A Compact Kähler Surface of Negative Curvature Not Covered by the Ball

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A compact Kähler surface of negative curvature not covered by the ball

By G. D. Mostow and Yum-Tong Siu

The uniformization theorem for Riemann surfaces says that a simply connected Riemann surface must be the Riemann sphere, the whole complex plane, or the open unit disc. In the higher dimensional case there is no such simple trichotomy, because generic slight perturbations of the ball give rise to complex manifolds no two of which are biholomorphic [1]. However, it is conjectured that with reasonable curvature assumptions a similar trichotomy exists in the higher dimensional case. Corresponding to the case of the Riemann sphere, one has the Frankel conjecture that a compact Kähler manifold of positive sectional curvature must be biholomorphic to the complex projective space. This conjecture was proved in the dimension 2 case by Andreotti-Frankel [2] and in the dimension 3 case by Mabuchi [5]. Very recently the general case was proved independently by Mori [6] using algebraic geometry of positive characteristic and by Siu-Yau [11] using the complex-analyticity of harmonic maps. (Mori's result is stronger than the result of Siu-Yau. Mori's result assumes only that the manifold has ample tangent bundle, whereas the result of Siu-Yau assumes that the manifold has positive holomorphic bisectional curvature.)

Corresponding to the case of the complex plane, one has the conjecture that a noncompact complete Kähler manifold of positive sectional curvature must be biholomorphic to some $\mathbb{C}^n$. Or, more generally, a noncompact simply connected complete Kähler manifold with sectional curvature $K \geq -A/r^{2+\epsilon}$ (or even with the weaker assumption $K \geq -k(r)$ with $k(r) \geq 0$ and $\int_0^\infty rk(r)dr < \infty$) must be obtained from some $\mathbb{C}^n$ by some proper modifications, where $A$, $\epsilon > 0$ and $r$ is the distance from some fixed point. Siu-Yau [10] proved that a noncompact simply connected complete Kähler manifold with $0 \geq K \geq -A/r^{2+\epsilon}$ must be biholomorphic to some $\mathbb{C}^n$. Greene-Wu [3] generalized their result to the case $0 \geq K \geq -k(r)$ with $k(r) \geq 0$ and...
\[ \int_0^\infty rk(r)dr < \infty. \]

Corresponding to the case of the ball, one has the conjecture that a simply connected complete Kähler manifold of negative sectional curvature must be biholomorphic to a bounded domain in \( \mathbb{C}^n \). A weaker version of this conjecture says that on such a manifold there should be enough bounded holomorphic functions to separate points and give local coordinates. Up to now no one has yet succeeded in producing a single nonconstant bounded holomorphic function on such a manifold, even if one assumes that the manifold is the universal covering of a compact Kähler manifold of negative sectional curvature.

There is a second conjecture which says that the universal covering of a compact Kähler manifold of negative sectional curvature should be biholomorphic to the ball. This second conjecture is encouraged by the following results of Wong [12] and Yang [13]. Wong's result is that a bounded domain in \( \mathbb{C}^n \) with smooth strongly pseudoconvex boundary whose automorphism group is noncompact must be the ball. His method of proof can also yield the result that a bounded domain in \( \mathbb{C}^n \) with smooth boundary which covers a compact manifold must be the ball. Yang's result is that a bounded symmetric domain of rank \( >1 \) cannot admit a complete Kähler metric with its holomorphic bisectional curvature bounded between two negative constants. In particular, a bounded symmetric domain of rank \( >1 \) cannot cover a compact Kähler manifold with negative bisectional curvature. In this paper we give a counterexample to the second conjecture by constructing a compact Kähler surface of negative sectional curvature whose universal covering is not biholomorphic to the open 2-ball. The surface is constructed by using the result of Mostow [7], [8] on groups generated by complex reflections. The Kähler metric on the surface is constructed by delicately piecing together the Poincaré metric of the 2-ball and the Bergman metric of the domain \( |z_1|^4 + |z_2|^2 < 1 \) in \( \mathbb{C}^2 \). The universal covering is shown to be non-biholomorphic to the ball by computing \( c_i/c_2 \) and verifying that it is not equal to 3. The ratio \( c_i/c_2 \) is 852/298 which is \( >2 \). Hence its index is positive. Besides the compact quotients of the complex 2-ball, the only known compact complex surface of positive index are the Kodaira surfaces [4]. Since the Kodaira surfaces admit nontrivial holomorphic deformation and the surface constructed in this paper is rigid, our surface is a new algebraic surface. Our surface is also the first known example of a negatively curved compact Riemannian fourfold which is not diffeomorphic to a locally symmetric manifold, because the Pontrjagin number \( p \), of our surface is
nonzero and the Pontrjagin number of any compact quotient of the real 4-ball, being proportional to that of the real projective 4-space which is covered by the 4-sphere cobordant to zero, is zero.

The method used to construct this surface actually can be used to construct an infinite discrete family of similar surfaces with negative sectional curvature not covered by the ball. This family is discussed in the last section of this paper. The maximum of $c_1/c_2$ for surfaces in this family is 2.9525 for the surface whose parameters are $p = 5$ and $\sigma = 6$ (for the meanings of $p$ and $\sigma$ see the last section of this paper).

All these surfaces have very strongly negative curvature tensor in the sense of [9]. Hence for them strong rigidity holds.

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1. Sectional curvature of Kähler metric

Suppose $M$ is a complex manifold of dimension $n$ with local holomorphic coordinate system $z^1, \ldots, z^n$. Let $2 \text{Re} \sum g_{\bar{a}\bar{b}} dz^a dz^b$ be a Kähler metric on $M$. The curvature tensor is given by

$$R_{\bar{a}\bar{b}\bar{c}\bar{d}} = \partial_{\bar{b}} \partial_{\bar{c}} g_{\bar{a}\bar{d}} - \sum_{\bar{a} \bar{b}} g_{\bar{b}\bar{c}} \partial_{\bar{a}} g_{\bar{c}\bar{d}} \partial_{\bar{b}} g_{\bar{a}\bar{d}}.$$ 

Let $p, q$ be two real vectors of the complexified tangent space at a point of $M$. We want to calculate the sectional curvature at the plane spanned by $p$ and $q$. We can write

$$p = 2 \text{Re} \xi,$$

$$q = 2 \text{Re} \eta$$

where

$$\xi = \sum_a \xi^a \frac{\partial}{\partial z^a},$$

$$\eta = \sum_a \eta^a \frac{\partial}{\partial z^a}.$$
We use italic lower case letters $h, i, j, k$, etc. to run through the range $1, \cdots, n$, $\tilde{1}, \cdots, \tilde{n}$ and keep the range of the Greek lower case letters $\alpha, \beta, \gamma, \delta$, etc. to be $1, \cdots, n$. Recall

$$p^a = \xi^a,$$

$$\tilde{p}^a = \tilde{\xi}^a,$$

$$q^a = \eta^a,$$

$$\tilde{q}^a = \tilde{\eta}^a,$$

and recall the following properties of the curvature tensor. The only nonzero components of $R_{hijk}$ are of the form

$$R_{\alpha\beta\gamma\delta}, \ R_{\alpha\beta\tilde{\gamma}\tilde{\delta}}, \ R_{\beta\alpha\gamma\delta}, \ R_{\beta\alpha\tilde{\gamma}\tilde{\delta}}$$

and they satisfy the symmetry relations

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = R_{\beta\alpha\gamma\delta},$$

$$R_{\gamma\beta\alpha\delta} = R_{\alpha\beta\tilde{\gamma}\tilde{\delta}} = R_{\alpha\beta\gamma\delta},$$

$$R_{\beta\alpha\tilde{\gamma}\tilde{\delta}} = R_{\alpha\beta\tilde{\gamma}\tilde{\delta}}.$$

The sectional curvature at the plane spanned by $p$ and $q$ is

$$\sum_{h,i,j,k} R_{hijk} p^h q^i p^j q^k \over \|p \wedge q\|^2.$$

Let $\xi^a = \tilde{\xi}^a$ and $\eta^a = \tilde{\eta}^a$. Then

$$\sum_{h,i,j,k} R_{hijk} p^h q^i p^j q^k = \sum_{\alpha, \beta, \gamma, \delta} (R_{\alpha\beta\gamma\delta} \xi^\alpha \eta^\beta \xi^\gamma \eta^\delta + R_{\alpha\beta\gamma\delta} \tilde{\xi}^\alpha \eta^\beta \xi^\gamma \eta^\delta + R_{\alpha\beta\gamma\delta} \xi^\alpha \eta^\beta \tilde{\xi}^\gamma \tilde{\eta}^\delta + R_{\alpha\beta\gamma\delta} \tilde{\xi}^\alpha \eta^\beta \tilde{\xi}^\gamma \tilde{\eta}^\delta)

= \sum_{\alpha, \beta, \gamma, \delta} (R_{\alpha\beta\gamma\delta} \xi^\alpha \eta^\beta \xi^\gamma \eta^\delta - R_{\alpha\beta\gamma\delta} \tilde{\xi}^\alpha \eta^\beta \xi^\gamma \eta^\delta - R_{\alpha\beta\gamma\delta} \xi^\alpha \eta^\beta \tilde{\xi}^\gamma \tilde{\eta}^\delta + R_{\alpha\beta\gamma\delta} \tilde{\xi}^\alpha \eta^\beta \tilde{\xi}^\gamma \tilde{\eta}^\delta)

= \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\beta\gamma\delta} (\xi^\alpha \eta^\beta - \eta^\alpha \xi^\beta)(\xi^\gamma \eta^\delta - \eta^\gamma \xi^\delta)

= -\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\beta\gamma\delta} (\xi^\alpha \eta^\beta - \eta^\alpha \xi^\beta)(\xi^\gamma \eta^\delta - \eta^\gamma \xi^\delta).$$

Hence negative sectional curvature means

$$\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\beta\gamma\delta} (\xi^\alpha \eta^\beta - \eta^\alpha \xi^\beta)(\xi^\gamma \eta^\delta - \eta^\gamma \xi^\delta) > 0$$

for all $\xi, \eta$ with $(\text{Re} \, \xi) \wedge (\text{Re} \, \eta) \neq 0$.

In the case $n = 2$, we have

$$\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\beta\gamma\delta} (\xi^\alpha \eta^\beta - \eta^\alpha \xi^\beta)(\xi^\gamma \eta^\delta - \eta^\gamma \xi^\delta) = R_{\text{1111}} |\xi^1 \eta^1 - \eta^1 \xi^1|^2$$

$$+ 4 \text{Re} (R_{\text{1111}} (\xi^1 \eta^1 - \eta^1 \xi^1)(\xi^2 \eta^2 - \eta^2 \xi^2))$$

$$+ 2 R_{\text{1122}} (|\xi^1 \eta^2|^2 + |\eta^1 \xi^2|^2 + \text{Re} (\xi^1 \eta^2 - \eta^1 \xi^2)(\xi^2 \eta^1 - \eta^2 \xi^1))$$

$$+ 2 \text{Re} (R_{\text{1212}} (\xi^1 \eta^2 - \eta^1 \xi^2)(\xi^2 \eta^1 - \eta^2 \xi^1))$$

$$+ 4 \text{Re} (R_{\text{2212}} (\xi^2 \eta^2 - \eta^2 \xi^2)(\xi^1 \eta^1 - \eta^1 \xi^1))$$

$$+ R_{\text{2222}} |\xi^2 \eta^2 - \eta^2 \xi^2|^2.$$
We now compute \( \| p \wedge q \| ^2 \). We will later need this only for the case \( n = 2 \) and \( g_{i \bar{j}} = 0 \). So we do our computation only in this case. (Computation in the general case is completely analogous.)

\[
p \wedge q = \left( \xi^1 \frac{\partial}{\partial z_1} + \xi^2 \frac{\partial}{\partial z_2} + \xi^3 \frac{\partial}{\partial z_3} \right) \\
\wedge \left( \eta^1 \frac{\partial}{\partial z_1} + \eta^2 \frac{\partial}{\partial z_2} + \eta^3 \frac{\partial}{\partial z_3} \right)
\]

\[
= (\xi^1 \eta^2 - \xi^2 \eta^1) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} + (\xi^1 \eta^3 - \xi^3 \eta^1) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_3} \\
+ (\xi^2 \eta^3 - \xi^3 \eta^2) \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_3}.
\]

\[
\| p \wedge q \| ^2 = g_{i \bar{j}}(\xi^i \eta^\bar{j} - \xi^\bar{j} \eta^i)^2 + 2g_{i \bar{k}}g_{\bar{k} \bar{j}}(\xi^i \eta^\bar{j} - \xi^\bar{j} \eta^i)^2 + (\xi^i \eta^\bar{j} - \xi^\bar{j} \eta^i)^2 + (\xi^\bar{j} \eta^i - \xi^i \eta^\bar{j})^2 + (\xi^i \eta^\bar{j} - \xi^\bar{j} \eta^i)^2.
\]

2. Bergman metric of \( | z_1 | ^{2m} + | z_2 | ^2 < 1 \)

Let \( D \) be the domain in \( \mathbb{C}^2 \) defined by \( | z_1 | ^{2m} + | z_2 | ^2 < 1 \). First we calculate its Bergman kernel function directly by using an orthonormal basis in the Hilbert space of \( L^2 \) holomorphic functions on \( D \). The set of all polynomials is dense in this Hilbert space.

Let \( z_\alpha = r_\alpha e^{i \theta_\alpha} \) (\( \alpha = 1, 2 \)). The Euclidean volume form \( dV \) of \( \mathbb{C}^2 \) is \( (r_r dr d\theta)(r_r dr d\theta) \). The inner product of \( z_1^p z_2^q \) and \( z_1^p z_2^q \) is given by

\[
(z_1^p z_2^q, z_1^p z_2^q) = \int_D (z_1^p z_2^q)(\bar{z}_1^p z_2^p) dV
\]

\[
= \int_{r_1=0}^{r_1=\infty} \int_{\theta_1=0}^{\theta_1=\pi} \int_{r_2=0}^{r_2=\infty} \int_{\theta_2=0}^{\theta_2=\pi} r_1^{k+p+1} r_2^{l+q+1} e^{i(k-p)\theta_1} e^{i(l-q)\theta_2} d\theta_1 d\theta_2 d\theta_2 d\theta_2,
\]

which is zero unless \( (k, l) = (p, q) \). Now consider the case \( (p, q) = (p, q) \).

\[
\frac{1}{4\pi^2} (z_1^p z_2^q, z_1^p z_2^q) = \int_{r_1=0}^{r_1=\infty} \int_{r_2=0}^{r_2=\infty} r_1^{2p+1} r_2^{2q+1} d\theta_1 d\theta_2
\]

\[
= \int_{r_1=0}^{r_1=\infty} r_1^{2p+1} dr_1 \int_{r_2=0}^{r_2=\infty} r_2^{2q+1} dr_2
\]

\[
= \frac{1}{2(q+1)} \int_{r_1=0}^{r_1=\infty} r_1^{2p+1}(1 - r_1^{2m})^{q+1} dr_1
\]

\[
= \frac{1}{2(q+1)} \int_{u=0}^{u=1} u^p(1 - u^m)^{q+1} du
\]

(\text{where } u = r_1^2)

\[
= \frac{m}{4(p+1)} \int_{u=0}^{u=1} u^{p+m}(1 - u^m)^q du
\]

