The action of Frobenius and rationality of zeta

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28 September, 2020

1 Introduction

These notes are for a lecture given at STAGE. The goal is to prove the rationality and functional equation of the zeta function of a nice curve \( X/\mathbb{F}_q \). However, these results are less important than the big idea we’ll encounter along the way: the étale cohomology groups \( H^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \) are \( \ell \)-adic representations of the Galois group \( G_{\overline{\mathbb{F}}_q} = \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \). Once this idea is thoroughly explored, the Lefschetz trace formula for étale cohomology will immediately show that the zeta function \( Z(X, T) \in \mathbb{Q}[T] \) can be written

\[
Z(X, T) = \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T)P_2(T) \cdots P_{2d}(T)}
\]

where each \( P_i(T) \in \mathbb{Q}_\ell[T] \) is a characteristic polynomial coming from the \( G_{\overline{\mathbb{F}}_q} \)-representation \( H^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \). After this, proving the rationality and functional equation of \( Z(X, T) \) is a pleasant exercise.

Throughout, \( p \) denotes a prime, and \( q \) is some power of \( p \). We will use \( \ell \) to denote a prime distinct from \( p \).

2 The Galois action on étale cohomology

2.1 Some generalities on étale cohomology

Recall that given a scheme \( X \), we have the étale site \( X_{\text{ét}} \) consisting of étale \( X \)-schemes, with the coverings being collections \( \{U_i \to V\} \) such that the images of the \( U_i \) cover \( V \). Then given an étale sheaf \( \mathcal{F} \in \text{Ab}(X_{\text{ét}}) \) of abelian groups, we define the étale cohomology groups \( H^i(X, \mathcal{F}) \) to be the right derived functors of the global sections functor \( \mathcal{F} \mapsto \mathcal{F}(X) \).

For a prime \( \ell \), we especially care about the étale cohomology \( H^i(X, \mathbb{Z}/\ell^n\mathbb{Z}) \) of the constant sheaves \( \mathbb{Z}/\ell^n\mathbb{Z} \) on \( X \). And we define

\[
H^i(X, \mathbb{Z}_\ell) = \lim_{\overset{\longrightarrow}{n \geq 1}} H^i(X, \mathbb{Z}/\ell^n\mathbb{Z}).
\]

(In particular, this is not the cohomology of the constant sheaf \( \mathbb{Z}_{\ell,X} \)!) And in the same vein, we define

\[
H^i(X, \mathbb{Q}_\ell) = H^i(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.
\]

If the contents of the last two paragraphs terrified you don’t worry, because none of it actually matters. All that matters for us is that we have a cohomology theory \( H^i(-, -) \) which is quite similar to the familiar singular cohomology from algebraic topology. Crucially, this cohomology
theory still supports things like Poincaré duality and some version of the Lefshetz trace formula (both of which we will use later). Outside of the discussion in this section, we won’t need to meddle with the homological algebraic details of étale cohomology, only its formal properties.

Being a cohomology theory, we expect \( H^i(-,-) \) to be covariant in the second variable and contravariant in the first. The covariance of the second variable is immediate from the definition of étale cohomology as a derived functor: if \( f: \mathcal{F} \to \mathcal{G} \) is a map of sheaves on \( X_{\acute{e}t} \), then we have maps \( H^i(f): H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G}) \). One must be more careful for contravariance in the first variable: a morphism \( f: X \to Y \) can’t induce a map \( H^i(Y, \mathcal{G}) \to H^i(X, \mathcal{G}) \) because if \( \mathcal{G} \in \text{Ab}(Y_{\acute{e}t}) \) then \( H^i(X, \mathcal{G}) \) doesn’t make sense. But we do have the following.

**Proposition 2.1.** Let \( f: X \to Y \) be a morphism of schemes. Then there are natural maps

\[
f^*: H^i(Y, \mathcal{G}) \to H^i(X, f^* \mathcal{G}) \quad \text{for} \; \mathcal{G} \in \text{Ab}(Y_{\acute{e}t}).
\]

If \( g: Y \to Z \) is another morphism of schemes, then \((g \circ f)^* = f^* \circ g^*\).

