Pontryagin Duality
(and Fourier inversion and the Plancherel theorem, oh my!)

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Let $G$ be an abelian locally compact topological group. Let $m$ be a Haar measure on $G$. For any measurable $f : G \to \mathbb{C}$ and $1 \leq p < \infty$, define the $L^p$-norm

$$
\|f\|_p = \left( \int_G \, dx \, |f(x)|^p \right)^{1/p} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.
$$

This gives $C_c(G)$ the structure of a pre-Banach space over $\mathbb{C}$.

Write $L^p(G)$ for the completion of $C_c(G)$ with respect to $\|\cdot\|_p$. Then $L^p(G)$ is a Banach space over $\mathbb{C}$, and recall we can identify it with

$$
\{f : G \to \mathbb{C} \mid f \text{ is measurable, and } \|f\|_p < \infty\}/\sim,
$$

where $f \sim g$ if and only if $f = g$ outside a subset of measure zero, i.e. almost everywhere.

**Definition**

Let $f : G \to \mathbb{C}$ be in $L^1(G)$. Its **Fourier transform** is the function $\hat{f} : \hat{G} \to \mathbb{C}$ given by $\chi \mapsto \int_G \, dx \, f(x) \chi^{-1}(x)$. 
Note that the triangle inequality yields

\[ |\hat{f}(\chi)| = \left| \int_G \, dx \, f(x)\chi^{-1}(x) \right| \leq \int_G \, dx \, |f(x)| = \|f\|_1. \]

Example

- Let \( G = \mathbb{Z}/n\mathbb{Z} \) with the discrete topology, and let \( m \) be the counting measure. Then every function \( f : G \rightarrow \mathbb{C} \) lies in \( L^1(G) \), and \( \hat{f}(\zeta) = \sum_{k=1}^{n} f(k)\zeta^{-k} \) for any \( n \)-th root of unity \( \zeta \).

- Let \( G = \mathbb{Z} \) with the discrete topology, and let \( m \) be the counting measure. Then \( f : G \rightarrow \mathbb{C} \) lies in \( L^1(G) \) if and only if \( \sum_{k=-\infty}^{\infty} |f(k)| \) is finite, and in that case \( \hat{f}(z) = \sum_{k=-\infty}^{\infty} f(k)z^{-k} \) for any \( z \) in \( S^1 \).

- Let \( G = S^1 \), and let \( m \) be the pushforward of the Lebesgue measure via \( \varphi : [0, 1] \rightarrow S^1 \). Let \( f : G \rightarrow \mathbb{C} \) be in \( L^1(G) \). Then

\[ \hat{f}(k) = \int_{S^1} \, dz \, f(z)z^{-k} = \int_{0}^{1} \, dx \, f(\varphi(x))e^{-2\pi kix}, \]

i.e. \( \hat{f}(k) \) is the \( k \)-th Fourier coefficient of the periodic function \( f \circ \varphi \).
Theorem (Plancherel)

There exists a Haar measure $\hat{m}$ on $\hat{G}$ such that, for all $f$ in $L^1(G) \cap L^2(G)$, its Fourier transform $\hat{f}$ lies in $L^2(\hat{G})$ and satisfies $\|f\|_2 = \|\hat{f}\|_2$. Furthermore, the set of all such $\hat{f}$ is dense in $L^2(\hat{G})$.

We call $\hat{m}$ the dual measure on $\hat{G}$. Note this implies that $f \mapsto \hat{f}$ extends to an isometry $L^2(G) \sim \to L^2(\hat{G})$, which we also denote using $\hat{\cdot}$.

Next, let $x$ be in $G$. Consider the group homomorphism $\text{ev}_x : \hat{G} \to S^1$ given by $\chi \mapsto \chi(x)$.

Proposition

The homomorphism $\text{ev}_x$ is continuous.

Proof.

We have to show $\text{ev}_x^{-1}(N(1)) = \{\chi \in \hat{G} \mid \chi(x) \subseteq N(1)\}$ is open. But this equals $W(\{x\}, 1, \sqrt{3})$, so it’s open.