(by using integration by parts to integrate \( u^p \) first)
The Bergman kernel function \( \Phi(z_1, z_2) \) is given by

\[
\Phi(z_1, z_2) = \sum_{p=0}^{\infty} \frac{1}{m^{p+1} \pi^2} \frac{(p+1)(p+m+1)(p+2m+1) \cdots (p+(q+1)m+1)}{q!} |z_1|^{2p} |z_2|^{2q}
\]

\[
= \sum_{p=0}^{\infty} \frac{(p+1)(p+m+1)}{m \pi^2} \left( \sum_{q=0}^{\infty} \frac{(p+2m+1)(p+2m+1)+1 \cdots (p+2m+1)+q-1}{q!} |z_1|^{2p} |z_2|^{2q} \right)
\]

\[
= \sum_{p=0}^{\infty} \frac{(p+1)(p+m+1)}{m \pi^2} \left( 1 - |z_2|^2 \right)^{(p+2m+1)/m} |z_1|^2 p
\]

\[
= \frac{1 - |z_2|^2}{m \pi^2} \sum_{p=0}^{\infty} (p+1)(p+m+1) \left( 1 - |z_2|^2 \right)^{-1/m} |z_1|^2 p
\]

\[
+ \sum_{p=0}^{\infty} (m-1)(p+1) \left( 1 - |z_2|^2 \right)^{-1/m} |z_1|^2 p
\]

\[
= \frac{1 - |z_2|^2}{m \pi^2} \left( 2 \sum_{p=0}^{\infty} (p+2m+1)/m \left( 1 - |z_2|^2 \right)^{-1/m} |z_1|^2 p \right)
\]

\[
+ (m-1) \sum_{p=0}^{\infty} \left( -2 \right)^{-3} \cdots (2 - p + 1) \left( 1 - |z_2|^2 \right)^{-1/m} |z_1|^2 p
\]

\[
= \frac{1 - |z_2|^2}{m \pi^2} \left( 2 \left( 1 - |z_2|^2 \right)^{-1/m} \right) + (m-1) \left( 1 - |z_1|^2 \right)^{-2/m}
\]

\[
= \frac{m+1}{m \pi^2} \left( 1 - |z_2|^2 \right)^{-1/m} \left( 1 - |z_2|^2 \right)^{1/m} - (m-1) |z_1|^2.
\]

Let

\[
\eta_{\alpha \beta} = \partial_{\alpha} \partial_{\beta} \log \Phi.
\]

Direct computation yields
\[
\begin{align*}
g_{i\bar{i}} &= \frac{3(1 - |z_2|^2)^{1/m}}{(1 - |z_1|^2)^{1/m} - |z_1|^2} - \frac{m - 1}{m + 1} \frac{(1 - |z_2|^2)^{1/m}}{(1 - |z_1|^2)^{1/m} - |z_1|^2}^2, \\
g_{i\bar{j}} &= \frac{-3z_1z_2(1 - |z_2|^2)^{(1/m) - 1}}{m((1 - |z_2|^2)^{1/m} - |z_1|^2)^2} - \frac{m - 1}{m(m + 1)} \frac{z_1z_2}{(1 - |z_2|^2)^{1/(1/m)} - |z_1|^2}^2, \\
g_{\bar{i}\bar{j}} &= \frac{2m - 1}{m} \frac{1}{(1 - |z_2|^2)^2} + \frac{3}{m(1 - |z_2|^2)^{2-(1/m)}} \frac{(1 - |z_1|^2)^{1/m} - |z_1|^2 + \frac{1}{m} |z_1|^2 |z_2|^2}{(1 - |z_2|^2)^{1/m} - |z_1|^2}^2 \\
&\quad - \frac{m - 1}{m(m + 1)} \frac{|z_1|^2 |z_2|^2}{(1 - |z_2|^2)^{1/m} - |z_1|^2}^2.
\end{align*}
\]

Moreover, at \((0, 0)\)

\[
R_{i\bar{i}i\bar{i}} = \frac{4m^2 + 16m + 4}{(m + 1)^2},
\]

\[
R_{i\bar{i}\bar{i}z} = -\frac{2m + 4}{m(m + 1)},
\]

\[
R_{\bar{i}\bar{i}zz} = -\frac{2m + 4}{m}.
\]

All other \(R_{\bar{\alpha}\bar{\beta}i\bar{i}} = 0\), since \(g_{i\bar{i}} = z_1z_2F_{i\bar{i}}(|z_1|^2, |z_2|^2)\) and thus \(R_{\bar{\alpha}i\bar{i}i\bar{i}} \neq 0\) only if \(\alpha = 2\) and \(\beta = 1\). Since

\[
\left(\frac{2m + 4}{m(m + 1)}\right)^2 < \frac{4m + 2}{m} \frac{4m^2 + 16m + 4}{(m + 1)^2},
\]

it follows from the following lemma that the sectional curvature of the Bergman metric of \(D\) is negative at \((0, 0)\).

**Lemma 1.** Suppose \(2 \text{Re} \sum_{\alpha, \beta = 1}^{2} g_{\alpha \bar{\beta}}dz^\alpha d\bar{z}^\beta\) is a Kähler metric on a neighborhood of 0 in \(\mathbb{C}^2\) and \(g_{i\bar{i}} = 0\) at 0. Suppose all the components \(R_{\alpha \bar{\beta}i\bar{i}}\) of the curvature tensor are zero at the origin except \(R_{i\bar{i}i\bar{i}}, R_{i\bar{i}\bar{i}z}, \) and \(R_{\bar{i}\bar{i}zz}\). Then all the sectional curvatures at the origin are negative if and only if \(R_{i\bar{i}i\bar{i}} > 0\), \(R_{i\bar{i}\bar{i}z} > 0\), \(R_{\bar{i}\bar{i}zz} > 0\), and \((R_{i\bar{i}i\bar{i}})^2 < R_{i\bar{i}i\bar{i}}R_{\bar{i}\bar{i}zz}\).

**Proof.** In this case

\[
\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \bar{\beta}i\bar{i}}(\xi^\alpha \eta^\beta - \eta^\alpha \xi^\beta)(\xi^\gamma \eta^\bar{\gamma} - \eta^\gamma \xi^\bar{\gamma})
\]

\[
= R_{i\bar{i}i\bar{i}}(|\xi^i \eta^{\bar{\gamma}} - \eta^i \xi^{\bar{\gamma}}|^2 + 2R_{i\bar{i}i\bar{i}}(|\xi^i \eta^{\bar{\gamma}} - \eta^i \xi^{\bar{\gamma}}|^2 + |\xi^i \eta^{\bar{\gamma}} - \eta^i \xi^{\bar{\gamma}}|^2) + (\xi^i \eta^{\bar{\gamma}} - \eta^i \xi^{\bar{\gamma}})(\xi^j \eta^{\bar{\gamma}} - \eta^j \xi^{\bar{\gamma}}) + R_{\bar{i}\bar{i}zz}(|\xi^\gamma \eta^{\bar{\gamma}} - \eta^\gamma \xi^{\bar{\gamma}}|^2).
\]
The sectional curvature is negative if and only if this expression is \( \geq \) some positive multiple of
\[
|\xi^i \eta^j - \eta^i \xi^j|^2 + |\xi^i \eta^2 - \eta^i \xi^2|^2 + |\xi^2 \eta^i - \eta^2 \xi^i|^2.
\]

Sufficiency follows from
\[
|2 R_{i12}(\xi^i \eta^j - \eta^i \xi^j)(\xi^2 \eta^2 - \eta^2 \xi^2)| \leq 2(R_{i111} R_{222})^{1/2}(\xi^i \eta^j - \eta^i \xi^j)(\xi^2 \eta^2 - \eta^2 \xi^2)
\leq R_{i11} |\xi^i \eta^j - \eta^i \xi^j|^2 + R_{222} |\xi^2 \eta^2 - \eta^2 \xi^2|^2
\]
and from
\[
|\xi^i \eta^j - \eta^i \xi^j|^2 = |\xi^i \eta^2 - \eta^i \xi^2|^2 - (\xi^i \eta^j - \eta^i \xi^j)(\xi^2 \eta^2 - \eta^2 \xi^2)
\leq |\xi^i \eta^j - \eta^i \xi^j|^2 + \frac{1}{2} |\xi^i \eta^j - \eta^i \xi^j|^2 + \frac{1}{2} |\xi^2 \eta^2 - \eta^2 \xi^2|^2.
\]

Necessity follows from considering the following special values of \( \xi^a \) and \( \eta^a \):
\[
i \xi^2 = \eta^2 = 0 \implies R_{i111} > 0,
\]
\[
i \xi^2 = \eta^2 = 0 \implies R_{i11} > 0,
\]
\[
i \xi^1 = \eta^1 = 0 \implies R_{i111} > 0,
\]
\[
i \xi^2 = \eta^2 = 0 \implies R_{i11} > 0,
\]
\[
i \xi^a = a \sqrt{-1} \eta^a = -\sqrt{-1} \eta^a \implies \eta^a = \eta^a = 1 \text{ for } a > 0 \implies R_{i11} 4a^2 - 2 R_{i111} 4a^2 + R_{i111} 4 > 0.
\]

Hence \( (R_{i111})^2 < R_{i111} R_{i111} \).

3. Curvature of the sum of metrics

As in Section 2 let \( D \subset C^1 \) be defined by \( |z_1|^2 + |z_2|^2 < 1 \). Let \( B \) be the open unit 2-ball with coordinates \( w_1, w_2 \). Let \( \sigma: D \to B \) be defined by
\[
\sigma(z_1, z_2) = (z_1^m, z_2).
\]

Let \( 2 \text{Re} \sum g_{a\bar{a}} dz^a \bar{dz}^{\bar{a}} \) be the Bergman metric of \( D \) from Section 2. Let \( 2 \text{Re} \sum h_{a\bar{a}} dw^a \bar{dw}^{\bar{a}} \) be the Poincaré metric of \( B \).

**Lemma 2.** There exists an open neighborhood \( U \) of 0 in \( D \) such that for \( \lambda \geq 0 \) the metric which is the sum of the Bergman metric of \( D \) and \( \lambda \) times the pullback of the Poincaré metric of \( B \) (i.e., the metric \( 2 \text{Re} \sum g_{a\bar{a}} dz^a \bar{dz}^{\bar{a}} + \lambda \sigma^* (\sum h_{a\bar{a}} dw^a \bar{dw}^{\bar{a}}) \) on \( D \)) has strictly negative sectional curvature on \( U \).

**Proof.** First we observe that for \( |b| < 1 \),
\[
\begin{align*}
  w'_1 &= \frac{w_1 \sqrt{1 - |b|^2}}{bw_2 + 1} \\
  w'_2 &= \frac{w_2 + b}{bw_2 + 1}
\end{align*}
\]
is a biholomorphism of the ball sending \( B \cap \{w_2 = -b\} \) to \( B \cap \{w_2 = 0\} \). (The fact that it is a biholomorphism follows from the simply identity
\[ |\bar{a}z + 1|^2 \left(1 - \frac{|z + a|}{|\bar{a}z + 1|}^2 \right) = (1 - |z|^2)(1 - |a|^2) \]

for \(a, z \in \mathbb{C}\) with \(|a| < 1, |z| < 1\). Moreover, this biholomorphism of \(B\) is covered by the following automorphism of \(D\):

\[
\begin{align*}
\begin{cases}
z'_1 = \frac{z_1(1 - |b|^2)^{1/2m}}{(b\bar{z}_2 + 1)^{1/m}} \\
z'_2 = \frac{z_2 + b}{b\bar{z}_2 + 1},
\end{cases}
\]

where \((b\bar{z}_2 + 1)^{1/m}\) is any branch. Hence instead of proving the existence of \(U\), it suffices to show that for \(\alpha \in \mathbb{C}\) sufficiently small the sectional curvature of the sum metric is negative at \((z_1, z_2) = (\alpha, 0)\). We will do our computation of the sectional curvature at \((z_1, z_2) = (\alpha, 0)\) with \(\alpha \neq 0\) and will use the coordinate system \((w_1, w_2)\) coming from \(B\) instead of using the coordinate system \((z_1, z_2)\). Let \(a = \alpha^m\). Then the \(\sigma\)-image of \((z_1, z_2) = (\alpha, 0)\) is \((w_1, w_2) = (a, 0)\). Let

\[
\sum \varphi_{\alpha \bar{\beta}} dw^\alpha dw^{\bar{\beta}} = \sum g_{\alpha \bar{\beta}} dz^\alpha dz^{\bar{\beta}}.
\]

Then

\[
\begin{align*}
\varphi_{ii} &= g_{ii} \frac{1}{m^2} w_{i}^{(1/m) - 1} \bar{w}_{i}^{(1/m) - 1} \\
\varphi_{i\bar{j}} &= g_{i\bar{j}} \frac{1}{m} w_{i}^{(1/m) - 1} \\
\varphi_{\bar{i}\bar{j}} &= g_{\bar{i}\bar{j}},
\end{align*}
\]

where \(g_{\alpha \bar{\beta}}\) is given by (*) of Section 2 and the \(m^{th}\) roots (and all other \(m^{th}\) roots from this point on) are taken in a way compatible with the point \((\alpha, 0)\).

Consider the following biholomorphism of \(B\)

\[
\begin{align*}
w_1 &= \frac{\zeta_1 + a}{\bar{a}\zeta_1 + 1} \\
w_2 &= \frac{\zeta_2 \sqrt{1 - |a|^2}}{\bar{a}\zeta_1 + 1}
\end{align*}
\]

which sends \((\zeta_1, \zeta_2) = (0, 0)\) to \((w_1, w_2) = (a, 0)\). We are going to calculate the sectional curvature of the sum metric using the coordinates \((\zeta_1, \zeta_2)\). Let

\[
\begin{align*}
\sum h_{\alpha \bar{\beta}} dw^\alpha dw^{\bar{\beta}} &= \sum H_{\alpha \bar{\beta}} d\zeta^\alpha d\zeta^{\bar{\beta}}, \\
\sum \varphi_{\alpha \bar{\beta}} dw^\alpha dw^{\bar{\beta}} &= \sum \psi_{\alpha \bar{\beta}} d\zeta^\alpha d\zeta^{\bar{\beta}}.
\end{align*}
\]

By \(\partial_\alpha\) and \(\partial_{\bar{\alpha}}\) we mean respectively \(\partial/\partial \zeta_\alpha\) and \(\partial/\partial \zeta^{\bar{\alpha}}\). Since \(2\text{Re} \sum h_{\alpha \bar{\beta}} dw_\alpha d\bar{w}_{\beta}\)

is the Poincaré metric of \(B\), it follows that
\[
\begin{align*}
\begin{cases}
H_{12} = 0 \\
dH_{a\bar{b}} = 0 \
\end{cases}
\text{at } (\zeta_1, \zeta_2) = (0, 0).
\end{align*}
\]

We claim that
\[
\begin{align*}
\psi_{1\bar{2}} &= 0 \\
\partial_{\bar{z}} \psi_{1\bar{2}} &= 0 \\
\partial_{\bar{z}} \psi_{1\bar{2}} &= 0, \quad \partial_{\bar{z}} \psi_{1\bar{2}} = 0 \\
\partial_{\bar{z}} \psi_{1\bar{2}} &= 0 \
\end{align*}
\text{at } (\zeta_1, \zeta_2) = (0, 0).
\]

To prove the claim, we express \( \psi_{a\bar{b}} \) in terms of \( \varphi_{a\bar{b}} \).

\[
egin{align*}
\psi_{1\bar{2}} &= \sum_{a, b} \varphi_{a\bar{b}} \frac{\partial w_a}{\partial \zeta_1} \frac{\partial w_b}{\partial \zeta_2} \\
&= \varphi_{1\bar{2}} \left( \frac{1 - |a|^2}{(a\zeta_1 + 1)^2} \right)^2 + \varphi_{1\bar{2}} \left( \frac{1 - |a|^2}{(a\zeta_1 + 1)^2} \right)^2 \\
&= \varphi_{1\bar{2}} \left( \frac{-a\zeta_2 \sqrt{1 - |a|^2}}{(a\zeta_1 + 1)^2} \right) \left( \frac{1 - |a|^2}{(a\zeta_1 + 1)^2} \right) + \varphi_{1\bar{2}} \left( \frac{-a\zeta_2 \sqrt{1 - |a|^2}}{(a\zeta_1 + 1)^2} \right)^2 \\
\end{align*}
\]