**Proof.** Let

\[
0 \to \mathcal{G} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \ldots,
\]

\[
0 \to f^* \mathcal{G} \to \mathcal{J}^0 \to \mathcal{J}^1 \to \ldots
\]

be injective resolutions of \( \mathcal{G} \) and \( f^* \mathcal{G} \). Since \( f^* \) exact, applying it to the injective resolution of \( \mathcal{G} \) gives a (not usually injective) resolution of \( f^* \mathcal{G} \). But then, since \( f^* \mathcal{G} \to \mathcal{J}^* \) is an injective resolution, the identity map \( f^* \mathcal{G} \to f^* \mathcal{G} \) extends to maps \( f^* \mathcal{I}^i \to \mathcal{J}^i: \)

\[
0 \to f^* \mathcal{G} \to f^* \mathcal{I}^0 \to f^* \mathcal{I}^1 \to \ldots
\]

Since \( f_* \) and \( f^* \) are adjoints, we have an adjunction morphism \( \mathcal{G} \to f_* f^* \mathcal{G} \). Recalling that \( f_* \mathcal{F}(U) = \mathcal{F}(U \times_Y X) \) by definition, we get maps \( f_* f^* \mathcal{T}^i(Y) = f_* f^* \mathcal{T}^i(Y \times_X X) = f^* \mathcal{T}^i(X) \). Composing this with the maps \( f^* \mathcal{T}^i(X) \to \mathcal{J}^i(X) \) defined above and taking cohomology gives the desired maps. \( \square \)

**Remark 2.2.** There is also a map

\[
H^i(Y, f_* \mathcal{F}) \to H^i(X, \mathcal{F})
\]

for \( \mathcal{F} \in \text{Ab}(X_{\acute{e}t}) \) obtained by composing the map of the proposition with the adjunction map \( f^* f_* \mathcal{F} \to \mathcal{F} \)

\[
H^i(Y, f_* \mathcal{F}) \xrightarrow{f^*} H^i(X, f^* f_* \mathcal{F}) \to H^i(X, \mathcal{F}).
\]

A key application of proposition 2.1 is when \( \mathcal{G} = \mathcal{A}_Y \) is the constant sheaf on \( Y \) associated to an abelian group \( A \). Then \( f^* \mathcal{G} = \mathcal{A}_X \) is the constant sheaf on \( X \), so the proposition gives a map

\[
f^*: H^i(Y, A) \to H^i(X, A).
\]

In particular, applying this with \( A = \mathbb{Z}/\ell^n \mathbb{Z} \) and taking an inverse limit gives maps

\[
H^i(Y, \mathbb{Z}_\ell) \to H^i(X, \mathbb{Z}_\ell)
\]

\[
H^i(Y, \mathbb{Q}_\ell) \to H^i(X, \mathbb{Q}_\ell).
\]
2.2 Galois representations

Let \( X/K \) be a scheme over a field \( K \) with separable closure \( K^s \). We let \( G_K = \text{Gal}(K^s/K) \) be the absolute Galois group of \( K \), and we write

\[
\overline{X} = X_{K^s} = X \times_{\text{Spec} K} \text{Spec} K^s
\]

for the base-change of \( X \) to \( K^s \). In this section we describe an action of \( G_K \) on the \( \mathbb{Q}_\ell \)-vector spaces \( H^i(\overline{X}, \mathbb{Q}_\ell) \).

This action is very easy to describe. Let \( \sigma \in G_K \). Since \( \text{Spec}(\sigma) : \text{Spec} K^s \to \text{Spec} K^s \) is a map of \( K \)-schemes, we get a map \( 1 \times \text{Spec}(\sigma) : X \to X \)

By proposition [2.1] we then have a map

\[
(1 \times \text{Spec}(\sigma))^* : H^i(\overline{X}, \mathbb{Q}_\ell) \to H^i(\overline{X}, \mathbb{Q}_\ell).
\]

This map defines the action of \( \sigma \) on \( H^i(\overline{X}, \mathbb{Q}_\ell) \). Note that this is a left action: the contravariance of \( \text{Spec} \) and \( f \mapsto f^* \) cancel out.

This action of \( G_K \) on \( H^i(\overline{X}, \mathbb{Q}_\ell) \) are examples of an all-important object in number theory: \( \ell \)-adic Galois representations.