Hence we get a map $\text{ev} : G \to \hat{G}$, which we see is a group homomorphism.
Theorem (Pontryagin duality)

The map $ev$ is an isomorphism of topological groups.

Example

- Let $G = \mathbb{Z}/n\mathbb{Z}$ with the discrete topology. Recall we identified $\hat{G} \sim \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$ via $\chi \mapsto \chi(1)$. By choosing a primitive $n$-th root of unity, we see that $\mathbb{Z}/n\mathbb{Z} \sim \hat{G}$ under this identification via $k \mapsto (\zeta \mapsto \zeta^k)$. As $\chi(1)^k = \chi(k)$, this shows ev is an isomorphism of groups. Since $\hat{G}$ is discrete, it’s a homeomorphism.

- Let $G = \mathbb{Z}$ with the discrete topology. Similarly, we have $\hat{G} \sim S^1$ via $\chi \mapsto \chi(1)$. Recalling also that $\mathbb{Z} \sim \hat{G}$ via $k \mapsto (z \mapsto z^k)$, the observation $\chi(1)^k = \chi(k)$ shows that ev is an isomorphism of topological groups here too.

We use ev to identify $\hat{G}$ with $G$. Thus $m$ yields a Haar measure on $\hat{G}$. 


Theorem (Fourier inversion)

Let \( f \) be in \( L^2(G) \). Then \( f(x) = \hat{f}(x^{-1}) \) almost everywhere on \( G \).

Example

Let \( G = S^1 \) with the usual measure \( m \), and let \( f \) be in \( L^2(G) \). Since \( \hat{m} \) is a Haar measure on \( \hat{G} = \mathbb{Z} \), it equals \( c \) times the counting measure for some \( c > 0 \). Taking \( f = 1 \) in the Fourier inversion formula yields

\[
1 = c \sum_{k=-\infty}^{\infty} \hat{f}(k)(z^{-1})^{-k} = c,
\]

since \( \hat{f} \) equals the indicator function on 0. Thus \( \hat{m} \) equals the counting measure. For general \( f \) in \( L^2(G) \), Fourier inversion then becomes

\[
f(\varphi(x)) = f(z) = \sum_{k=-\infty}^{\infty} \hat{f}(k)(z^{-1})^{-k} = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ki x},
\]

where we set \( z = \varphi(x) \). This is precisely the Fourier expansion of \( f \circ \varphi \).
Next, let’s discuss Pontryagin duality relates to closed subgroups.

Proposition

Let \( H \) be a closed subgroup of \( G \).

1. \( H \) is an abelian locally compact topological group.
2. \( G/H \) is an abelian locally compact topological group.

Proof.

1. Now \( H \) is immediately an abelian Hausdorff topological group. For any open subset \( U \) of \( G \) with compact closure, \( W = U \cap H \) is open in \( H \), and \( \text{cl}_H W = \text{cl}_G U \cap H \) is a closed subset of \( \text{cl}_G U \), thus compact.

2. Since \( H \) is closed, we see \( G/H \) is an abelian Hausdorff topological group. Let \( U \) be a neighborhood of \( 1 \) in \( G \) with compact closure. Because the quotient map \( \pi : G \to G/H \) is open, we see \( \pi(U) \) is a neighborhood of \( 1 \) in \( G/H \). Now \( \pi(\overline{U}) \) is compact and hence closed, as \( G/H \) is Hausdorff. Thus \( \overline{\pi(U)} \subseteq \pi(\overline{U}) \) is also compact.

Note that we can identify \( \hat{G}/H \) with \( \{ \chi \in \hat{G} \mid \chi(H) = 1 \} \) as groups.
**Proposition**

This identifies \( \hat{G}/H \) as a closed subgroup of \( \hat{G} \), and we have a short exact sequence of topological groups \( 1 \to \hat{G}/H \to \hat{G} \to \hat{H} \to 1 \), where \( \hat{G} \to \hat{H} \) is given by restriction.