Since \( g_{1\bar{2}} = 0 \) when \( z_2 = 0 \), it follows that \( \psi_{1\bar{2}} = 0 \) at \( (\zeta_1, \zeta_2) = (0, 0) \). Observe that from (*) of Section 2 we have

\[
\begin{align*}
g_{1\bar{2}}(z_1, z_2) &= F_{1\bar{2}}(|z_1|^2, |z_2|^2) \text{ with } F_{1\bar{2}}(0, 0) > 0 \\
g_{1\bar{2}}(z_1, z_2) &= \overline{z}_1 z_2 F_{1\bar{2}}(|z_1|^2, |z_2|^2) \\
g_{2\bar{2}}(z_1, z_2) &= F_{2\bar{2}}(|z_1|^2, |z_2|^2)
\end{align*}
\]

for some smooth functions, \( F_{1\bar{2}}, F_{1\bar{2}}, F_{2\bar{2}} \) of two real variables. Hence by (\#) of Section 3,

\[
\begin{align*}
\varphi_{1\bar{2}}(w_1, w_2) &= |w_1|^{\frac{1}{2}(m-2)} \Phi_{1\bar{2}}(|w_1|^{2/m}, |w_2|^2) \text{ with } \Phi_{1\bar{2}}(0, 0) > 0 \\
\varphi_{1\bar{2}}(w_1, w_2) &= \frac{w_2}{w_1} \Phi_{1\bar{2}}(|w_1|^{2/m}, |w_2|^2) \\
\varphi_{2\bar{2}}(w_1, w_2) &= \Phi_{2\bar{2}}(|w_1|^{2/m}, |w_2|^2)
\end{align*}
\]

for some smooth functions \( \Phi_{1\bar{2}}, \Phi_{1\bar{2}}, \Phi_{2\bar{2}} \) of two real variables. It follows that

\[
\begin{align*}
\psi_{1\bar{2}} &= |\zeta_1 + a|^{\frac{1}{2}(m-2)} \psi_{1\bar{2}}(\zeta_1, |\zeta_1 + a|^{2/m}, |\zeta_2|^2) \text{ with } \psi_{1\bar{2}}(0, 0, 0) > 0 \\
\psi_{\bar{2}} &= \psi_{\bar{2}}(\zeta_1, |\zeta_1 + a|^{2/m}, |\zeta_2|^2) \\
\psi_{\bar{2}} &= \psi_{\bar{2}}(\zeta_1, |\zeta_1 + a|^{2/m}, |\zeta_2|^2)
\end{align*}
\]
for some smooth functions \( \psi_{i1}, \psi_{i\hat{z}}, \psi_{\hat{z}\hat{z}} \) of a complex variable and two real variables. It is now obvious that all the equations in \((\dagger)\) of Section 3 are satisfied. Let

\[
G_{a\bar{b}} = \psi_{a\bar{b}} + \lambda H_{a\bar{b}}
\]

where \( \lambda \) is a positive number. Let \( R^a_{\hat{b}\hat{r}\hat{t}}, R^\psi_{a\hat{b}\hat{r}\hat{t}}, R^U_{a\hat{b}\hat{r}\hat{t}} \) be the curvature tensors of respectively the metrics \( 2 \text{Re} \sum G_{a\bar{b}} d\zeta_a d\bar{\zeta}_b, 2 \text{Re} \sum \psi_{a\bar{b}} d\zeta_a d\bar{\zeta}_b, \) and \( 2 \text{Re} \sum H_{a\bar{b}} d\zeta_a d\bar{\zeta}_b \). From \((*)\) and \((\dagger)\) of Section 3 it follows that at \((\zeta_1, \zeta_2) = (0, 0)\)

\[
\begin{align*}
R^a_{1i1} &= R^\psi_{1i1} + \lambda R^U_{1i1} + \left( \frac{1}{\psi_{i1}} - \frac{1}{\psi_{i1} + \lambda H_{i1}} \right) |\partial_1 \psi_{i1}|^2 \\
R^a_{1i\hat{z}} &= R^\psi_{1i\hat{z}} + \lambda R^U_{1i\hat{z}} + \left( \frac{1}{\psi_{i\hat{z}}} - \frac{1}{\psi_{i\hat{z}} + \lambda H_{i1}} \right) |\partial_1 \psi_{i\hat{z}}|^2 \\
R^a_{a\hat{b}\hat{r}} &= R^\psi_{a\hat{b}\hat{r}} + \lambda R^U_{a\hat{b}\hat{r}} \quad \text{for} \quad (\alpha, \beta, \gamma, \delta) = (1, \hat{1}, 1, \hat{2}), (1, \hat{2}, 1, \hat{2}), \\
& \quad (1, 2, 2, \hat{2}) \quad \text{and} \quad (2, \hat{2}, 2, 2).
\end{align*}
\]

Hence

\[
\sum_{a, \beta, \gamma, \delta} R^a_{a\beta\gamma\delta}(\zeta^a \eta^\beta - \eta^a \zeta^\beta)(\zeta^\gamma \eta^\delta - \eta^\gamma \zeta^\delta)
= \sum_{a, \beta, \gamma, \delta} R^\psi_{a\beta\gamma\delta}(\zeta^a \eta^\beta - \eta^a \zeta^\beta)(\zeta^\gamma \eta^\delta - \eta^\gamma \zeta^\delta)
+ \lambda \sum_{a, \beta, \gamma, \delta} R^U_{a\beta\gamma\delta}(\zeta^a \eta^\beta - \eta^a \zeta^\beta)(\zeta^\gamma \eta^\delta - \eta^\gamma \zeta^\delta)
+ \left( \frac{1}{\psi_{i1}} - \frac{1}{\psi_{i1} + \lambda H_{i1}} \right) |\partial_1 \psi_{i1}|^2 |\zeta^1 \eta^1 - \eta^1 \zeta^1|^2
+ 2 \left( \frac{1}{\psi_{i\hat{z}}} - \frac{1}{\psi_{i\hat{z}} + \lambda H_{i\hat{z}}} \right) |\partial_1 \psi_{i\hat{z}}|^2 |\zeta^1 \eta^\hat{z} - \eta^1 \zeta^\hat{z}|^2
+ 2 \left( \frac{1}{\psi_{\hat{z}\hat{z}}} - \frac{1}{\psi_{\hat{z}\hat{z}} + \lambda H_{\hat{z}\hat{z}}} \right) |\partial_1 \psi_{\hat{z}\hat{z}}|^2 |\zeta^\hat{z} \eta^\hat{z} - \eta^\hat{z} \zeta^\hat{z}|^2.
\]

We want to show that the sectional curvature of the metric \( 2 \text{Re} \sum G_{a\bar{b}} d\zeta_a d\bar{\zeta}_b \) is \( \leq -c \) at \( (\zeta_1, \zeta_2) = (0, 0) \) for \( |a| < \varepsilon \) and \( \lambda \geq 0 \), where \( c, \varepsilon \) are suitable positive numbers. Only the last term on the right-hand side is an undesirable term and we have to dominate it by other terms. The absolute value of \( \partial_1 \psi_{\hat{z}\hat{z}} \) at \( (\zeta_1, \zeta_2) = (0, 0) \) is \( \leq C_1 |a|^{(4/m) - 1} \) where \( C_1 \) is a positive constant independent of \( a \) when \( |a| < \varepsilon \) and \( \varepsilon \) is a fixed positive number \( < 1 \). Hence the absolute value of

\[
2 \left( \frac{1}{\psi_{\hat{z}\hat{z}}} - \frac{1}{\psi_{\hat{z}\hat{z}} + \lambda H_{\hat{z}\hat{z}}} \right) |\partial_1 \psi_{\hat{z}\hat{z}}|^2 |\zeta^\hat{z} \eta^\hat{z} - \eta^\hat{z} \zeta^\hat{z}|^2
\]

is

\[
\leq C_2 |a|^{(4/m) - 2} |\zeta^1 \eta^\hat{z} - \eta^1 \zeta^\hat{z}| |\zeta^\hat{z} \eta^\hat{z} - \eta^\hat{z} \zeta^\hat{z}|,
\]

where \( C_2 \) is again a positive constant independent of \( a \) and \( \lambda \), because

\[
\frac{1}{\psi_{\hat{z}\hat{z}}} - \frac{1}{\psi_{\hat{z}\hat{z}} + \lambda H_{\hat{z}\hat{z}}} \leq \frac{1}{\psi_{\hat{z}\hat{z}}}.
\]
Since the sectional curvature of the Bergman metric of $D$ is negative at the center of $D$, it follows that for $|a| \leq \varepsilon$ and $\varepsilon$ sufficiently small, there exists a positive number $s_2$ such that

$$\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \beta \gamma \delta} (\xi^\alpha \xi^\beta) (\xi^\gamma \xi^\delta) (\xi^\gamma \xi^\delta) \geq s_2 |a|^{(4/m)-2} |\xi^\gamma \xi^\delta| |\xi^\gamma \xi^\delta|$$

where $s_2$ is a positive constant independent of $a$. Since

$$C_2 |a|^{(4/m)-2} |\xi^\gamma \xi^\delta| \leq \frac{1}{2} C_2 |a|^{2/m} (|a|^{(4/m)-4} |\xi^\gamma \xi^\delta| + |\xi^\gamma \xi^\delta|)$$

it follows that

$$\frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \beta \gamma \delta} (\xi^\alpha \xi^\beta) (\xi^\gamma \xi^\delta) (\xi^\gamma \xi^\delta) \geq C_2 |a|^{(4/m)-2} |\xi^\gamma \xi^\delta| |\xi^\gamma \xi^\delta|$$

when $|a| \leq (s_2 C_2^{-1})^{m/2}$. Hence for $a$ sufficiently small and for all $\lambda \geq 0$

$$\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \beta \gamma \delta} (\xi^\alpha \xi^\beta) (\xi^\gamma \xi^\delta) (\xi^\gamma \xi^\delta) \geq \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \beta \gamma \delta} (\xi^\alpha \xi^\beta) (\xi^\gamma \xi^\delta) (\xi^\gamma \xi^\delta) \cdot$$

Now the lemma follows from the fact that the sectional curvature of the Bergman metric of $D$ is negative at the center of $D$.

The sum of the two metrics actually satisfies a curvature condition stronger than the negativity of the sectional curvature on a neighborhood of 0 for all $\lambda \geq 0$. First we introduce this curvature condition. Define

$$A_{\alpha \beta, \gamma \delta} = R_{\alpha \beta \gamma \delta} .$$

Then

$$\overline{A_{\alpha \beta, \gamma \delta}} = \overline{R_{\alpha \beta \gamma \delta}} = R_{\gamma \delta \alpha \beta} = A_{\gamma \delta, \alpha \beta} .$$

Hence, as a matrix with row index $(\alpha, \beta)$ and column index $(\gamma, \delta)$, $A_{\alpha \beta, \gamma \delta}$ is a hermitian matrix. We say that the curvature tensor is very strongly negative if this hermitian matrix is strictly positive definite. In other words,

$$\sum_{\alpha, \beta, \gamma, \delta} A_{\alpha \beta, \gamma \delta} \theta_{\alpha \beta} \theta_{\gamma \delta} \geq c \sum_{\alpha, \beta} |\theta_{\alpha \beta}|^2$$

for some $c > 0$; that is,

$$\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \beta \gamma \delta} \theta_{\alpha \beta} \theta_{\gamma \delta} \geq c \sum_{\alpha, \beta} |\theta_{\alpha \beta}|^2 .$$
We recall that the negativity of the sectional curvature means that
\[
\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \beta \gamma \delta} (\xi^\alpha \eta^\beta - \gamma^\alpha \zeta^\beta)(\xi^\gamma \eta^\delta - \gamma^\gamma \zeta^\delta) \geq c \sum_{\alpha, \beta} |\xi^\alpha \eta^\beta - \gamma^\alpha \zeta^\beta|^2,
\]
for some \(c > 0\). Clearly the negativity of the sectional curvature follows from the very strong negativity of the curvature tensor, because one can set
\[
\theta_{\alpha \bar{\beta}} = \xi^\alpha \eta^\beta - \gamma^\alpha \zeta^\beta.
\]
To prove that the sum of the two metrics has a very strongly negative curvature tensor, one observes that
\[
\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \beta \gamma \delta} \theta_{\alpha \bar{\beta}} \theta_{\gamma \bar{\delta}} = \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \beta \gamma \delta} \theta_{\alpha \bar{\beta}} \theta_{\gamma \bar{\delta}} + \lambda \sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \beta \gamma \delta} \theta_{\alpha \bar{\beta}} \theta_{\gamma \bar{\delta}}
\]
\[
+ \left( \frac{1}{\psi_{\bar{1}1}} - \frac{1}{\psi_{11} + \lambda H_{11}} \right) |\partial_1 \psi_{11}|^2 |\theta_{11}|^2
\]
\[
+ \left( \frac{1}{\psi_{\bar{2}2}} - \frac{1}{\psi_{22} + \lambda H_{22}} \right) |\partial_2 \psi_{22}|^2 (|\theta_{22}|^2 + |\theta_{21}|^2)
\]
\[
+ 2 \left( \frac{1}{\psi_{\bar{2}2}} - \frac{1}{\psi_{22} + \lambda H_{22}} \right) \text{Re}(\theta_{11} \theta_{22}).
\]
Moreover, one observes that the Bergman metric of \(D\) has a very strongly negative curvature tensor at 0, because the very strong negativity of the curvature tensor is equivalent to the positive definiteness of the matrix
\[
(A_{\alpha \bar{\beta}, \gamma \bar{\delta}}) = \begin{pmatrix}
R_{1111} & R_{1121} & R_{1112} & R_{1122} \\
R_{1211} & R_{1221} & R_{1212} & R_{1222} \\
R_{2111} & R_{2121} & R_{2112} & R_{2122} \\
R_{2211} & R_{2221} & R_{2212} & R_{2222}
\end{pmatrix}
\]
which becomes in this case
\[
\begin{pmatrix}
R_{1111} & 0 & 0 & R_{1122} \\
0 & R_{1122} & 0 & 0 \\
0 & 0 & R_{1122} & 0 \\
R_{1122} & 0 & 0 & R_{2222}
\end{pmatrix}
\]
which is positive definite because
\[
R_{1111} > 0,
\]
\[
\det \begin{pmatrix}
R_{1111} & 0 \\
0 & R_{1122}
\end{pmatrix} = R_{1111} R_{1122} > 0,
\]
\[
\det \begin{pmatrix}
R_{1111} & 0 \\
0 & R_{1122}
\end{pmatrix} = R_{1111} (R_{1122})^2 > 0,
\]

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\[
\det \begin{pmatrix}
R_{111} & 0 & 0 & R_{112} \\
0 & R_{112} & 0 & 0 \\
0 & 0 & R_{122} & 0 \\
R_{122} & 0 & 0 & R_{222}
\end{pmatrix} = (R_{112})^3(R_{111}R_{222} - (R_{122})^2) > 0.
\]

Since the ball is the special case of \( D \) with \( m = 1 \), it follows that the curvature tensor of the invariant metric of the ball is very strongly negative. Borrowing earlier notations, we have, when \( |a| \leq (s_2C^{-1})^{m/2} \),

\[
2\left(\frac{1}{\psi_{\overline{z}z}} - \frac{1}{\psi_{zz} + \lambda H_{zz}}\right)|\partial_z\psi_{\overline{z}z}|^2 \text{Re} \theta_{\overline{z}z} = 0,
\]

\[
\leq C_2 |a|^{(4/m) - 2} |\theta_{\overline{z}z}| \theta_{\overline{z}z}|
\leq \frac{1}{2} C_2 |a|^{2/m} \left(|a|^{(4/m) - 4} |\theta_{\overline{z}z}|^2 + |\theta_{zz}|^2\right)
\leq \frac{1}{2} s_2 (|a|^{(4/m) - 4} |\theta_{\overline{z}z}|^2 + |\theta_{zz}|^2)
\leq \frac{1}{2} R_{\overline{z}z} \theta_{\overline{z}z} \theta_{\overline{z}z}.
\]

Hence

\[
\sum_{\alpha, \beta, \gamma, \delta} R^g_{\alpha\beta\overline{\gamma}\overline{\delta}} \theta_{\alpha\beta} \overline{\theta_{\overline{\gamma}\overline{\delta}}} \leq \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} R^*_\alpha \theta_{\alpha\beta} \theta_{\gamma\delta}.
\]

The significance of the very strong negativity of the curvature tensor is that Siu [9] proved that any compact Kähler manifold having the same homotopy type as a compact Kähler manifold of dimension \( \geq 2 \) with a very strongly negative curvature tensor must be either biholomorphic or conjugate biholomorphic to it. In particular, strong rigidity holds for compact Kähler manifolds of dimension \( \geq 2 \) with a very strongly negative curvature tensor.

4. Construction of the surface

The main idea of the construction is to construct first a subgroup \( \Gamma \) of \( \text{Aut} B \) generated by three complex reflections. \( \Gamma \) is not discrete, but is almost discrete in the sense that there is a complex surface \( Y \) and a holomorphic map \( \sigma \) from \( Y \) onto \( B \) such that \( \Gamma \) lifts to a discrete subgroup \( \bar{\Gamma} \) of \( \text{Aut} Y \), and the only kind of singularity of \( \sigma \) is simple winding singularity along an infinite number of disjoint complex curves whose images are complex lines in \( B \). The compact surface is \( Y/\bar{\Gamma}_0 \) for some subgroup \( \bar{\Gamma}_0 \) of \( \bar{\Gamma} \) of finite index chosen for the sole purpose of making \( Y/\bar{\Gamma}_0 \) nonsingular.
The investigation of subgroups of Aut $B$ generated by three complex reflections of order $p$ for $p = 3, 4, 5$ was extensively carried out by Mostow [7], [8]. We will describe below the construction of the subgroup we need, but will refer to [8] for detailed proofs of the properties of this subgroup. We will need only the case $p = 5$.