**Definition 2.3.** An \( n \)-dimensional \( \ell \)-adic Galois representation of \( K \) is a continuous group homomorphism \( \rho : G_K \to \text{GL}_n(\mathbb{Q}_\ell) \).

In fact, \( H^i(\overline{X}, \mathbb{Q}_\ell) \) are the most important examples of Galois representations. You’ve probably even encountered them before: the \( \ell \)-adic Tate module of an elliptic curve \( E/K \) is (the dual of) \( H^1(E, \mathbb{Q}_\ell) \).

We now briefly describe one other very important Galois representation called \( \mathbb{Q}_\ell(1) \), which will show up later in our statement of Poincaré duality. Suppose that \( \ell \neq \text{char}(K) \), and let

\[
\mu_m = \{ \zeta \in K^s : \zeta^m \}
\]

denote the group of \( m \)th roots of unity, equipped with the natural action of \( G_K \). For a prime \( \ell \), we have \( G_K \)-equivariant \( \ell \)-power maps

\[
\cdots \to \mu_{\ell^3} \xrightarrow{\ell} \mu_{\ell^2} \xrightarrow{\ell} \mu_{\ell}.
\]

As \( \mu_{\ell^n} \cong \mathbb{Z}/\ell^n\mathbb{Z} \) for each \( n \geq 1 \), we have that the inverse limit

\[
\mathbb{Z}_\ell(1) = \varprojlim_{n \geq 1} \mu_{\ell^n}
\]

is isomorphic to \( \mathbb{Z}_\ell \) as a \( \mathbb{Z}_\ell \)-module, but also carries a \( G_K \)-action. This is not quite a \( \ell \)-adic representation because \( \mathbb{Z}_\ell \) is not a field, so we tensor by \( \mathbb{Q}_\ell \):

\[
\mathbb{Q}_\ell(1) = \mathbb{Z}_\ell(1) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.
\]

**Example 2.4.** (1) Let \( K = \mathbb{F}_q \). Then \( G_K \) is topologically generated by the \( q \)-power Frobenius map \( \sigma_q : \alpha \mapsto \alpha^q \). We see that \( \sigma_q \) acts on \( \mathbb{Q}_\ell(1) \) as multiplication by \( q \):

\[
\varphi_q((\mu_n)_{n \geq 1}) = (\mu_n^q)_{n \geq 1} = q(\mu_n)_{n \geq 1}.
\]
(2) Let $K = \mathbb{Q}$. For a prime $p \neq \ell$, let $\sigma_p$ be a lift of Frobenius under the map $D_p \to G_{\bar{\mathbb{F}}_p}$ for some absolute decomposition group $D_p \leq G_{\mathbb{Q}}$ at $p$. Using that $\sigma_p|_{\mathbb{Q}(\mu_{\ell^n})}$ is the element of $\text{Gal}(\mathbb{Q}(\mu_{\ell^n})/\mathbb{Q}) \cong (\mathbb{Z}/\ell^n\mathbb{Z})^\times$ corresponding to $p$, we find that $\sigma_p$ acts on $\mathbb{Q}_\ell(1)$ as multiplication by $p$.

$\mathbb{Q}_\ell(1)$ is important because we use it to modify every Galois representation in sight by twisting.

**Definition 2.5.** Let $V$ be an $\ell$-adic Galois representation. The Tate twists of $V$ are defined to be

$$V(n) = V \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(1)^{\otimes n}$$

for $n \in \mathbb{Z}$. Here, if $n < 0$ we define $\mathbb{Q}_\ell(1)^{\otimes n} = \text{Hom}(\mathbb{Q}_\ell(1)^{\otimes (-n)}, \mathbb{Q}_\ell)$ to be the dual of $\mathbb{Q}_\ell(-n)$.

So Tate twists leave the vector space structure unchanged, but modify the Galois action. As an example, here is the statement of Poincaré duality for étale cohomology with $\mathbb{Q}_\ell$-coefficients.