**Example**

Let \( G = F \) be a local field, and let \( \psi : G \to S^1 \) be a nontrivial continuous homomorphism. For any \( a \) in \( G \), the homomorphism \( \psi_a : G \to S^1 \) given by \( x \mapsto \psi(ax) \) is continuous, since multiplication by \( a \) is continuous. I claim this yields an isomorphism \( \psi : G \to \hat{G} \) of topological groups.

It is injective because if \( \psi(ax) = 1 \) for all \( x \) in \( G \), the nontriviality of \( \psi \) implies that \( a = 0 \). Next, consider the closed subgroup \( H = \overline{\psi(G)} \) of \( \hat{G} \). We can identify \( \hat{G}/H \) with the group \( \{ \chi \in \hat{G} \mid \chi(H) = 1 \} \). This group is trivial, since \( H \supseteq \overline{\psi(G)} \), and if \( \psi(ax) = 1 \) for all \( a \) in \( G \), then \( x = 0 \) as before. Thus the proposition shows \( \hat{G} \sim \hat{H} \), and Pontryagin duality gives \( H = \hat{G} \).
Example (continued)

If we could show \( \psi \) is a homeomorphism onto its image, we’d be done, because \( \psi(G) \) would be locally compact and hence closed. For continuity, let \( a \) be in \( G \), and consider the neighborhood \( W(B_c(0, r), 1, \sqrt{3})\psi_a \) of \( \psi_a \). As \( \psi \) is continuous, we see \( \psi(VB_c(0, r)) \) lies in \( N(1) \) for a small enough neighborhood \( V \) of \( 1 \). Thus \( \psi(V) \) lies in \( W(B_c(0, r), 1, \sqrt{3}) \), implying that \( \psi \) sends \( V + a \) to \( W(B_c(0, r), 1, \sqrt{3})\psi_a \).

For openness, let \( x_0 \neq 0 \) in \( G \) satisfy \( \psi(x_0) \neq 1 \), and consider the neighborhood \( B_o(a, \epsilon) \) of \( a \). Any \( \psi_b \) in \( W(B_c(0, |x_0|/\epsilon), 1, |\psi(x_0) - 1|) \) must not have \( x_0 \) in \( bB_c(0, |x_0|/\epsilon) \). Therefore \( |x_0| > |b|(|x_0|/\epsilon) \) and hence \( \epsilon > |b| \), implying that \( \psi^{-1} \) sends \( W(B_c(0, |x_0|/\epsilon), 1, |\psi(x_0) - 1|) \) to \( B_o(a, \epsilon) \).

Our flexibility in choosing \( \psi \) for this isomorphism is convenient for making calculations.
Example

Let $G = \mathbb{R}$, and let $m$ be the Lebesgue measure on $G$. Choose $\psi = \varphi$, and let $f$ be in $L^1(G)$. Under the above identification, the Fourier transform of $f$ is given by

$$\hat{f}(a) = \int_{\mathbb{R}} dx \, f(x)\psi_a(x)^{-1} = \int_{-\infty}^{\infty} dx \, f(x)e^{-2\pi ai x},$$

i.e. it’s the usual Fourier transform. Since $\hat{m}$ is a Haar measure on $\hat{G} \cong G$, it equals $c$ times $m$ for some $c > 0$. Taking $f(x) = e^{-\pi x^2}$ in the above yields $\hat{f}(a) = e^{-\pi a^2}$. Thus $c = 1$, i.e. the Lebesgue measure on $\mathbb{R}$ is self-dual with respect to this choice of $\psi$.

Suppose now that $f$ lies in $L^1(G) \cap L^2(G)$. Fourier inversion then becomes

$$f(x) = \int_{\mathbb{R}} da \, \hat{f}(a)\psi_a(-x)^{-1} = \int_{-\infty}^{\infty} da \, \hat{f}(a)e^{2\pi ai x},$$

i.e. it’s the classic Fourier inversion formula.