Let $a = 1/(2 \sin (\pi/5))$ and $\varphi$ be a complex number of modulus 1 whose argument will be given later. Let $e_i$, $e_2$, $e_3$ be a basis of $C^3$ over $C$. Consider the hermitian form $\langle \cdot, \cdot \rangle$ defined with respect to the basis $e_i$, $e_2$, $e_3$ by the matrix

\[
\begin{pmatrix}
1 & -a\varphi & -a\bar{\varphi} \\
-a\bar{\varphi} & 1 & -a\varphi \\
-a\varphi & -a\bar{\varphi} & 1
\end{pmatrix}
\]

whose determinant is $\Delta = 1 - 3a^2 - a^2(\varphi^3 + \varphi^{-3})$. We consider only values of arg $\varphi$ for which $\Delta$ is negative. Let $V$ be the set of all $x \in C^3$ with $\langle x, x \rangle < 0$. The quotient of $V$ by the $C^*$ action is biholomorphic to $B$.

Let $e_j$ be the set of all $x \in C^3$ with $\langle x, e_j \rangle = 0$. Define the complex reflection $R_j$ in $C^3$ of order 5 about the complex plane $e_j$ as follows.

$R_j(x) = x + (e^{2\pi i/j} - 1)\langle x, e_j \rangle e_j$.

Since $\langle \cdot, \cdot \rangle$ is invariant under $R_j$ and $R_j$ commutes with the $C^*$ action, it follows that $R_j$ defines an element $R_j$ of Aut $B$.

Let $\Gamma_{ij}$ be the subgroup of Aut $B$ generated by $R_i$ and $R_j$. The only relations of $R_i$ and $R_j$ are $R_i^2 = R_j^2 = 1$ and $R_iR_jR_i = R_jR_iR_j$. The group $\Gamma_{ij}$ has order 600. Let $p_{ij}$ be the image of $e_i \cap e_j$ in $B$. Then $p_{ij}$ is the only fixed point of $\Gamma_{ij}$. We give $B$ the invariant metric. Then $p_{i2}, p_{23}, p_{13}$ form the vertices of an equilateral triangle. Let $p_0$ be the center of the circumscribed ball of this equilateral triangle.

For $\gamma \in$ Aut $B$ let

$\tilde{\gamma} = \{x \in B \mid d(x, p_0) = d(x, \gamma^{-1}p_0)\}$,

$X(\gamma) = \{x \in B \mid d(x, p_0) \leq d(x, \gamma^{-1}p_0)\}$

where $d(\cdot, \cdot)$ is the invariant distance in $B$. For any subset $D_*$ of Aut $B$ let

$X(D_*) = \bigcap_{\gamma \in D_*} X(\gamma)$,

$\tilde{\gamma} = \gamma \cap X(D_*)$.

For $i \neq j$ let $D_{ij}$ be the subset of $\Gamma_{ij}$ consisting of the ten elements

$D_{ij} = \{R_i^{\pm 1}, R_j^{\pm 1}, (R_iR_j)^{\pm 1}, (R_jR_i)^{\pm 1}, (R_iR_jR_i)^{\pm 1}\}$.

Then $X(D_{ij})$ is a fundamental domain of $\Gamma_{ij}$.

Let $\Gamma$ denote the subgroup of Aut $B$ generated by $R_i$, $R_2$, $R_3$. Let
\[ D_\ast = \{ R_i^{\pm 1}, (R_i R_j)^{\pm 1}, (R_i R_j R_k)^{\pm 1}, i \neq j, i, j = 1, 2, 3 \} . \]

That is, \( \Gamma \) is generated by \( \Gamma_{12}, \Gamma_{23}, \Gamma_{13} \) and \( D_\ast = D_{12} \cup D_{23} \cup D_{13} \). Let \( \Omega = X(D_\ast) \). \( \Gamma \) is discrete for certain values of \( \arg \varphi \). In these cases \( \Omega \) is a fundamental domain for \( \Omega \) modulo the finite subgroup of \( \Gamma \) which stabilizes \( \Omega \). In general, \( \Gamma \) is not discrete. For any \( \varphi \) such that \( |\arg \varphi^3| < \pi/2 - \pi/5 \), \( \Omega \) has twenty-four 3-faces given by

\[ \tilde{R}_i^{\pm 1}, (\tilde{R}_i \tilde{R}_j)^{\pm 1}, (\tilde{R}_i \tilde{R}_j \tilde{R}_k)^{\pm 1}; i \neq j, i, j = 1, 2, 3 \]

(note that \( R_i R_j R_k = R_j R_i R_k \)). Each of the twenty-four 3-faces of \( \Omega \) is oriented so that together with the inner normal to \( \Omega \), it carries the orientation of the ambient oriented space \( B \). Inasmuch as each \( \gamma \in \Gamma \) is an orientation-preserving homeomorphism of \( B \) which carries \( \tilde{\gamma} \) into \( \tilde{\gamma}^{-1} \) and an inner normal to an outer normal, we see that for each 3-face \( \tilde{\gamma} \) of \( \Omega \), the map \( \gamma: \tilde{\gamma} \to \tilde{\gamma}^{-1} \) reverses the assigned orientation of the 3-face \( \tilde{\gamma} \). Next, we orient each 2-face of each \( \tilde{\gamma} \) by the inner-normal rule, using the assigned orientation of the ambient orientable 3-manifold \( \partial \Omega \). Since \( \partial \Omega \) is an oriented manifold without boundary, each 2-face appears in exactly two 3-faces and is assigned two opposite orientations. We adopt the convention: whenever we represent a 2-face as the intersection of two 3-faces \( F' \cap F'' \), it carries the orientation induced by the first 3-face \( F' \).

The vertices of \( \Omega \) are \( p_{ij}, s_{ij}, \tilde{s}_{ij}, t_{ij} \) \((i \neq j, i, j = 1, 2, 3)\) defined by

\[ s_{ij} = s_{ji} = \tilde{R}_i \cap \tilde{R}_j \cap \tilde{R}_i \tilde{R}_j \cap \tilde{R}_i \tilde{R}_k \cap \tilde{R}_j \tilde{R}_k \cap \tilde{R}_k^{-1} \cap (\tilde{R}_k \tilde{R}_j)^{-1} \cap (\tilde{R}_k \tilde{R}_i)^{-1} , \]

\[ \tilde{s}_{ij} = \tilde{s}_{ji} = \tilde{R}_i^{-1} \cap \tilde{R}_j^{-1} \cap (\tilde{R}_i \tilde{R}_j)^{-1} \cap (\tilde{R}_i \tilde{R}_k)^{-1} \cap \tilde{R}_i \tilde{R}_j \tilde{R}_k \cap \tilde{R}_j \tilde{R}_k \;
\]

\[ t_{ij} = \tilde{R}_k \cap \tilde{R}_k^{-1} \cap \tilde{R}_k \tilde{R}_j \cap (\tilde{R}_k \tilde{R}_j)^{-1} \cap \tilde{R}_j \tilde{R}_k \cap (\tilde{R}_j \tilde{R}_k \tilde{R}_k)^{-1} , \]

where \( \{i, j, k\} = \{1, 2, 3\} \). For \( i \neq j \) there are two 1-faces joining \( p_{ij} \) and \( s_{ij} \). To distinguish them we select a point \( s_{ij}^* \) on one and a point \( s_{ij}^* \) on the other. The actions of \( R_j \) on some of the vertices are

\[ R_j s_{ij} = \tilde{s}_{ik} , \]

\[ R_k \tilde{s}_{ij} = s_{ij} , \]

\[ R_i R_j t_{kj} = t_{ik} \]

where \( \{i, j, k\} = \{1, 2, 3\} \).

Later we will use the fact that for \( \varphi = 1 \), in homogeneous coordinates of \( \mathbb{P}_2 \),

\[ s_{23} = [1, e^{-(\pi \sqrt{-1}/5)} \sqrt{-1}, e^{-(\pi \sqrt{-1}/5)} \sqrt{-1}] , \]

\[ \tilde{s}_{21} = [-e^{\pi \sqrt{-1}/5} \sqrt{-1}, -e^{\pi \sqrt{-1}/5} \sqrt{-1}, 1] . \]

When \( |\arg \varphi^3| < \pi/2 - \pi/5 \), the only 2-faces not passing through any of
the vertices $p_{ij}$ are the ones with vertices $t_{ik}, s_{ik}, \bar{s}_{ij}$. We denote it by $\Delta_{ijk}$. It lies in exactly two 3-faces: $\tilde{R}_i \tilde{R}_k$ and $(\tilde{R}_i \tilde{R}_j)^{-1}$. Moreover, it lies in a complex line pointwise fixed under $(R_i R_j R_k)^2$. $(R_i R_j R_k)^2$ is a complex reflection about this line. Its action can best be described by looking at a complex line perpendicular to $\Delta_{ijk}$. There are six 2-faces $\Delta_{ijk}$ and the cyclic permutation $1 \to 2 \to 3$ yields the isometries $\Delta_{123} \to \Delta_{231} \to \Delta_{321}$ and $\Delta_{231} \to \Delta_{123} \to \Delta_{321}$. So we consider only $\Delta_{312}$ and $\Delta_{313}$. The geodesic triangle $p_{12} p_{31} p_{32}$ intersects $\Delta_{312}$ at $t_{32}$ and is perpendicular to $\Delta_{312}$. The angle at $t_{32}$ of the triangle $p_{12} t_{32} p_{31}$ is $\pi/2 - \pi/5 + \arg \varphi^3$. $(R_j R_i R_k)^2$ rotates the triangle $p_{12} t_{32} p_{31}$ on its plane about the vertex $t_{32}$ by an angle equal to $3\pi/5 + \pi/2 - \arg \varphi^3$. The geodesic triangle $p_{12} t_{23} p_{31}$ intersects $\Delta_{313}$ at $t_{32}$ and is perpendicular to $\Delta_{313}$. The angle at $t_{32}$ of the triangle $p_{12} t_{23} p_{31}$ is $\pi/2 - \pi/5 + \arg \varphi^3$. $(R_j R_i R_k)^2$ rotates the triangle $p_{12} t_{23} p_{31}$ on its plane about the vertex $t_{32}$ by an angle equal to $3\pi/5 + \pi/2 + \arg \varphi^3$.

For later use we list also here the magnitudes of certain angles. In the geodesic triangles $\Delta_{312}$ and $\Delta_{313}$,

$$\angle s_{12} = \angle \bar{s}_{31} = \frac{1}{2} \left( \frac{\pi}{2} - \frac{\pi}{5} - \arg \varphi^3 \right),$$

$$\angle s_{12} = \angle \bar{s}_{31} = \frac{1}{2} \left( \frac{\pi}{2} - \frac{\pi}{5} + \arg \varphi^3 \right).$$

The angle at $p_{31}$ of the triangle $t_{23} p_{21} t_{32}$ and the angle at $p_{31}$ of the triangle $t_{23} p_{31} t_{32}$ are both $\pi/10$.

In [8] it is proved that $\Gamma$ is discrete and $\Omega$ is a fundamental domain for $\Gamma$ modulo the stabilizer in $\Gamma$ of $\Omega$ if and only if for all distinct $i, j, k$ the interior of the triangle $p_{jk} t_{ik} p_{ij}$ intersects its image under $(R_i R_j R_k)^{2\nu}$ only when $(R_i R_j R_k)^{2\nu} = 1$. In other words, a necessary and sufficient condition is that the following two conditions are satisfied.

Let $\nu_+ = \text{order } (R_i R_j R_k)^2$ and $\nu_- = \text{order } (R_i R_j R_k)^3$.

i) $2\pi$ is an integral multiple of $(\pi/2 - \pi/5 + \arg \varphi^3)\nu_+$,

ii) $2\pi$ is an integral multiple of $(\pi/2 - \pi/5 - \arg \varphi^3)\nu_-.$

The reason for this criterion of discreteness is the following: Clearly $\Gamma$ is discrete and $\Omega$ is its fundamental domain modulo the stabilizer in $\Gamma$ of $\Omega$ if for $\gamma \in \Gamma$ the interior of $\Omega$ intersects its image under $\gamma$ only when $\gamma$ belongs to the stabilizer. The elements $(R_i R_j R_k)^{2\nu}$ such that $(\text{Interior } \Omega) \cap (R_i R_j R_k)^{2\nu} \Omega = \emptyset$ are the obstructions to the fulfillment of these requirements. The above criterion stipulates the removal of these obstructions.

Now we choose $\arg \varphi^3 = \pi/20,$
\[
\begin{align*}
< p_{12} t_{23} p_{31} &= \frac{\pi}{2} - \frac{\pi}{5} + \frac{\pi}{20} = \frac{7\pi}{20} . \\
\text{Angle of rotation of } (R_3 R_1 R_2)^2 &= \frac{3\pi}{5} + \frac{\pi}{2} - 3 \arg \varphi^3 = \frac{19\pi}{20} . \\
< p_{12} t_{23} p_{31} &= \frac{\pi}{2} - \frac{\pi}{5} - \frac{\pi}{20} = \frac{\pi}{4} . \\
\text{Angle of rotation of } (R_2 R_1 R_3)^2 &= \frac{3\pi}{5} + \frac{\pi}{2} + 3 \arg \varphi^3 = \frac{5\pi}{4} .
\end{align*}
\]

In this example, condition ii) is satisfied, but condition i) is not satisfied. It is proved in [8] that \( \Gamma \) is not discrete. Nevertheless it is not far from acting discretely. The obstruction to \( \Omega \) being a fundamental domain near \( \Delta_{312} \) can be described as follows. In \( C \) we have a closed sector \( S \) of angle \( 7\pi/20 \) with the origin as vertex and we have the action of a cyclic group \( G \) whose generator \( g \) is the rotation of angle \( 19\pi/20 \). \( S \) is not a fundamental domain for \( G \). However, we can take a 7-sheeted analytic cover \( \alpha: \tilde{C} \to C \) with the origin as the branching locus. Then \( \tilde{C} \) is the union of 40 sets \( \tilde{S}_\nu (1 \leq \nu \leq 40) \) so that \( \alpha \) maps each \( \tilde{S}_\nu \) bijectively onto a sector in \( C \) of angle \( 7\pi/20 \) with the origin as vertex and \( \alpha(\tilde{S}_\nu) = S \). The generator \( g \) is the same as the rotation of angle \(-21\pi/20\). The cyclic group action of \( G \) on \( C \) can be lifted to a cyclic group action of \( \tilde{G} \) on \( \tilde{C} \) whose generator \( \tilde{g} \) is the rotation of angle \(-21\pi/20\) with the angle measured from \( C \). \( \tilde{S} \) is a fundamental domain for \( \tilde{G} \). Observe that \( \tilde{C} \) is the same as the quotient \( S \times G/\sim \) where \( \sim \) is defined as follows: \((s, g) \sim (s', g')\) if and only if \( gs = g's' \) and \( s, s' \in \partial S \). This description fits precisely the situation at hand with \( C \) being the complex line containing \( \Delta_{312} \), the sector being defined by the two geodesics \( t_{23} p_{12} \) and \( t_{32} p_{21} \), and \( g \) being the restriction of \( (R_3 R_1 R_2)^2 \) to the complex line containing \( \Delta_{312} \). Besides the obstruction to discreteness of \( \Gamma \) due to \( (R_3 R_1 R_2)^2 \), there are similar obstructions due to \( (R_1 R_2 R_3)^2 \) and \( (R_2 R_1 R_3)^2 \). Hence to remove all the obstructions to discreteness, it takes more than construction of a 7-sheeted analytic cover of \( B \) with branching locus along the complex line containing \( \Delta_{312} \). One has to construct also 7-sheeted analytic covers with branching locus along the complex lines containing respectively \( \Delta_{132} \) and \( \Delta_{231} \). Moreover, the set of 3 complex lines containing \( \Delta_{312}, \Delta_{123}, \Delta_{231} \) is far from invariant under the action \( \Gamma \). This necessitates the construction of analytic covers with branching locus along their images under \( \Gamma \). There is a natural way of constructing a \( \Gamma \)-space lying over \( B \) with \( \Omega \) as a fundamental domain modulo \( \text{Aut} \Omega \); on such a space \( \Gamma \) will necessarily be discrete.