**Theorem 2.6 (Poincaré duality).** If $X/K$ is smooth, proper, and pure of dimension $d$. There is a $G_K$-equivariant homomorphism

$$tr : H^{2d}(X, \mathbb{Q}_\ell)(d) \to \mathbb{Q}_\ell$$

($G_K$ acting trivially on the right) so that the cup product pairing

$$H^i(X, \mathbb{Q}_\ell)(j) \times H^{2d-i}(X, \mathbb{Q}_\ell)(d-j) \to H^{2d}(X, \mathbb{Q}_\ell)(d) \xrightarrow{tr} \mathbb{Q}_\ell$$

is perfect for all $i \geq 0$, $j \in \mathbb{Z}$.

Note that this is richer than a statement solely about vector spaces.

## 3 Frobenii

We now specialize to the case that $X$ is defined over $\mathbb{F}_q$. In this case, $G_{\mathbb{F}_q}$ has a distinguished element, the $q$-power Frobenius $\sigma_q$. We give special names to the endomorphisms $1 \times \sigma_q$ and $1 \times \sigma_q^{-1}$ of $\overline{X}$.

**Definition 3.1.**

(1) The morphism $1 \times \sigma_q : \overline{X} \to \overline{X}$ is the arithmetic Frobenius.

(2) The morphism $1 \times \sigma_q^{-1}$ is the geometric Frobenius.

Why is one Frobenius morphism arithmetic and one geometric? It ends up that the geometric Frobenius is “the same as” the endomorphism $\text{Fr} \times 1$ where $\text{Fr}$ is the Frobenius endomorphism of the scheme $X$. This identification between the geometric object $\text{Fr} \times 1$ and the arithmetic object $1 \times \sigma_q^{-1}$ will be useful later.

### 3.1 Absolute and relative Frobenii

**Definition 3.2.** Let $X/\mathbb{F}_q$ be a scheme. The absolute Frobenius $\text{Fr}_X : X \to X$ of $X$ is the morphism which is the identity on underlying topological spaces and the $q$th power map on structure sheaves (i.e.

$$\text{Fr}_X : \mathcal{O}_X(U) \to \mathcal{O}_X(U) \quad \alpha \mapsto \alpha^q$$

for $U \subseteq X$ open).
The word “absolute” is to contrast $\text{Fr}_X$ with a relative Frobenius defined below. If $\varphi : X \to Y$ is any morphism of $\mathbb{F}_q$-schemes then

$$
\begin{array}{ccc}
X & \xrightarrow{\text{Fr}_X} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\text{Fr}_Y} & Y
\end{array}
$$

commutes because $\alpha \mapsto \alpha^q$ commutes with any ring homomorphism. When $\varphi$ is étale more is true.

**Proposition 3.3.** If $\varphi$ is étale then

$$
\begin{array}{ccc}
X & \xrightarrow{\text{Fr}_X} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\text{Fr}_Y} & Y
\end{array}
$$

is cartesian (i.e. $X \cong Y \times_{Y, \text{Fr}_Y} X$).

**Proof (sketch).** Since the given diagram commutes, the universal property of fiber products gives a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\text{Fr}_X} & X \\
\downarrow & & \downarrow \\
Y \times_{Y, \text{Fr}_Y} X & \xrightarrow{\text{pr}_X} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\text{Fr}_Y} & Y
\end{array}
$$

Since $\varphi$ and $\text{pr}_Y$ are étale ($\text{pr}_Y$ is a base-change of an étale map), so is $\text{Fr}_X/Y$. Moreover, $\text{Fr}_Y$, $\text{Fr}_X$, and $\text{pr}_X$ are universally bijective ($\text{pr}_X$ is a base-change of the universally bijective map $\text{Fr}_Y$); hence, $\text{Fr}_X/Y$ is as well. It ends up that étale universally bijective morphisms are all isomorphisms, giving the result. 

**Corollary 3.4.**

1. The morphism $\text{Fr}_X : X_{\text{ét}} \to X_{\text{ét}}$ of the étale site is naturally isomorphic to the identity.
2. For any $\mathcal{F} \in \text{Ab}(X_{\text{ét}})$, the maps $\text{Fr}_X^* : H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F})$ are the identity.

**Proof (sketch).** Proposition 3.3 gives an isomorphism $\mathcal{F} \cong (\text{Fr}_X)_* \mathcal{F}$. The claims follow by repeatedly using the adjunction of $\text{Fr}_X^*$ and $(\text{Fr}_X)_*$ and computing.

