( p \) for a prime split in \( F \) is empty. For a prime \( p \) of \( F \) lying over a nonsplit rational prime \( p \), let \( \kappa(p) \) be the algebraic closure of \( \mathcal{O}_F/p \). Then we see that the fibre of \( Z(t) \) over \( M(\kappa(p)) \) is nonempty if and only if \( V(E, \iota) \) represents \( t \).

As in section 1.3, suppose that \( (p \ell) = 1 \) so that the fibre of \( Z(t) \) over \( p \) is empty. We are ready to compute the \( p \)-adic degree of \( Z(t) \), viewed as a \( p \)-adic Arakelov divisor. Define

\[ \rho(t) = |\{a \subset \mathcal{O}_F : N_{F/Q}(a) = t\}| \]

to be the number of integral ideals with norm \( t \).

**Proposition 2.2.1.** Let \( q \) be a rational prime not split in \( F \). Let \( q|q \) be a prime of \( F \).

(a) Let \( z \in Z(t)(\kappa(q)) \). Then \( |\mathcal{O}_{Z(t), z}| = q^{\text{ord}_q t + 1} \).

(b) \( |Z(t)(\kappa(q))| = 2 \cdot \begin{cases} \rho(t/q), & q \neq \ell \\ \rho(t), & q = \ell. \end{cases} \)

**Proof.** Part (a) follows from [14, Theorem 5.11]. Part (b) follows from [14, Corollary 5.14] and the discussion before Theorem 5.15 loc. cit. \( \Box \)

**Corollary 2.2.2.** The \( p \)-adic Arakelov degree of \( Z(t) \) is

\[ \deg Z(t) = 2 \log(\ell) \cdot (\text{ord}_\ell t + 1) \rho(t) + 2 \sum_{q} \log(q) \cdot (\text{ord}_q t + 1) \rho(t/q) \]

where \( \log \) denotes the \( p \)-adic logarithm and the sum is over primes \( q \) inert in \( F \).

Note also that corollary 2.2.2 could have been obtained much more easily from [14, Eq. 6.10] which expresses \( Z(t) \) as a divisor on \( \text{Spec}(\mathcal{O}_H) \). However, the steps carried out here mimic those
used to obtain said equation, so we included them for the sake of completeness.

We proceed to the task of relating $\deg Z(t)$ to the central derivative of the $p$-adic Eisenstein series computed in section 1.3.

**Proposition 2.2.3.** $\rho(t) = \sigma_0'(t, \chi)$, where $\sigma$ and $\chi$ are as defined in section 1.3.

**Proof.** Factor $t$ as $t = \prod q^e_q$ with $e_q = \text{ord}_q t$. If $q$ is split then $\chi_q(q) = 1$ so that

$$\rho(q^{e_q}) = e_q + 1 = \sigma_0(q^0(t, \chi_q))$$

since $q^{e_q}$ can factor as $q_1^k q_2^{e_q-k}$ for each $0 \leq k \leq e_q$ where $q_i$ are the primes of $F$ lying over $q$. If $q$ is inert so that $\chi_q(q) = -1$, we have

$$\rho(q^{e_q}) = \begin{cases} 1, & e_q \mid \text{ord}_q t \\ 0, & e_q \nmid \text{ord}_q t \end{cases} = \sigma_0(q^0(t, \chi_q)).$$

And when $q = \ell$ we have $\rho(\ell^{e_\ell}) = 1$. Thus

$$\rho(t) = \prod q \rho(q^{e_q}) = \sigma_0'(t, \chi)$$

as desired. $\square$

**Corollary 2.2.4.** Let $a_t = \frac{d}{dk} E_{l,k}^*|_{k=1}$ be the $t^{th}$ Fourier coefficient of the $p$-adic Eisenstein series computed in section 1.3. Then we have

$$(10) \quad a_t = \frac{1}{2(\text{ord}_p t + 1)} \deg Z(t).$$

In particular, when $p \nmid t$, the two quantities agree up to a factor of 2.