We observe first that \( \text{Aut} \Omega \) is induced by the 3 cyclic permutations
1 \to 2 \to 3 \text{ of the coordinates of } \mathbb{C}^3. \text{ It is shown in [8] that for } \arg \varphi^3 = \pi/20, \text{ these 3 elements are in } \Gamma. \text{ Define } Y = \Gamma \times \Omega/\sim, \text{ where the relation } \sim \text{ is generated by the conditions: } (\gamma, x) \sim (\gamma', x') \text{ if}

(1) \quad \gamma x = \gamma' x' \text{ with } x, x' \in \partial \Omega \text{ and } \gamma^{-1}\gamma' \in D^* \text{ or}

(2) \quad \gamma^{-1}\gamma' \in \text{Aut } \Omega.

Define } \sigma : Y \to B \text{ as the map induced by } \Gamma \times \Omega \to B \text{ which sends } (\gamma, x) \text{ to } \gamma x. \text{ It is proved in [8] that } Y \text{ can be given the structure of a complex manifold such that } \sigma \text{ is holomorphic and } \Gamma \text{ acts holomorphically on } M. \text{ Let } E \text{ be the set of all points of } Y \text{ where the Jacobian determinant of } \sigma \text{ is zero. } E \text{ is a disjoint union of a countable number of nonsingular complex curves } E_i. \text{ Each } E_i \text{ is biholomorphic to } \sigma(E_i) \text{ under } \sigma. \text{ The set of all } \sigma(E_i) \text{ is precisely the set of all images of the 3 complex lines containing } \Delta_{123}, \Delta_{231}, \Delta_{312} \text{ under elements of } \Gamma. \text{ Note that, though the } E_i \text{'s are mutually disjoint and are locally finite, the } \sigma(E_i) \text{'s are not mutually disjoint and their union is dense in } B. \text{ It is true though that the 3 complex lines in } B \text{ containing } \Delta_{123}, \Delta_{231}, \text{ and } \Delta_{312} \text{ respectively are disjoint. For every } i, \text{ there exist an open neighborhood } Q_i \text{ of } E_i \text{ in } Y \text{ and an open neighborhood } U_i \text{ of } \sigma(E_i) \text{ in } B \text{ such that we have the following commutative diagram}

\[
\begin{array}{ccc}
Q_i & \xrightarrow{\theta} & \pi^{-1}(\tau(U_i)) \subseteq D \\
\sigma \downarrow & & \downarrow \pi \\
U_i & \xrightarrow{\tau} & \tau(U_i)
\end{array}
\]

where } \sigma(Q_i) = U_i, \text{ } \theta \text{ is biholomorphic, } \tau \in \text{Aut } B \text{ sends } \sigma(E_i) \text{ to the complex line } \{w_i = 0\} \cap B \text{ with } B \text{ realized as } \{(w_1, w_2) \in \mathbb{C}^2 | |w_1|^2 + |w_2|^2 < 1\}, \text{ } D = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^4 + |z_2|^2 < 1\}, \text{ and } \pi(w_i, w_2) = (w_1^\sigma, w_2). \text{ For } \gamma_0 \in \Gamma, \text{ the map } \Gamma \times \Omega \to \Gamma \times \Omega \text{ sending } (\gamma, x) \text{ to } (\gamma^*_0 \gamma, x) \text{ induces an element } \gamma_0^* \text{ of Aut } Y. \text{ } \gamma_0^* \text{ covers } \gamma_0 \text{ in the sense that } \sigma \gamma_0^* = \gamma_0 \sigma. \text{ The group } \Gamma^* \subseteq \text{Aut } Y \text{ of all such } \gamma_0^* \text{ is discrete. The quotient } Y/\Gamma^* \text{ is compact, but in general has singularities. We can remove the singularities by using, instead of } \Gamma^*, \text{ a normal subgroup } \Gamma_0^* \text{ of finite index in } \Gamma \text{ which contains no element of finite order except the identity. For example, } \Gamma \text{ can be regarded as a subgroup of } \text{PSL}(3, \mathbb{Z}(e^{\pi(1-1/20)})) \text{ and it suffices to take } \Gamma_0 \text{ as the subgroup of all elements } \equiv 1 \text{ modulo some prime divisor of 3 in } \mathbb{Z}(e^{\pi(1-1/20)}), \text{ and take as } \Gamma_0^* \text{ the subgroup of } \Gamma^* \text{ corresponding to } \Gamma_0. \text{ Let } M = Y/\Gamma_0^*. \text{ Then } M \text{ is a compact nonsingular complex surface and is the surface we want to construct.}
5. Construction of the Kähler metric

Let $\kappa: Y \to M$ be the quotient map. $\kappa(E)$ is the disjoint union of a finite number of nonsingular complex curves $C_\nu (1 \leq \nu \leq l)$ in $M$. Let $\{E_\nu\}$ be the set of all $E_\nu$ such that $\kappa(E_\nu) = C_\nu$. Each $E_\nu$ is a universal covering of $C_\nu$. Let $\Gamma_\nu \subset \Gamma_\alpha^*$ be the stabilizer of $E_\nu$. Then $C_\nu = E_\nu^\nu/\Gamma_\nu^\nu$.

Let $W_\nu$ be an open neighborhood of $C_\nu$ in $M$ such that the $W_\nu$'s are mutually disjoint. Take a $C^\infty$ function $0 \leq \rho_\nu \leq 1$ on $W_\nu$ with compact support such that $\rho_\nu \equiv 1$ on an open neighborhood of $C_\nu$. For $1 \leq \nu \leq l$, fix some $E_\nu^\nu$. We have an open neighborhood $Q_\nu$ of $E_\nu^\nu$ and an open neighborhood $U_\nu$ of $\sigma(E_\nu^\nu)$ such that $\sigma(Q_\nu) = U_\nu$ and

$$
\begin{array}{ccc}
Q_\nu & \xrightarrow{\theta_\nu} & \pi^{-1}(\tau_\nu^{-1}(U_\nu)) \subset D \\
\sigma & \downarrow & \pi \\
U_\nu & \xrightarrow{\tau_\nu} & \tau_\nu(U_\nu) \subset B
\end{array}
$$

is commutative, where $\theta_\nu$ is biholomorphic, $\tau_\nu \in \text{Aut} B$ sends $\sigma(E_\nu^\nu)$ to the complex line $\{w_1 = 0\} \cap B$ with $B$ realized as $\{(w_\nu, w_i) \in \mathbb{C}^l | |w_\nu|^2 + |w_i|^2 < 1\}$, $D = \{(z_\nu, z_i) \in \mathbb{C}^l | |z_\nu|^2 + |z_i|^2 < 1\}$, and $\pi(w_\nu, w_i) = (w_\nu, w_i)$. By replacing $Q_\nu$ by a possibly smaller neighborhood, we can assume that $Q_\nu$ is a component of $\kappa^{-1}(W_\nu)$ and is mapped bijectively onto $W_\nu$ by $\kappa$. $Q_\nu$ is stabilized by $\Gamma_\nu^\nu$.

Let $\Phi$ be the Bergman kernel form on $D$. Let $\Phi_\nu^\nu = \theta_\nu^* \Phi$. So $\Phi_\nu^\nu$ is a $(2, 2)$-form on $Q_\nu$ and is invariant under $\Gamma_\nu^\nu$. Define $Q_\nu^\nu = \gamma^* \Phi_\nu^\nu$ and $Q_i^\nu = \gamma^{-1}(Q_\nu^\nu)$, where $\gamma \in \Gamma_\nu^\nu$ maps $E_i^\nu$ to $E_\nu^\nu$. These definitions are independent of the choice of $\gamma$. The $Q_i^\nu$'s are mutually disjoint. Let $\bar{Q} = \bigcup_{i, \nu} Q_i^\nu$ and let $\bar{\Phi}$ be the $(2, 2)$-form on $\bar{Q}$ which agrees with $\Phi_\nu^\nu$ on $Q_i^\nu$. Then both $\bar{Q}$ and $\bar{\Phi}$ are $\Gamma_\alpha^*$-invariant.

Let $\rho$ be the function on $\bar{Q}$ which agrees with $\rho_\nu \circ \kappa$ on $Q_i^\nu$. Then $\rho$ has compact support on $\bar{Q}$ mod $\Gamma_\alpha^*$, is identically 1 on an open neighborhood of $E$, and is $\Gamma_\alpha^*$-invariant. Let $\Psi$ be the Bergman kernel form on $B$ and let $\bar{\Psi} = \sigma^* \Psi$. Define $\Theta = \rho \bar{\Phi} + (1 - \rho) \bar{\Psi}$. At every point of $E$ the $(2, 2)$-form $\bar{\Phi}$ is a positive multiple of the volume form. At every point of $Y$ the $(2, 2)$-form $\bar{\Psi}$ is a nonnegative multiple of the volume form. At every point of $Y - E$ the $(2, 2)$-form $\bar{\Psi}$ is a positive multiple of the volume form. It follows that at every point of $Y$ the $(2, 2)$-form $\Theta$ is a positive multiple of the volume form. $\Theta$ is clearly $\Gamma_\alpha^*$-invariant. Consider the Kähler metric $\partial \bar{\partial} \log \Theta$ on $Y$ (i.e., $\partial \bar{\partial}$ of the log of the coefficient of $\Theta$). This Kähler metric is $\Gamma_\alpha^*$-invariant and it agrees with $\partial \bar{\partial} \log \bar{\Phi}$ on some open neighborhood of $E$. It follows from the result of Section 3 that on some open neighborhood of $E$ the Kähler metric $\partial \bar{\partial} \log \bar{\Phi} + \lambda \sigma^*(\partial \bar{\partial} \log \Psi)$ has negative sectional curvature (and actually
satisfies a stronger negative curvature condition as defined in § 3) for all \( \lambda \geq 0 \). Take a compact subset \( A \) of \( Y \) with \( \kappa(A) = M \). Then for \( \lambda \) sufficiently large, the Kähler metric \( \partial \bar{\partial} \log \Phi + \lambda \sigma^*(\partial \bar{\partial} \log \psi) \) has negative sectional curvature (actually satisfies a stronger negative curvature condition) at every point of \( A \). As a consequence this \( \Gamma^*_\sigma \)-invariant Kähler metric for sufficiently large \( \lambda \) induces a Kähler metric on \( M \) which has negative sectional curvature (and actually satisfies a stronger negative curvature condition).

6. Computation of \( c_3(M) \)

We will show that the universal covering of \( M \) is not biholomorphic to the open 2-ball \( B \) by calculating \( c_3 \) and \( c_2 \) for \( M \) and verifying that \( c_2 \neq 3c_3 \).

We first calculate \( c_3(M) \) which is simply the Euler characteristic of \( M \). We do it by choosing a suitable cellular decomposition of \( M \). For this purpose we have to understand how \( \Gamma \) identifies the \( \nu \)-faces of \( \Omega \) for \( \nu = 0, 1, 2, 3 \). For detailed proofs of this identification, we again refer to [8]. Let \( p = \) the order of \( R_{12} \), \( N = \) the order of the group \( \Gamma_{12} \), \( r = \) the order of \( (R_1R_2)^c \) and \( s = \) the order of \( (R_1R_2R_3)^c \). In our case \( p = 5, N = 600, r = 40, \) and \( s = 8 \).

i) The 3 cyclic permutations 1 \( \rightarrow \) 2 \( \rightarrow \) 3 for the coordinates of \( C^3 \) define 3 elements of \( \Gamma \) and these 3 elements form precisely the stabilizer in \( \Gamma \) of \( \Omega \).

ii) Under \( \Gamma \) every 3-face of \( \Omega \) is identified with one of the four 3-faces \( \tilde{R}_1, \tilde{R}_2R_3, \tilde{R}_3R_1, \tilde{R}_2R_1 \). There is no identification among these four 3-faces. The stabilizer in \( \Gamma \) of any one of them is trivial.

iii) Under \( \Gamma \) every 2-face of \( \Omega \) is identified with one of the nine 2-faces

\[
\tilde{R}_1 \cap (\tilde{R}_1R_2), \tilde{R}_2 \cap (\tilde{R}_1R_2), \tilde{R}_1 \cap (\tilde{R}_1R_3R_1), \\
\tilde{R}_2 \cap (\tilde{R}_1R_3R_1), \tilde{R}_3 \cap (\tilde{R}_2R_1), (\tilde{R}_2R_1) \cap (\tilde{R}_1R_3),
\]

the complex line cell in \( \Omega \) defined by \( e^i_1 \) (which we will simply call \( e^i_1 \)), \( \Delta_{123} \), \( \Delta_{231} \). There is no identification among these nine 2-faces. The stabilizers in \( \Gamma \) of the first six 2-faces are all trivial. The stabilizer in \( \Gamma \) of \( e^i_1 \) is of order \( p \). The stabilizer in \( \Gamma \) of \( \Delta_{123} \) is of order \( r \). The stabilizer in \( \Gamma \) of \( \Delta_{231} \) is of order \( s \).

iv) Under \( \Gamma \) every 1-face of \( \Omega \) is identified with one of the following eight 1-faces among which there is no identification. \( p_{23}t_{13} \) (order of stabilizer \( = p \)), \( p_{23}t_{23} \) (order of stabilizer \( = p \)), \( p_{12}s_{12} \) via \( s^*_1 \) (order of stabilizer \( = 1 \)), \( p_{12}s_{12} \) via \( s^*_1 \) (order of stabilizer \( = 1 \)), \( s_{23}\tilde{s}_{12} \) (order of stabilizer \( = 2r \)), \( s_{21}\tilde{s}_{32} \) (order of stabilizer \( = 2s \)), \( s_{23}t_{13} \) (order of stabilizer \( = r \)), \( s_{21}t_{23} \) (order of stabilizer \( = s \)).

v) Under \( \Gamma \) every 0-face of \( \Omega \) can be identified with one of the follow-
ing four 0-faces among which there is no identification: \( p_{12} \) (order of stabilizer \( = N \)), \( t_{13} \) (order of stabilizer \( = pr \)), \( t_{31} \) (order of stabilizer \( = ps \)), \( s_{12} \) (order of stabilizer \( = rs \)).

\( \Omega \) can be lifted to a domain \( \tilde{\Omega} \) in \( Y \). We construct a cellular decomposition \( L \) of \( Y \) by using \( \tilde{\Omega} \) and its \( \nu \)-faces (\( \nu = 0, 1, 2, 3 \)) corresponding to the \( \nu \)-faces of \( \Omega \) and by using all their images under \( \Gamma^* \). This cellular decomposition \( L \) induces a cellular decomposition \( K \) of \( M \). (Note that since the action of \( \Gamma^*_0 \) on \( Y \) is fixed point free, it follows that no element of \( \Gamma^*_0 \) other than the identity element stabilizes any cell of \( L \).) Let \( k \) be the index of \( \Gamma^*_0 \) in \( \Gamma^* \). Then the number of distinct \( \nu \)-cells in \( K \) which each \( \nu \)-face on our list above gives rise to is equal to \( k \) divided by the order of the stabilizer in \( \Gamma^* \) of that \( \nu \)-face. Let \( \varepsilon \), be the number of \( \nu \)-cells in \( K \). Then

\[
\begin{align*}
\varepsilon_4 &= \frac{k}{3} \\
\varepsilon_3 &= 4k \\
\varepsilon_2 &= k \left( 6 + \frac{1}{p} + \frac{1}{r} + \frac{1}{s} \right) \\
\varepsilon_1 &= k \left( \frac{2}{p} + 2 + \frac{1}{2r} + \frac{1}{2s} + \frac{1}{r} + \frac{1}{s} \right) \\
\varepsilon_0 &= k \left( \frac{1}{N} + \frac{1}{pr} + \frac{1}{ps} + \frac{1}{rs} \right).
\end{align*}
\]

Hence the Euler characteristic is

\[
k \left( \frac{1}{3} - 4 + \left( 6 + \frac{1}{p} + \frac{1}{r} + \frac{1}{s} \right) - \left( \frac{2}{p} + 2 + \frac{1}{2r} + \frac{1}{2s} + \frac{1}{r} + \frac{1}{s} \right) \right. \\
+ \left( \frac{1}{N} + \frac{1}{pr} + \frac{1}{ps} + \frac{1}{rs} \right)
\]

\[
= k \left( \frac{1}{3} + \frac{1}{N} + \frac{1}{pr} + \frac{1}{ps} + \frac{1}{rs} - \frac{1}{p} - \frac{1}{2r} - \frac{1}{2s} \right)
\]

\[
= k \left( \frac{1}{3} + \frac{1}{600} + \frac{1}{200} + \frac{1}{40} + \frac{1}{320} - \frac{1}{5} - \frac{1}{80} - \frac{1}{16} \right)
\]

\[
= \frac{149k}{1600}.
\]

This method of calculating the Euler characteristic works also for other values of \( \arg \varphi^3 \).