On $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ we still have the absolute Frobenius $\text{Fr}_{\overline{X}}$, but we also have a more interesting Frobenius map that leverages the origin of $\overline{X}$ as a scheme coming from $X/\mathbb{F}_q$.

**Definition 3.5.** Let $X/\mathbb{F}_q$ be a scheme. The relative Frobenius is the morphism

$$
\text{Fr}_X \times 1 : X \to \overline{X}.
$$

We now have four different Frobenii on $\overline{X}$: the arithmetic, geometric, absolute, and relative Frobenii.
Example 3.6. Let $X = \text{Spec} \mathbb{F}_q[t]$. Then $\overline{X} = \text{Spec} \mathbb{F}_q[t]$ and our Frobenii act as:

$$\begin{align*}
\text{Fr}_X &: \sum a_n t^n \mapsto \sum a_n^p t^n \quad \text{("raise everything to the $p$th power")}, \\
\text{Fr}_X \times 1 &: \sum a_n t^n \mapsto \sum a_n t^{pn} \quad \text{("raise variables to the $p$th power")}, \\
1 \times \sigma_q &: \sum a_n t^n \mapsto \sum a_n^p t^n \quad \text{("raise coefficients to the $p$th power")}, \\
1 \times \sigma_q^{-1} &: \sum a_n t^n \mapsto \sum a_n^{1/p} t^n \quad \text{("take $p$th roots of coefficients")}.
\end{align*}$$

From this example, we expect there to be relationships among the various Frobenii. Indeed, we can compute

$$(\text{Fr}_X \times 1) \circ (1 \times \sigma_q) = \text{Fr}_X \times \sigma_q = \text{Fr}_{\overline{X}}$$

where the last equality is since $\text{Fr}_X \times \sigma_q$ is, by definition, the unique dotted arrow that makes

\[ \begin{array}{ccc}
X & \xrightarrow{\text{Fr}_X} & X \\
\downarrow & & \downarrow \\
X & \xrightarrow{1 \times \sigma_q} & X \\
\downarrow & & \downarrow \\
\overline{X} & \xrightarrow{\sigma_q} & \overline{X}
\end{array} \]

commute, which $\text{Fr}_{\overline{X}}$ does. Applying corollary 3.4 gives the following.

**Proposition 3.7.** Let $X/\mathbb{F}_q$ be a scheme.

1. $\text{Fr}_X \times 1$ and $1 \times \sigma_q^{-1}$ are naturally isomorphic as functors $X_{\text{ét}} \to X_{\text{ét}}$.
2. For any $\mathcal{F} \in \text{Ab}(X_{\text{ét}})$ we have $(\text{Fr}_X \times 1)^* = (1 \times \sigma_q^{-1})^*$ as maps $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F})$.

4 Applications to $Z(X, T)$

Recall that for a scheme $X/\mathbb{F}_q$ we have the zeta function

$$Z(X, T) = \exp \left( \sum_{n=1}^{\infty} \frac{|X(\mathbb{F}_{q^n})| T^n}{n} \right) \in \mathbb{Q}[T]$$

Our goal is to prove the rationality and functional equation of $Z(X, T)$ (see corollary 4.4 below). But more important is the following fundamental structural result which relates $Z(X, T)$ to the representation theory of $G_{\mathbb{F}_q}$.

**Theorem 4.1** (Grothendieck). Let $X/\mathbb{F}_q$ be a smooth proper variety of dimension $d$. Then for any prime $\ell \neq p$ we have

$$Z(X, T) = \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T)P_2(T) \cdots P_{2d}(T)}$$

where for each $i = 0, \ldots, 2d$ we write

$$P_i(T) = \det(1 - FT|H^i(X, \mathbb{Q}_\ell))$$

for the characteristic polynomial of $F = (\text{Fr}_X \times 1)^*$ acting on $H^i(X, \mathbb{Q}_\ell)$.
Remark 4.2. If $X$ is geometrically irreducible, then $F$ acts as the identity on $H^0(X, \mathbb{Q}_\ell)$. And by Poincaré duality we have $H^{2d}(X, \mathbb{Q}_\ell) \cong H^0(X, \mathbb{Q}_\ell)^\vee(d)$ as $G_{\mathbb{F}_q}$-modules, so that $F = (1 \times \sigma_q^{-1})^*$ acts on $H^{2d}(X, \mathbb{Q}_\ell)$ as multiplication by $q^d$. Thus in this case we find

$$P_0(T) = 1 - T, \quad P_{2d}(T) = 1 - q^d T.$$  

The key input to this theorem is the Lefschetz trace formula, which gives the number of fixed points of a map in terms of an alternating sum of Betti numbers.