**Proof.** Since $\rho(t) = 0$ if $2 \nmid \text{ord}_q t$ for any prime $q$ inert in $F$ we see that at most one of the summands in corollary 2.2.2 is nonzero:

- if $\chi_r(t) = 1$ for all primes $r$ then only $\rho(t) \neq 0$,
- if $\chi_\ell(t) = -1, \chi_q(t) = -1$ for some $q$, and $\chi_r(t) = 1$ for all other $r$, then only $\rho(t/q)$ is nonzero, and
- $\deg Z(t) = 0$ in all other cases.
These are exactly the cases given in Theorem 1.3.5. And since \( \sigma_0, p(t, \chi_p) = \text{ord}_p t + 1 \), applying proposition 2.2.3 completes the proof.

### 2.3 The case \( n = 3 \)

Let \( V \) be a quadratic space of signature \((2,2)\) over \( \mathbb{Q} \). In [12], Kudla and Rapoport associated to \( V \) an incoherent genus 3 weight 2 Siegel Eisenstein series and related its central derivative to 0-cycles on the Shimura variety associated to \( \text{GSpin}(V) \). In section 1.4 we formed a \( p \)-adic analogue of the \( T \)-th Fourier coefficient \( E^*_{T,k} \) under suitable hypotheses on \( T \) and \( p \), and we computed the derivative \( \frac{d}{dk} E^*_{T,k} \big|_{k=2} \). In this section, we will compute \( p \)-adic degrees of the 0-cycles defined in [12] and compare the result to the value of \( \frac{d}{dk} E^*_{T,k} \big|_{k=2} \). As in section 1.4, these computations rely heavily on the results in loc. cit., so it will be infeasible to keep the exposition self-contained. In particular, we will do little in the way of describing the relevant moduli space, and proposition 2.3.1 will involve constants not defined here but rather in loc. cit. However, these constants are the same constants appearing in proposition 1.4.1, so that a comparison of the formulas will be possible.

Let \( C(V) = C^+(V) \oplus C^-(V) \) be the Clifford algebra of \( V \). The center \( F \) of \( C^+(V) \) is either a real quadratic field or \( \mathbb{Q} \oplus \mathbb{Q} \), and we can write \( C^+(V) = B_0 \otimes_{\mathbb{Q}} F \) for an indefinite quaternion algebra \( B_0 \) over \( \mathbb{Q} \). In [12, §1] (see also [8] for more details), the authors define a moduli space \( X \) with points parameterizing polarized abelian varieties \( A \) of dimension 8 with an action \( \iota \) of a maximal order in \( C(V) \otimes_{\mathbb{Q}} F \). In particular, we will do little in the way of describing the relevant moduli space, and proposition 2.3.1 will involve constants not defined here but rather in loc. cit. However, these constants are the same constants appearing in proposition 1.4.1, so that a comparison of the formulas will be possible.

Let \( \Lambda \subset V \) be a \( \mathbb{Z} \)-lattice and \( N \) an integer as in section 1.4 so that \( \Lambda_p \subset V_p \) is self-dual for \( p \nmid N \). By adding more factors to \( N \) if necessary, one can ensure that \( \chi = \chi_V \) does not ramify at any \( p \nmid N \), and one can define a model of \( X \) over \( \text{Spec} \mathbb{Z}[\frac{1}{N}] \); from now on \( X \) will refer to this model. As in section 2.2, we will define 0-cycles on \( X \) by considering abelian varieties with special endomorphisms. We define the set of special endomorphisms of a given \((A, \iota)\), \( A \) an abelian variety of dimension 8 and \( \iota : \mathcal{O} \to \text{End}(A) \) an action of a maximal order \( \mathcal{O} \subset C(V) \otimes_{\mathbb{Q}} F \) by

\[
V(A, \iota) = \{x \in \text{End}(A) : x^* = x \text{ and } x \circ \iota(c \otimes a) = \iota(c \otimes \bar{a}) \circ x \text{ for all } c \otimes a \in \mathcal{O}\}
\]

where \( * \) denotes the Rosati involution on \( A \). As before, for \( x \in V(A, \iota) \) we have \( x^2 \in \mathbb{Z} \), giving a \( \mathbb{Z} \)-valued quadratic form \( Q(x) = x^2 \) on the \( \mathbb{Z} \)-module \( V(A, \iota) \).