7. Noneuclidean volume of a fundamental domain

The calculation of \( c_2 \) for \( M \) is much more complicated than that of \( c_1 \). At one point the noneuclidean volume of \( \Omega \) is needed in the calculation. So
we have to calculate first the noneuclidean volume of $\Omega$.

Let the open 2-ball $B$ be realized by $\{ w = (w_1, w_2) \in C^2 | |w_1|^2 + |w_2|^2 < 1 \}$. Let $\bar{E}(w) = \log(1/(1 - |w_1|^2 - |w_2|^2))$. Then the invariant Kähler metric of $B$ with holomorphic sectional curvature $-1$ is given by $2 \text{Re} \left( \sum \frac{\partial^2 \bar{E}}{\partial w_\alpha \partial \bar{w}_\beta} dw_\alpha d\bar{w}_\beta \right)$. We have the following transformation law for $\bar{E}$:

$$\bar{E}(\gamma w) + \frac{2}{3} \log |J_\gamma(w)|^3 = \bar{E}(w)$$

for any automorphism $w \mapsto \gamma w$ of $B$, where $J_\gamma$ is the Jacobian determinant for $\gamma$ (calculated with respect to the coordinates $w = (w_1, w_2)$).

We introduce a new coordinate system for $B$. We realize $B$ as the set of all points $[x_1, x_2, x_3] \in P_3$ such that

$$\begin{pmatrix}
1 & -a\varphi & -a\bar{\varphi} \\
-a\bar{\varphi} & 1 & -a\varphi \\
-a\varphi & -a\bar{\varphi} & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
< 0.$$ 

Recall that $a = 1/(2 \sin(\pi/5))$ and $\varphi$ is a complex number of modulus 1 so that the determinant of the matrix $\Delta = 1 - 3a^2 - a^3(\varphi^3 + \bar{\varphi}^3)$ is negative. Introduce the coordinate system $z = (z_1, z_2)$ on $B$ defined by

$$\begin{cases}
z_1 = \frac{x_1}{x_3} \\
z_2 = \frac{x_2}{x_3}.
\end{cases}$$

This is indeed a global coordinate system on $B$, because the positive definiteness of the matrix

$$\begin{pmatrix}
1 & -a\varphi \\
-a\bar{\varphi} & 1
\end{pmatrix}$$

implies that $B$ is disjoint from $\{ x_3 = 0 \}$. Let $w = f(z)$ be the function relating the two coordinate systems $w$ and $z$ on $B$. Define

$$\Xi(z) = \bar{E}(f(z)) + \frac{2}{3} \log |J_f(z)|^3.$$ 

Clearly the invariant metric of $B$ of holomorphic sectional curvature $-1$ is given by

$$2 \text{Re} \left( \sum \frac{\partial^2 \Xi}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha d\bar{z}_\beta \right).$$

We claim that $\Xi$ obeys the following transformation law for any automorphism $z \mapsto \theta z$ of $B$. 
\[ \Xi(\theta z) + \frac{2}{3} \log \left| J_\theta(z) \right|^2 = \Xi(z). \]

The verification is straightforward. In the rest of this section, we will use only the coordinate system \( z \) and all the Jacobian determinants of automorphisms of \( B \) will be calculated with respect to this coordinate system.

The twenty-four 3-faces of \( \Omega \) occur in pairs. We denote them by \( F_j^\pm \) (\( j = 1, \ldots, 12 \)). More precisely we agree that

\[ \{ F_j^+ \}_{j=1}^{12} = \{ \bar{R}_i, \bar{R}_i \bar{R}_i, \bar{R}_i \bar{R}_i \bar{R}_i \}_{i,j=1,2,3,i \neq j}, \]

\[ \{ F_j^- \}_{j=1}^{12} = \{ \bar{R}_i^{-1}, \bar{R}_i \bar{R}_i^{-1}, \bar{R}_i \bar{R}_i \bar{R}_i^{-1} \}_{i,j=1,2,3,i \neq j}. \]

(Recall that \( R_i R_j R_i = R_j R_i R_j \).) \( \Omega \) is oriented. Its orientation is described in Section 4. \( R_i \) maps \( \bar{R}_i \) to \( \bar{R}_i^{-1} \), \( R_i R_j \) maps \( \bar{R}_i \bar{R}_j \) to \( (R_i R_j)^{-1} \), and \( R_i R_j R_i \) maps \( \bar{R}_i \bar{R}_j \bar{R}_i \) to \( (R_i R_j R_i)^{-1} \). These maps precisely reverse the orientations of the 3-faces. We use \( f_j : F_j^+ \to F_j^- \) to denote these automorphisms. When we take orientations into account, \( f_j \) actually maps \( F_j^+ \) to \( -F_j^- \).

\[ \pm \text{vol}(\Omega) = \int_{\Omega} \left( \frac{\sqrt{1 - \bar{\partial} \bar{\partial} \Xi}}{2} \right)^2 = -\frac{1}{2} \int_{\partial \Omega} \bar{\partial} \Xi \wedge \bar{\partial} \Xi \]

\[ = -\frac{1}{2} \sum_{j=1}^{12} \left( \int_{F_j^+} \bar{\partial} \Xi \wedge \bar{\partial} \Xi + \int_{F_j^-} \bar{\partial} \Xi \wedge \bar{\partial} \Xi \right) \]

\[ = -\frac{1}{2} \sum_{j=1}^{12} \int_{F_j^+} (\bar{\partial} \Xi \wedge \bar{\partial} \Xi) - (\bar{\partial} \Xi \wedge \bar{\partial} \Xi) \circ f_j. \]

We use \( \pm \text{vol}(\Omega) \), because we do not know whether the orientation we have chosen for \( \partial \Omega \) in Section 4 is compatible with the orientation induced by the volume form. Since \( \Xi \circ f_j = \Xi - 2/3 \log |J_{f_j}|^2 \), it follows that

\[ (\bar{\partial} \bar{\partial} \Xi) \circ f_j = \bar{\partial} \bar{\partial} \Xi, \]

\[ (\bar{\partial} \Xi) \circ f_j = \bar{\partial} \Xi - \frac{2}{3} \bar{d} \log J_{f_j}. \]

Hence

\[ \int_{F_j^+} (\bar{\partial} \Xi \wedge \bar{\partial} \Xi) - (\bar{\partial} \Xi \wedge \bar{\partial} \Xi) \circ f_j \]

\[ = \frac{2}{3} \int_{F_j^+} \bar{d} \log J_{f_j} \wedge \bar{\partial} \bar{\partial} \Xi \]

\[ = \frac{2}{3} \int_{\partial F_j^+} \log J_{f_j} \bar{\partial} \bar{\partial} \Xi \]

after we choose a branch for \( \log J_{f_j} \) on \( B \), and
\[ \pm \text{vol}(\Omega) = -\frac{1}{3} \sum_{j=1}^{12} \int_{\partial F_j^+} \log \tilde{J}_{f_j} \overline{\partial} \Xi \]
\[ = \frac{\sqrt{-1}}{3} \sum_{j=1}^{12} \int_{\partial F_j^+} (\log \tilde{J}_{f_j}) \omega , \]
where \( \omega = \sqrt{-1} \overline{\partial} \Xi \) is the Kähler form for the invariant metric on \( B \) with holomorphic sectional curvature \(-1\).

As is well-known, \( J_{\tau_1 \tau_2} = (J_{\tau_1} \circ \gamma_2) J_{\tau_2} \). From this we get for \( x \in e_j^i \),
\[ 1 = J_{R_j^i}(x) = \left( J_{R_j}(x) \right)^5 . \]

We define the branch of \( \log J_{f_j} \) in the following way. \( R_j \) is a complex reflection fixing \( e_j^i \) pointwise. The order of \( R_j \) is 5. We define the branches:
\[ \log J_{R_j} = \frac{2\pi \sqrt{-1}}{5} + 2\pi n \sqrt{-1} \]

at \( e_j^i \) for \( j = 1, 2, 3 \), where \( n \) is an indeterminate integer;
\[ \log J_{R_j R_j} = (\log J_{R_j}) \circ R_j + \log J_{R_j} ; \]
\[ \log J_{R_j R_j R_j} = (\log J_{R_j}) \circ (R_j R_j) + (\log J_{R_j}) \circ R_j + \log J_{R_j} . \]

Note that the last two are defined from the first one according to the chain rule. Moreover,
\[ \log J_{R_j R_j R_j}(p_{ij}) = 2\pi \sqrt{-1} \left( \frac{3}{5} + 3n \right) = \log J_{R_j R_j R_j}(p_{ij}) . \]

Since \( R_j R_j R_j = R_j R_j R_j \), it follows that \( \log J_{R_j R_j R_j} = \log J_{R_j R_j R_j} \) and is well-defined.

To compute \( \sum_{j=1}^{12} \int_{\partial F_j^+} (\log \tilde{J}_{f_j}) \omega \), we first observe that
\[ \partial F_j^+ = \left( \bigcup_{i=1}^{12} F_j^+ \cap F_i^+ \right) \cup \left( \bigcup_{i=1}^{12} F_j^+ \cap F_i^- \right) \]
(where of course some of the \( F_j^+ \cap F_i^+ \) and \( F_j^+ \cap F_i^- \) are empty). Every 2-face of \( \Omega \) belongs precisely to two 3-faces and therefore is of the form \( F_j^+ \cap F_i^+ \), \( F_j^+ \cap F_i^- \), or \( F_j^- \cap F_i^- \). We recall that whenever we represent a 2-face as the intersection of two 3-faces \( F'' \cap F''' \), it carries the orientation induced by the first 3-face \( F'' \). So \( F'' \cap F''' = -F''' \cap F'' \). Using this orientation convention we have
\[ \sum_{j=1}^{12} \int_{\partial F_j^+} (\log \tilde{J}_{f_j}) \omega \]
\[ = \sum_{j=1}^{12} \left( \sum_{i=1}^{12} \int_{F_j^+ \cap F_i^+} (\log \tilde{J}_{f_j}) \omega + \sum_{i=1}^{12} \int_{F_j^+ \cap F_i^-} (\log \tilde{J}_{f_j}) \omega \right) \]
\[ = \sum_{i} \sum_{S_i} \tilde{\gamma_i} \omega \]
where \( \{S_i\} \) is the set of all 2-faces of \( \Omega \) and
i) when $S_\nu$ is of the form $F^+_j \cap F^+_i$, $\eta_\nu = \log J_{f_j} - \log J_{f_i}$;
ii) when $S_\nu$ is of the form $F^+_j \cap F^-_i$, $\eta_\nu = \log J_{f_j}$;
iii) when $S_\nu$ is of the form $F^-_j \cap F^-_i$, $\eta_\nu = 0$.

So in order to compute $\sum_{j=1}^{12} \int_{x_{F_j}} (\log J_{f_j}) \omega$, we have to look at the set of all 2-faces of $\Omega$. There are the following 3 types of 2-faces of $\Omega$.

i) The eighteen 2-faces passing through one of the three points $p_{13}$, $p_{23}$, $p_{12}$.

ii) The three 2-faces $\hat{R}_1 \cap \hat{R}_1^{-1}$, $\hat{R}_2 \cap \hat{R}_2^{-1}$, $\hat{R}_3 \cap \hat{R}_3^{-1}$ (i.e., $e^+_{1}$, $e^+_{2}$, $e^+_{3}$).

iii) The six 2-faces $\Delta_{ijk}$ with $(i, j, k) = (1, 2, 3)$ ($\Delta_{ijk}$ = the triangle $t_{ijk8}9_{ij} = \hat{R}_j \hat{R}_k \cap (\hat{R}_i \hat{R}_j)^{-1}$).

First, consider the eighteen 2-faces passing through some $p_{ij}$. They can be grouped in six groups of three each. We give them below in the following six diagrams.
In these diagrams the arrows give elements of \( \Gamma \) mapping a 2-face to another 2-face. These maps preserve orientations when the 2-faces are given the orientations in the diagrams. A triple of points in the diagrams denotes the triangle with the three points as vertices. These triples of points are ordered and they give also the orientations of the triangles. The equality signs mean not just equality as sets but also equality in orientation. We have

\[ \sum_{\mathcal{S}_v \text{ in one of the above groups}} \int_{\mathcal{S}_v} \gamma_v \omega = 0. \]

We prove this only for the first group. The proofs for the other five groups are completely analogous.
\[
\sum_{\delta_v \text{ in the first group}} \int_{S_v} \gamma_v \omega \\
= \int_{\tilde{R}_1 \cap \tilde{R}_2 \cap \tilde{R}_3} (\log \tilde{J}_{R_1 R_2} - \log \tilde{J}_{R_1 R_3}) \omega + \int_{\tilde{R}_1 \cap (\tilde{R}_2 \cap \tilde{R}_3)} (\log \tilde{J}_{R_1}) \omega \\
= \int_{\tilde{R}_1 \cap \tilde{R}_2 \cap \tilde{R}_3} (\log \tilde{J}_{R_1 R_2} - \log \tilde{J}_{R_1 R_3}) \omega + \int_{\tilde{R}_2 \cap \tilde{R}_1 \cap \tilde{R}_3} (\log \tilde{J}_{R_1}) \circ (R_1 R_3) \omega \\
= \int_{\tilde{R}_1 \cap \tilde{R}_2 \cap \tilde{R}_3} (\log \tilde{J}_{R_1 R_2} - \log \tilde{J}_{R_1 R_3} + (\log \tilde{J}_{R_1}) \circ (R_1 R_3)) \omega \\
= 0
\]

from the definition of the branch of \( \log \tilde{J}_{R_i} \). Thus we have

\[
\pm \text{vol} (\Omega) = \frac{\sqrt{-1}}{3} \sum_{i=1}^3 \int_{\tilde{R}_i \cap \tilde{R}_i^{-1}} (\log \tilde{J}_{R_i}) \omega \\
+ \frac{\sqrt{-1}}{3} \sum_{(i, j, k) = (1, 2, 3)} \int_{\tilde{R}_i \cap \tilde{R}_j \cap \tilde{R}_k} (\log \tilde{J}_{R_i R_j R_k}) \omega.
\]

Now

\[
\int_{\tilde{R}_i \cap \tilde{R}_i^{-1}} (\log \tilde{J}_{R_i}) \omega = \left( -\frac{2\pi}{5} \sqrt{-1} - 2\pi n \sqrt{-1} \right) \text{signed area of } \tilde{R}_i \cap \tilde{R}_i^{-1}.
\]

\(\tilde{R}_i \cap \tilde{R}_i^{-1}\) is the quadrilateral \(t_{32} p_{31} t_{32} p_{21}\).

The area of \(\tilde{R}_i \cap \tilde{R}_i^{-1}\) is therefore

\[
2\pi - \left( \left( \frac{\pi}{2} - \frac{\pi}{5} + \text{arg } \varphi \right) + \left( \frac{\pi}{2} - \frac{\pi}{5} - \text{arg } \varphi \right) + \frac{\pi}{5} \right) = \frac{6\pi}{5}.
\]

Recall that the cyclic permutations \(1 \rightarrow 2 \rightarrow 3\) of the coordinates of \(C^3\) define elements of \(\Gamma\). Hence some element of \(\Gamma\) maps \(\tilde{R}_i \cap \tilde{R}_i^{-1}\) to \(\tilde{R}_2 \cap \tilde{R}_2^{-1}\) with orientation preserved. It follows that

\[
(\dagger) \quad \frac{\sqrt{-1}}{3} \sum_{i=1}^3 \int_{\tilde{R}_i \cap \tilde{R}_i^{-1}} (\log \tilde{J}_{R_i}) \omega = \pm \frac{1}{3} \left( \frac{6\pi}{5} + 6\pi n \right) \frac{6\pi}{5}.
\]

We have the \(\pm\) sign on the right-hand side, because we do not know whether the signed area of \(\tilde{R}_i \cap \tilde{R}_i^{-1}\) is \(+\) or \(-\) the area of \(\tilde{R}_i \cap \tilde{R}_i^{-1}\).
The six $\Delta_{ij}$s can be divided into two groups. We give them in the following two diagrams.

$$\Delta_{123} = \overset{\sim}{R_1 R_2 \cap (R_1 R_2)^{-1}} = (t_{123}, s_{123}, \bar{s}_{12})$$

$$\Delta_{231} = \overset{\sim}{R_2 R_3 \cap (R_2 R_3)^{-1}} = (t_{231}, s_{231}, \bar{s}_{231})$$

$$\Delta_{312} = \overset{\sim}{R_3 R_1 \cap (R_3 R_1)^{-1}} = (t_{312}, s_{312}, \bar{s}_{312})$$

These two diagrams are to be interpreted in the same way as the preceding six diagrams.