Theorem 4.3 (Grothendieck-Lefschetz trace formula). Let $X$ be a smooth proper variety of dimension $d$ over an algebraically closed field $K$, and let $\ell \neq \text{char}(K)$ be prime. Then for any $f : X \to X$ we have

$$\Delta \cdot \Gamma_f = \sum_{i=0}^{2d} (-1)^i \text{Tr}(f^*|H^i(X, \mathbb{Q}_\ell))$$

where $\Delta \cdot \Gamma_f$ is the intersection number of the graph of $f$ with the diagonal $\Delta \subseteq X \times X$. In particular, if $\Gamma_f$ and $\Delta$ intersect transversely, then $\Delta \cdot \Gamma_f$ is the number of fixed points of $f$.

We will apply this theorem with $f = \text{Fr}_X \times 1$ the relative Frobenius of $X/\mathbb{F}_q$. Since $d \text{Fr}_X = 0$, this intersects the diagonal transversely. Moreover, the fixed points of $(\text{Fr}_X \times 1)^n$ on $X$ are exactly $X(\mathbb{F}_q^n)$. So in this case, the trace formula says

(1) \quad $|X(\mathbb{F}_q^n)| = \sum_{i=0}^{2d} (-1)^i \text{Tr}(F^n|H^i(X, \mathbb{Q}_\ell))$.

Proof (of Theorem 4.4). Using (1) we get

$$Z(X, T) = \exp \left( \sum_{n=1}^{\infty} |X(\mathbb{F}_q^n)| \frac{T^n}{n} \right)$$

$$= \exp \left( \sum_{n=1}^{\infty} \sum_{i=0}^{2d} (-1)^i \text{Tr}(F^n|H^i(X, \mathbb{Q}_\ell)) \frac{T^n}{n} \right)$$

$$= \prod_{i=0}^{2d} \exp \left( \sum_{n=0}^{\infty} \text{Tr}(F^n|H^i(X, \mathbb{Q}_\ell)) \frac{T^n}{n} \right)^{(-1)^i}$$

$$= \prod_{i=0}^{2d} \det(1 - F|H^i(X, \mathbb{Q}_\ell))^{(-1)^i+1}$$

where we’ve use the formula

$$\det(1 - AT)^{-1} = \exp \left( \sum_{n=0}^{\infty} \text{Tr}(A^n) \frac{T^n}{n} \right)$$

for any endomorphism $A$ of a finite-dimensional vector space. \hfill \Box

With this established, we easily obtain the rationality and functional equation of $Z(X, T)$.

Corollary 4.4. Let $X/\mathbb{F}_q$ be a smooth proper variety of dimension $d$.

(1) (Rationality) We have $Z(X, T) \in \mathbb{Q}(T)$. 

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(2) (Functional equation) We have

\[ Z \left( X, \frac{1}{q^{dT}} \right) = \pm q^{dX/2}T^X Z(X, T) \]

where \( \chi = \sum_{i=0}^{2d} (-1)^i \dim_{\mathbb{Q}_\ell} H^i(X, \mathbb{Q}_\ell) \) is the Euler characteristic of \( X \).

Proof. For (1), Theorem 4.1 gives \( Z(X, T) \in \mathbb{Q}_\ell(T) \). But also \( Z(X, T) \in \mathbb{Q}[T] \) by definition. As \( \mathbb{Q}_\ell(T) \cap \mathbb{Q}[T] = \mathbb{Q}(T) \), we are done.

Remark 4.5. This does not show that each \( P_\ell(T) \in \mathbb{Q}[T] \). This is true, however, by an important theorem of Deligne.