Thus for \( T \in \text{Sym}^3(T)_{>0} \), one can define a moduli space \( Z(T) \) parameterizing \((A, \iota, x)\) where \((A, \iota)\) corresponds to a point of \( X \) and \( x \in V(A, \iota)^3 \) with \( Q[x] = T \). As before we have a forgetful map \( \text{pr} : Z(T) \to X \). The structure of \( Z(T) \) is subject to the following constraints [12, Theorems 1,2].
• If $|\text{Diff}(T, C)| \geq 2$ then $Z(T)$ is empty.

• If $\text{Diff}(T, C) = \{\ell\}$ then $Z(T)$ is supported in characteristic $\ell$ and consists entirely of supersingular points. If $\ell$ is split in $F$ or $\ell \nmid \det T$ then $Z(T)$ is a 0-cycle on $X$.

From now on we will fix an $\ell$ inert in $F$ and a $T \in \text{Sym}_3(T)_{>0}$ with $\text{Diff}(T, C) = \{\ell\}$ and $\ell \nmid \det T$. We give here the relevant formulas from [12] which will allow us to compute the $p$-adic degree of $Z(T)$; the relevant formulas for $\ell$ split in $F$ can be found in [12, §11]

**Proposition 2.3.1.**  (a) Let $z$ be a point of $Z(T)$ in characteristic $\ell$. Then $|O_{Z(T), z}| = \ell^{e_T(\ell)}$ where $e_T$ is the polynomial of proposition 1.4.1 ([12, Proposition 6.2]).

(b) The number of points of $Z(T)$ in characteristic $\ell$ is the natural number $\Delta_T$ of proposition 1.4.1 ([12, Proposition 7.1]).

**Corollary 2.3.2.** The $p$-adic degree of the 0-cycle $Z(T)$ is given by

$$\deg Z(T) = \Delta_T e_T(\ell) \log \ell$$

where $\ell$ denotes the $p$-adic logarithm.

Finally, we can compare this with the formula found in section 1.4 for $\frac{d}{dk} E^*_T, k|_k=2$ where $E^*_T, k$ is the normalized $T$th Fourier coefficient of an Eisenstein series, and $p \nmid T$ so that $E^*_T, k$ varies $p$-adically continuously with $k$.

**Corollary 2.3.3.** If $p \nmid T$ and $E^*_T, k$ is as defined in section 1.4 then

$$\frac{d}{dk} E^*_T, k|_k=2 = -C \cdot G(2) \deg Z(T)$$

where $C$ and $G(2)$ are as in proposition 1.4.1.

**Remark 2.3.4.**  (a) It is reasonable to expect there to be a relationship along the lines of (10) of which corollary 2.3.3 is the special case for $p \nmid T$. Recall that the obstruction to obtaining such a result is that it is not clear how to $p$-adically stabilize the Eisenstein series so that the Fourier coefficients $E^*_T, k$ for $p|T$ vary $p$-adically continuously as well.

(b) Even supposing a relationship similar to (10) in the $n = 3$ case, note that we still would not have a simple relationship connecting $E^*_k$ and degrees of 0-cycles along the lines of

$$\frac{d}{dk} E^*_k|_k=k_0 = C \sum_T \deg(Z(T)) q_T$$
because of these issues when $p | T$. This is in contrast to Kudla’s work, in which such simple relations do hold.

(c) Despite the concern raised in (b), corollaries 2.2.4 and 2.3.3 nevertheless provide weak evidence that Kudla’s program could have a $p$-adic analogue.

References