From the first diagram it follows that

$$\int_{\Delta_{123}} (\log J_{R_2 R_3}) \omega + \int_{\Delta_{231}} (\log J_{R_2 R_3}) \omega + \int_{\Delta_{312}} (\log J_{R_2 R_3}) \omega$$

$$= \int_{\Delta_{123}} (\log J_{R_2 R_3}) \omega + \int_{\Delta_{231}} ((\log J_{R_2 R_3}) \circ (R_2 R_3)) \omega$$

$$+ \int_{\Delta_{312}} ((\log J_{R_2 R_3}) \circ (R_2 R_3 R_2 R_3)) \omega$$

$$= \int_{\Delta_{123}} (\log J_{(R_1 R_2 R_3)^2}) \omega$$

where the branch $\log J_{(R_1 R_2 R_3)^2}$ is defined as

$$\log J_{R_2 R_3} + (\log J_{R_2 R_3}) \circ (R_2 R_3) + (\log J_{R_2 R_3}) \circ (R_3 R_2 R_3 R_3),$$

that is, according to the chain rule.

Likewise from the second diagram it follows that
\[
\int_{\Delta_{123}} (\log J_{R_3 R_2}) \omega + \int_{\Delta_{321}} (\log J_{R_2 R_3}) \omega + \int_{\Delta_{213}} (\log J_{R_1 R_3}) \omega = \int_{\Delta_{321}} (\log J_{(R_3 R_2 R_1)^2}) \omega
\]

where the branch \( \log J_{(R_3 R_2 R_1)^2} \) is defined according to the chain rule. We know that

\[
\begin{align*}
J_{(R_1 R_2 R_3)^2} &= \sqrt{-1} e^{(3\pi/5) \sqrt{-1} \varphi^3} \quad \text{at } \Delta_{123}, \\
J_{(R_3 R_2 R_1)^2} &= \sqrt{-1} e^{(3\pi/5) \sqrt{-1} \varphi^3} \quad \text{at } \Delta_{321}.
\end{align*}
\]

Hence

\[
\begin{align*}
\log J_{(R_1 R_2 R_3)^2} &= \sqrt{-1} \left( -\frac{3\pi}{5} - \frac{\pi}{2} + 3 \arg \varphi^3 - 12\pi n + 2\pi q \right) \quad \text{at } \Delta_{123}, \\
\log J_{(R_3 R_2 R_1)^2} &= \sqrt{-1} \left( -\frac{3\pi}{5} - \frac{\pi}{2} - 3 \arg \varphi^3 - 12\pi n + 2\pi q' \right) \quad \text{at } \Delta_{321},
\end{align*}
\]

where \( q \) and \( q' \) are integers independent of \( n \), because the branches \( \log J_{(R_1 R_2 R_3)^2} \) and \( \log J_{(R_3 R_2 R_1)^2} \) are defined from the branches \( \log J_{R_1}, \log J_{R_2}, \log J_{R_3} \) by the chain rule. We will determine \( q \) and \( q' \) later.

\[
\frac{1}{3} \sum_{(i, j, k) = (1, 2, 3)} \int_{\Delta_{ijk}} (\log J_{R_j R_k}) \omega
\]

\[
= -\frac{1}{3} \left( \left( -\frac{3\pi}{5} - \frac{\pi}{2} + 3 \arg \varphi^3 - 12\pi n + 2\pi q \right) \text{signed area of } \Delta_{123} \right.
\]

\[
\left. + \left( -\frac{3\pi}{5} - \frac{\pi}{2} - 3 \arg \varphi^3 - 12\pi n + 2\pi q' \right) \text{signed area of } \Delta_{321} \right).
\]

![Diagram](image)

The area of \( \Delta_{123} \) is

\[
\pi - \left( \frac{2\pi}{5} + \left( \frac{\pi}{2} - \frac{\pi}{5} - \arg \varphi^3 \right) \right) = \frac{3\pi}{10} + \arg \varphi^3.
\]

The area of \( \Delta_{321} \) is

\[
\pi - \left( \frac{2\pi}{5} + \left( \frac{\pi}{2} - \frac{\pi}{5} + \arg \varphi^3 \right) \right) = \frac{3\pi}{10} - \arg \varphi^3.
\]

Hence
$$\frac{\sqrt{-1}}{3} \sum_{\{i, j, k\} = \{1, 2, 3\}} \int_{\nu_{ijk}} (\log J_{R^2 u_k}) \omega$$

$$= -\frac{1}{3} \left( \pm \left( -\frac{11\pi}{10} + 3\arg \varphi^3 - 12\pi n + 2\pi q \right) \left( \frac{3\pi}{10} + \arg \varphi^3 \right) \right)$$

$$\pm \left( -\frac{11\pi}{10} - 3\arg \varphi^3 - 12\pi n + 2\pi q' \right) \left( \frac{3\pi}{10} - \arg \varphi^3 \right) \right).$$

Combining this with (†), we obtain

$$\pm \text{vol} (\Omega) = \pm \frac{1}{3} \left( 6\pi \frac{5}{n} + 6\pi n \right) \frac{6\pi}{5}$$

$$- \frac{1}{3} \left( \pm \left( -\frac{11\pi}{10} + 3\arg \varphi^3 - 12\pi n + 2\pi q \right) \left( \frac{3\pi}{10} + \arg \varphi^3 \right) \right)$$

$$\pm \left( -\frac{11\pi}{10} - 3\arg \varphi^3 - 12\pi n + 2\pi q' \right) \left( \frac{3\pi}{10} - \arg \varphi^3 \right) \right).$$

The left-hand side is independent of $n$. The right-hand side should also be independent of $n$. Since $|\arg \varphi^3| < \pi/2 - \pi/5$, it follows from

$$0 = \pm \frac{6\pi}{5} n \pm 2 \left( \frac{3\pi}{10} + \arg \varphi^3 \right) n \pm 2 \left( \frac{3\pi}{10} - \arg \varphi^3 \right) n$$

that the sign for the first term should be opposite to the signs of the last two terms. Hence

$$\pm \text{vol} (\Omega) = \frac{1}{3} \left( \frac{6\pi}{5} \right)^2 + \frac{1}{3} \left( -\frac{11\pi}{10} + 3\arg \varphi^3 + 2\pi q \right) \left( \frac{3\pi}{10} + \arg \varphi^3 \right)$$

$$+ \frac{1}{3} \left( -\frac{11\pi}{10} - 3\arg \varphi^3 + 2\pi q' \right) \left( \frac{3\pi}{10} - \arg \varphi^3 \right)$$

$$= \frac{13\pi^2}{50} + 2(\arg \varphi^3)^2 + \frac{2\pi q}{3} \left( \frac{3\pi}{10} + \arg \varphi^3 \right)$$

$$+ \frac{2\pi q'}{3} \left( \frac{3\pi}{10} - \arg \varphi^3 \right).$$

The integer $n$ is introduced only to determine the signs of the signed areas and we have no further use for it. From this point on, we choose $n = 0$.

Now we compute the integers $q$ and $q'$. By continuity considerations, it suffices to compute $q$ and $q'$ for the special case $\arg \varphi^3 = 0$. $q$ and $q'$ will then have the same values for a general $\arg \varphi^3$. Assume $\varphi = 1$. Let $\eta = e^{i \sqrt{-1} \theta}$. Then in terms of the homogeneous coordinates $[x_1, x_2, x_3]$,

$$s_{23} = [1, \eta \sqrt{-1}, \bar{\eta} \sqrt{-1}],$$

$$\bar{s}_{23} = [-\eta \sqrt{-1}, \eta \sqrt{-1}, 1].$$

We first calculate $q$. Since $s_{23}$ belongs to $\Delta_{123}$ and since the branch
\[ \log J_{(R_1 R_2 R_3)^2} \] is defined from the branches \( \log J_{R_j} \) by the chain rule, it suffices
to calculate the following values:

\[
\begin{align*}
\log J_{R_3} & \text{ at } s_{23} \\
\log J_{R_2} & \text{ at } R_3 s_{23} = \tilde{s}_{13} \\
\log J_{R_1} & \text{ at } R_2 \tilde{s}_{13} = s_{31} \\
\log J_{\bar{R}_3} & \text{ at } R_3 s_{31} = \tilde{s}_{21} \\
\log J_{\bar{R}_2} & \text{ at } R_3 \tilde{s}_{21} = s_{12} \\
\log J_{\bar{R}_1} & \text{ at } R_3 s_{12} = \tilde{s}_{32} .
\end{align*}
\]

The value of \( \log J_{(R_1 R_2 R_3)^2} \) at \( s_{23} \) will then be the sum of these six values. Recall that \( a = 1/(2 \sin(\pi/5)) \). In the homogeneous coordinate system
\([x_1, x_2, x_3], \)

\[
\begin{align*}
R_1 x_1 & = \eta^2 x_1 - a(\eta^2 - 1)x_2 - a(\eta^2 - 1)x_3 \\
R_1 x_2 & = x_2 \\
R_1 x_3 & = x_3 \\
R_2 x_1 & = x_1 \\
R_2 x_2 & = -a(\eta^2 - 1)x_1 + \eta^2 x_2 - a(\eta^2 - 1)x_3 \\
R_2 x_3 & = x_3 \\
R_3 x_1 & = x_1 \\
R_3 x_2 & = x_2 \\
R_3 x_3 & = -a(\eta^2 - 1)x_1 - a(\eta^2 - 1)x_2 + \eta^2 x_3 .
\end{align*}
\]

Hence in the coordinate system \( z = (z_1, z_2) , \)

\[
\begin{align*}
R_1 z_1 & = \eta^2 z_1 - a(\eta^2 - 1)z_2 - a(\eta^2 - 1) \\
R_1 z_2 & = z_2 \\
R_2 z_1 & = z_1 \\
R_2 z_2 & = -a(\eta^2 - 1)z_1 + \eta^2 z_2 - a(\eta^2 - 1) \\
R_3 z_1 & = \frac{z_1}{l(z_1, z_2)} \\
R_3 z_2 & = \frac{z_2}{l(z_1, z_2)} ,
\end{align*}
\]

where \( l(z_1, z_2) = -a(\eta^2 - 1)z_1 - a(\eta^2 - 1)z_2 + \eta^2 . \)

\[
\begin{align*}
J_{R_1} & \equiv \eta^2 \\
J_{R_2} & \equiv \eta^2 \\
J_{R_3} & \equiv \frac{\eta^2}{l(z_1, z_2)} .
\end{align*}
\]

It follows that the branches \( \log J_{R_1} , \log J_{R_2} \) will be identically \( \log \eta^2 \) on the
whole ball $B$. As multi-valued functions

$$
\log J_{R_3} = \log \frac{\eta^3}{l(z_1, z_2)^3} = \log \frac{\eta^2}{(-a(\eta^2 - 1))^3} - 3 \log \left( z_1 + z_2 - \frac{\eta^2}{a(\eta^2 - 1)} \right)
= \frac{13}{10} \pi \sqrt{-1} - 3 \log (z_1 + z_2 + \eta \sqrt{-1}) \text{ because } a(\eta^2 - 1) = \eta \sqrt{-1}.
$$

We have to obtain at $s_{23}$ and $s_{21}$ the values of the branch of $\log J_{R_3}$ which assumes the value $\log \eta^3$ at every point of $e_{3}^{*}$. Now $e_{3}^{*}$ is defined by

$$
(x_{1}, x_{2}, x_{3}) \begin{pmatrix}
1 & -a & -a \\
-a & 1 & -a \\
-a & -a & 1
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} = 0;
$$

that is,

$$
-a(x_{1} + x_{2}) + x_{3} = 0
$$
or equivalently

$$
z_{1} + z_{2} = \frac{1}{a}.
$$

At $s_{23}$, $z_{1} + z_{2} = -\eta \sqrt{-1} + 1$.
At $s_{21}$, $z_{1} + z_{2} = -2\eta \sqrt{-1}$.

A branch of $\log J_{R_3}$ corresponds to a branch of $\log (z_{1} + z_{2} + \eta \sqrt{-1})$. Our task is now reduced to calculating at $z_{1} + z_{2} = -\eta \sqrt{-1} + 1$ and $-2\eta \sqrt{-1}$ the values of the branch of $\log (z_{1} + z_{2} + \eta \sqrt{-1})$ on $B$ which assumes at $z_{1} + z_{2} = 1/a$ the value

$$
\frac{\sqrt{-1}}{3} \left( \frac{13\pi}{10} - \frac{2\pi}{5} \right) = \frac{3\pi}{10} \sqrt{-1}.
$$

Let $w = z_{1} + z_{2}$. In $B$ we join a point on $e_{3}^{*}$ to $s_{23}$ by a straight line segment and join a point on $e_{3}^{*}$ to $s_{21}$ by a straight line segment. Their images in the $w$-plane are straight line segments $[1/a, -\eta \sqrt{-1} + 1]$ and $[1/a, -2\eta \sqrt{-1}]$.
It is clear from the picture that the branch of \( \log (z_1 + z_2 + \eta \sqrt{-1}) = \log (w + \eta \sqrt{-1}) \) on \( B \), which assumes at \( w = 1/a \) the value \((3\pi/10)\sqrt{-1}\), should assume 0 at \( w = -\eta \sqrt{-1} + 1 \) and assume \(-(3\pi/10)\sqrt{-1}\) at \( w = -2\eta \sqrt{-1} \). Hence

\[
\log J_{R_2}(s_{23}) = \frac{13}{10} \pi \sqrt{-1}
\]

\[
\log J_{R_2}(s_{21}) = \frac{13}{10} \pi \sqrt{-1} - 3 \left( -\frac{3\pi}{10} \sqrt{-1} \right) = \frac{11\pi}{5} \sqrt{-1}
\]

and

\[
\log J_{(R_1 R_2 R_3)^2}(s_{23}) = 4 \frac{2\pi \sqrt{-1}}{5} + \frac{13\pi \sqrt{-1}}{10} + \frac{11\pi \sqrt{-1}}{5} = \frac{51\pi \sqrt{-1}}{10}.
\]

Comparing this with

\[
\log J_{(R_1 R_2 R_3)^2} = \sqrt{-1} \left( -\frac{3\pi}{5} - \frac{\pi}{2} + 2\pi q \right) \text{ at } \varDelta_{123}
\]

for the case \( \arg \varphi = 0 \), we conclude that \( q = -2 \).

In the case \( \arg \varphi = 0 \), the roles of the coordinates \( x_1, x_2 \) can be interchanged. Hence we conclude also that \( q' = -2 \).

Since \( \operatorname{vol}(\Omega) > 0 \) for \( \arg \varphi = 0 \), the sign in front of \( \operatorname{vol} \Omega \) in equation (8) should be +. Finally we have the formula

\[
\operatorname{vol}(\Omega) = \frac{27\pi}{50} - 2(\arg \varphi^3)^2.
\]

In the case \( \arg \varphi^3 = \pi/20 \), \( \operatorname{vol}(\Omega) = 107\pi^2/200 \).

8. Chern class of the normal bundle of a curve

In our computation of \( c^i \) for \( M \) we have to calculate the Chern class of the normal bundle of a curve. We carry out below the calculation of such Chern classes.

**Lemma 3.** Let the open 2-ball \( B \) be realized as \( \{(w_1, w_2) \in \mathbb{C}^2 | |w_1|^2 + |w_2|^2 < 1\} \). Then any element of \( \operatorname{Aut} B \) which stabilizes \( \{w_1 = 0\} \) must be of the form

\[
(w_1, w_2) \mapsto \left( e^{-\alpha} \frac{w_1 \sqrt{1 - |b|^2}}{bw_2 + 1}, e^{-\alpha'} \frac{w_2 + b}{bw_2 + 1} \right)
\]

with \( b \in \mathbb{C}, |b| < 1 \), and \( \alpha, \alpha' \in \mathbb{R} \).