For (2), we simplify notation by writing \( H^i \) for \( H^i(X, \mathbb{Q}_\ell) \) and \( b_i = \dim_{\mathbb{Q}_\ell} H^i \) for the \( i \)th Betti number. The key idea is to use Poincaré duality to relate each \( P_\ell(T) \) to \( P_{2d-i}(T) \) and multiply them together pairwise. Indeed, let \( \langle c, d \rangle = \text{tr}(c \cup d) \) be the perfect pairing \( H^i \times H^{2d-i} \to \mathbb{Q}_\ell \) of Theorem 2.6. Since \( F = (1 \times \sigma_q^{-1})^* \) by proposition 3.7 and the \( G_{\mathbb{Z}_q} \)-action on \( \mathbb{Q}_\ell \) is trivial, we have

\[ \langle Fc, d \rangle = F(\langle c, F^{-1}d \rangle) = \langle c, F^{-1}d \rangle. \]

In other words, the diagram

\[
\begin{array}{ccc}
H^{2d-i} & \xrightarrow{\cong} & (H^i(d))^\vee \\
\downarrow \quad \phi_{2d-i} & & \downarrow \\
H^{2d-i} & \xrightarrow{\cong} & (H^i(d))^\vee
\end{array}
\]

commutes, where the horizontal arrows are the isomorphism \( d \mapsto (c \mapsto \langle c, d \rangle) \) and the right vertical arrow is the dual of the action of \( F = (1 \times \sigma_q^{-1})^* \) on \( H^i(d) \). By example 2.4 and proposition 3.7, \( F \) acts on \( \mathbb{Q}_\ell(d) \) by multiplication by \( q^{-d} \). Thus, if \( A \) is the matrix of \( F \) acting on \( H^i \), then the matrix of \( F \) acting on \( H^{2d-i} \) is

\[ (q^{-d}A^T)^{-1} = q^d(A^T)^{-1}. \]

This enables the following computation.

\[
P_\ell \left( \frac{1}{q^{dT}} \right) = \det \left( 1 - F \frac{1}{q^{dT}} | H^i \right) \\
= \det \left( -\frac{1}{q^{dT}} | H^i \right) \det(F | H^i) \det(1 - q^dF^{-1}T | H^i) \\
= (-1)^{b_i} \left( \frac{1}{q^{dT}} \right)^{b_i} \det(F | H^i) \det(1 - FT | H^{2d-i})
\]

The rest of the proof is bookkeeping. Let \( \alpha_1, \ldots, \alpha_{b_i} \) be the eigenvalues of \( F \) on \( H^i \). Then the eigenvalues of \( F \) on \( H^{2d-i} \) are \( q^d\alpha_1^{-1}, \ldots, q^d\alpha_{b_i}^{-1} \). So

\[ \det(F | H^i) \det(F | H^{2d-i}) = q^{db_i}. \]

So multiplying \( P_\ell(T) \) with \( P_{2d-i} \) for \( i \neq d \) takes care of nearly all terms.

For \( i = d \) we need a slightly finer analysis. Let

\[
N_+ = \# \{ j : \alpha_j = q^{d/2} \} \\
N_- = \# \{ j : \alpha_j = -q^{d/2} \}.
\]
Then all $b_d - N_+ - N_-$ eigenvalues $\alpha$ of $F$ on $H^d$ such that $\alpha \neq \pm q^{d/2}$ form pairs whose product is $q^d$. Hence
\[
\det(F|H^d) = (-1)^{N_-} q^{d(N_+ + N_-)} q^{d(b_d - N_+ - N_-)/2} = (-1)^{N_-} q^{db_d/2}.
\]

All in all, we have
\[
Z \left( X, \frac{1}{q^{dT}} \right) = \left[ P_0 \left( \frac{1}{q^{dT}} \right) P_{2d} \left( \frac{1}{q^{dT}} \right) \right]^{-1} \cdot \left[ P_1 \left( \frac{1}{q^{dT}} \right) P_{2d-1} \left( \frac{1}{q^{dT}} \right) \right] \cdots P_d \left( \frac{1}{q^{dT}} \right) \pm
\]
\[
= \left[ (-1)^{b_0+b_{2d}} q^{dT T} - b_0 - b_{2d} q^{d(b_0+b_{2d})/2} P_0(T) P_{2d}(T) \right]^{-1} \cdots (-1)^{N_-} q^{db_d/2} P_d(T)
\]
\[
= (-1)^{N_-} q^{dT T} X Z(X,T)
\]
as desired. \qed