**Proof.** Any element of \( \operatorname{Aut} B \) is of the form

\[
(w_1, w_2) \mapsto \left( \frac{a_{11} w_1 + a_{12} w_2 + a_{13}}{a_{31} w_1 + a_{32} w_2 + a_{33}}, \frac{a_{21} w_1 + a_{22} w_2 + a_{23}}{a_{31} w_1 + a_{32} w_2 + a_{33}} \right)
\]
where \((a_{ij})_{1 \leq i, j \leq 3}\) belongs to \(SU(2, 1)\). If this element of \(\text{Aut } B\) stabilizes \(\{w_i = 0\}\), then we have \(a_{12} = a_{13} = 0\). It follows that \(a_{21} = a_{31} = 0\), \(|a_{11}| = 1\), and \((a_{22}, a_{23})\) belongs to \(U(1, 1)\). The lemma follows from the fact that every element of \(U(1, 1)\) is of the form

\[
\begin{pmatrix}
\frac{e^{\sqrt{-1}\beta}}{\sqrt{1 - |b|^2}} & \frac{be^{\sqrt{-1}\beta'}}{\sqrt{1 - |b|^2}} \\
\frac{e^{\sqrt{-1}\beta'} \overline{b}}{\sqrt{1 - |b|^2}} & \frac{e^{\sqrt{-1}\beta}}{\sqrt{1 - |b|^2}}
\end{pmatrix}
\]

for some \(b \in \mathbb{C}\) with \(|b| < 1\) and some \(\beta, \beta' \in \mathbb{R}\).

Suppose \(H\) is a discrete subgroup of \(\text{Aut } B\) such that \(H\) is fixed point free and every element of \(H\) stabilizes \(\{w_i = 0\}\). Let \(S = B/H\) and let \(C\) be the image of \(\{w_i = 0\}\) in \(S\). We assume that \(C\) is compact. Let \(N\) be the normal bundle of \(C\) in \(M\).

**Lemma 4.** \(c_i(N) = (1/2)c_i(T_c)\), where \(T_c\) is the tangent bundle of \(C\).

**Proof.** Let \(L = B \cap \{w_i = 0\}\). Then \(L\) is the universal cover of \(C\) and \(C = L/H\). Let \(\pi: L \to C\) be the quotient map.

Let \(N^*\) be the dual bundle of \(N\). The sheaf of germs of holomorphic sections of \(N^*\) over \(C\) is isomorphic to the analytic restriction to \(C\) of the ideal-sheaf of \(C\). In other words, \(N^* = [C]_c^{-1}\).

Take a connected and simply connected open subset \(U\) of \(C\). Let \(\pi^{-1}(U) = \bigcup_i U_i\). A holomorphic section of \(N^*\) over \(U\) corresponds to \(\{f_i\}\), where \(f_i(w_i)\) is a holomorphic function on \(U_i\) such that, if \(\gamma \in H\) maps \(U_i\) to \(U_j\), then \(\gamma\) maps \(f_i(w_i)w_i\) to \(f_j(w_j)w_j\). That is, if \(\gamma(\zeta_1, \zeta_2) = (w_1, w_2)\) with

\[
\begin{align*}
w_1 &= \frac{e^{\sqrt{-1}a}\overline{\zeta_1}}{b\overline{\zeta_2} + 1} \\
w_2 &= \frac{e^{\sqrt{-1}a}(\overline{\zeta_2} + b)}{b\overline{\zeta_2} + 1}
\end{align*}
\]

(*)

then

\[
f_i(\zeta_2)\zeta_1 = f_j\left(\frac{e^{\sqrt{-1}a}(\overline{\zeta_2} + b)}{b\overline{\zeta_2} + 1}\right)\frac{e^{\sqrt{-1}a}\sqrt{1 - |b|^2}}{b\overline{\zeta_2} + 1}\zeta_1,
\]

or equivalently

\[
f_i(\zeta_2) = f_j\left(\frac{e^{\sqrt{-1}a}(\overline{\zeta_2} + b)}{b\overline{\zeta_2} + 1}\right)\frac{e^{\sqrt{-1}a}\sqrt{1 - |b|^2}}{b\overline{\zeta_2} + 1}.
\]

Now we look at the tangent bundle of \(C\). A holomorphic section of \(T_c\) over \(U\) corresponds to \(\{g_i\}\), where \(g_i(w_i)\) is a holomorphic function on \(U_i\) such that, if \(\gamma \in H\) maps \(U_i\) to \(U_j\), then \(\gamma\) sends \(g_i(w_i)\partial/\partial w_i\) to \(g_j(w_j)\partial/\partial w_j\).
That is, if $\gamma(\zeta_1, \zeta_2) = (w_1, w_2)$ is given by (*), then
\[
g_{ij}(\zeta_2) \frac{\partial}{\partial \zeta_2} = g_j \left( \frac{e^{\frac{-i\alpha}{b} (\zeta_2 + b)}}{b\zeta_2 + 1} \right) \frac{\partial}{\partial w_2} = g_j \left( \frac{e^{\frac{-i\alpha}{b} (\zeta_2 + b)}}{b\zeta_2 + 1} \right) \frac{1}{\partial w_2} \frac{\partial}{\partial \zeta_2},\]
or equivalently,
\[
g_{ij}(\zeta_2) = g_j \left( \frac{e^{\frac{-i\alpha}{b} (\zeta_2 + b)}}{b\zeta_2 + 1} \right) \frac{1}{e^{\frac{-i\alpha}{b} (1 - |b|^2)}} \frac{1}{(b\zeta_2 + 1)^2}.
\]

Let us look at the transition functions of $N$ and $T_c$. Cover $C$ by an open covering $\{W_v\}$ such that both $W_v$ and $W_v \cap W_v'$ are connected and simply connected. Select a component $\tilde{W}_v$ of $\pi^{-1}(W_v)$. When $W_v$ intersects $W_v'$, $\tilde{W}_v \cap \pi^{-1}(W_v \cap W_v')$ and $\tilde{W}_v \cap \pi^{-1}(W_v \cap W_v')$ are components of $\pi^{-1}(W_v \cap W_v')$. There exists a unique $\gamma_{\mu \nu} \in H$ mapping $\tilde{W}_v \cap \pi^{-1}(W_v \cap W_v')$ to $\tilde{W}_v \cap \pi^{-1}(W_v \cap W_v')$. By identifying holomorphic sections of $N^*$ (respectively $T_c$) over open subsets of $W_v$ with holomorphic functions on open subsets of $\tilde{W}_v$, and by applying the preceding arguments to $U = W_v \cap W_v'$, we conclude that if $\gamma_{\mu \nu}$ sends $(w_1, w_2)$ to
\[
\left( \frac{e^{\frac{-i\alpha}{\mu \nu} \sqrt{1 - |b_{\mu \nu}|^2}}}{b_{\mu \nu} w_2 + 1}, \frac{e^{\frac{-i\alpha}{\mu \nu} (w_2 + b_{\mu \nu})}}{b_{\mu \nu} w_2 + 1} \right),
\]
then the transition function from $W_v$ to $W_v'$ for $N^*$ is
\[
e^{\frac{-i\alpha}{\mu \nu}} \sqrt{1 - |b_{\mu \nu}|^2} \frac{1}{b_{\mu \nu} w_2 + 1}
\]
and the transition function from $W_v$ to $W_v'$ for $T_c$ is
\[
\frac{1}{(b_{\mu \nu} w_2 + 1)^2} \frac{e^{\frac{-i\alpha}{\mu \nu} (1 - |b_{\mu \nu}|^2)}}{b_{\mu \nu} w_2 + 1}.
\]
Hence $T_cN^{-2}$ is a flat bundle over $C$, i.e., the transition functions of $T_cN^{-2}$ can be chosen to be locally constant. Since the Chern class of a flat bundle is zero, it follows that $c_1(N) = (1/2)c_1(T_c)$.

9. Computation of $c_i(M)$

Denote by $K_B$ (respectively $K_Y, K_M$) the canonical line bundle of $B$ (respectively $Y, M$). Let $h$ be the Hermitian metric for $K_B$ defined by the reciprocal of the Bergman function of $B$. Let $\theta$ be the curvature form of $h$. Then $\theta$ equals $3/(4\pi)$ times the Kähler form of the invariant metric of $B$ (by
the invariant metric we mean the metric of \( B \) of holomorphic sectional curvature \(-1\)). The metric \( h \) for \( K_B \) defines a metric \( \tilde{h} \) for the line bundle \( K_Y - 6[E] \) on \( Y \), because the nonsingular curve \( E \) in \( Y \) is the singular set of \( \sigma: Y \to B \) and the vanishing order of the Jacobian determinant of \( \sigma \) along \( E \) is 6. Since \( h \) is invariant under \( \Gamma \), \( \tilde{h} \) is invariant under \( \Gamma^* \) and \( \tilde{h} \) defines a metric \( \tilde{h} \) for the line bundle \( K_M - 6 \sum_{\nu=1}^{l} [C_{\nu}] \) on \( M \). The curvature form of \( \tilde{h} \) is equal to \( \sigma^* \theta \) and the curvature form of \( h \) is therefore equal to the form \( \tilde{\theta} \) induced by \( \sigma^* \theta \) on \( M \).

\[
c_i^2(K_M - 6 \sum_{\nu=1}^{l} [C_{\nu}]) = \int_M \tilde{\theta}^2,
\]

which is equal to \( \int_Q (\sigma^* \theta)^2 \) for any open subset \( Q \) which is mapped one-to-one onto a set of \( M \) whose complement is of measure zero. Since the index of \( \Gamma^*_\sigma \) in \( \Gamma^* \) is \( k \) and since the stabilizer in \( \Gamma \) of \( \Omega \) is of order 3, it follows that

\[
\int_Q (\sigma^* \theta)^2 = \frac{k}{3} \int_Q \theta^2 = \frac{2k}{3} \left( \frac{3}{4\pi} \right)^2 \mathrm{vol}(\Omega).
\]

On the other hand,

\[
c_i^2(K_M - 6 \sum_{\nu=1}^{l} [C_{\nu}])
= c_i^2(K_M) - 12(K_M - 6 \sum_{\nu=1}^{l} [C_{\nu}]) \cdot \sum_{\nu=1}^{l} [C_{\nu}] - 36(\sum_{\nu=1}^{l} [C_{\nu}])^2
= c_i^2(K_M) - 12 \sum_{\nu=1}^{l} \int_{C_{\nu}} \tilde{\theta} - 36(\sum_{\nu=1}^{l} [C_{\nu}])^2.
\]

Since the curves \( C_{\nu} \) are mutually disjoint and the self-intersection number of \( C_{\nu} \) equals the Chern number of its normal bundle \( N_{\nu} \) in \( M \), it follows that

\[
c_i^2(K_M - 6 \sum_{\nu=1}^{l} [C_{\nu}]) = c_i^2(K_M) - 12 \sum_{\nu=1}^{l} \int_{C_{\nu}} \tilde{\theta} - 36 \sum_{\nu=1}^{l} c_i(N_{\nu}).
\]

First we calculate \( c_1(N_{\nu}). \) Recall that \( E_{i}^{(\nu)} \) is a universal covering of \( C_{\nu} \) and \( \sigma(E_{i}^{(\nu)}) \) is a complex line in \( B \) which is the image under an element of \( \Gamma \) of the complex line \( L \) containing \( \Delta_{1,2} \). After changing the position of \( \Omega \) by an element of \( \Gamma \) we can assume without loss of generality that \( \sigma(E_{i}^{(\nu)}) \) is the complex line \( L \) containing \( \Delta_{1,2} \). Let \( (\Gamma_{\nu})_L \) (respectively \( \Gamma_L ) \) be the stabilizer in \( \Gamma_\nu \) (respectively \( \Gamma \)) of \( L \). The stabilizer \( (\Gamma^*_{\nu})_L \) in \( \Gamma_{\nu} \) of \( E_{i}^{(\nu)} \) corresponds to \( (\Gamma_{\nu})_L \) in the correspondence \( \Gamma^* \to \Gamma \). Since the quotient space of \( L \) by \( (\Gamma_{\nu})_L \) is biholomorphic to \( C_{\nu} \), \( \Gamma_{\nu} \) is a fixed point free discrete subgroup of \( \text{Aut} B \). Let \( T = Y/(\Gamma_{\nu}^*E_{i}^{(\nu)}) \) and \( S = B/(\Gamma_L ) \). Let \( F \) (respectively \( A \)) be the image of \( E_{i}^{(\nu)} \) (respectively \( L \)) in \( T \) (respectively \( S \)). The map \( \sigma: Y \to B \) induces a map \( \tilde{\sigma}: T \to S \). \( \tilde{\sigma} \) maps \( F \) biholomorphically onto \( A \). There exists some open neighborhood \( U \) of \( F \) in \( T \) such that \( F \) is the singular set of \( \sigma|U \) whose Jacobian determinant vanishes to order 6 along \( F \) and \( U \) is biholomorphic
to an open neighborhood of $C_v$ in $M$ under some map which sends $F$ to $C_v$. Hence $N_\ell$ is isomorphic to the normal bundle $N_F$ of $F$ in $T$ and the normal bundle $N_A$ of $A$ in $S$ is isomorphic to the tensor product of 6 copies of $N_F$. It follows that

$$c_1(N_\ell) = c_1(N_F) = \frac{1}{6} c_1(N_A).$$

By Section 8, $c_1(N_A) = (1/2)c_1(T_A)$, where $T_A$ is the tangent bundle of $A$. Hence $c_1(N_\ell) = (1/12)c_1(T_A)$.

Let $n$ be the index of $(\Gamma_0)_L$ in $\Gamma_L$. Since $\Delta_{123}$ is a fundamental domain in $L$ for $\Gamma_L$ modulo the stabilizer in $\Gamma$ of $\Delta_{123}$ which is of order 40, it follows that the area of a fundamental domain $W$ in $L$ for $(\Gamma_0)_L$ (calculated with respect to the invariant metric of $B$) is $n/40$ times the area of $\Delta_{123}$, i.e., $(7\pi/40 \times 20)n$. On the other hand, we know that $c_1(T_A)$ equals $-1/(2\pi)$ times the area of $A$ (calculated with respect to the metric of holomorphic sectional curvature $-1$). This area of $A$ is simply the area of $W$. Hence

$$c_1(N_\ell) = -\frac{7n}{24 \times 40 \times 20}.$$

Recall that $k$ is the index of $\Gamma_0$ in $\Gamma$. From the diagram

$$\begin{array}{c}
\Gamma \\
\downarrow \quad \quad \downarrow \\
\Gamma_0 \quad \Gamma_L \\
\downarrow \\
(\Gamma_0)_L = \Gamma_0 \cap \Gamma_L
\end{array}$$

it follows that the index of $\Gamma_0/(\Gamma_0)_L$ in $\Gamma/\Gamma_L$ is $k/n$. It is easily seen that this index equals $l$ (which is the number of $C_v$'s). Hence

$$\sum_{\ell=1}^l c_1(N_\ell) = -\frac{7k}{24 \times 40 \times 20}.$$

Finally we have to calculate $\int_{C_v} \tilde{\theta}$. Clearly we have

$$\int_{C_v} \tilde{\theta} = \int_w \theta = \frac{n}{40} \int_{\Delta_{123}} \theta = \frac{n}{40} \cdot \frac{3}{4\pi} \text{(area of } \Delta_{123}) = \frac{3 \times 7n}{4 \times 40 \times 20}$$

and

$$\sum_{\ell=1}^l \int_{C_v} \tilde{\theta} = \frac{3 \times 7k}{4 \times 40 \times 20}.$$

Combining all these calculations we have
\[ c_t^2(K) = \frac{k}{3} \left( \frac{3}{4\pi} \right)^{2} \frac{214\pi^2}{200} + 12 \frac{3 \times 7k}{4 \times 40 \times 20} - \frac{7k}{24 \times 40 \times 20} = \frac{852k}{3200} \]

and

\[ \frac{c_t^2(M)}{c_s(M)} = \frac{852}{298}. \]

10. An infinite discrete family of similar surfaces

The surface we have constructed above with \( p = 5 \) (where \( p \) is the order of the three complex reflections) and \( \arg \varphi^3 = \pi/20 \) is only one member of an infinite discrete family of negatively compact Kähler surfaces not covered by the ball which can be constructed in a similar way with \( p = 3, 4, 5 \) and a discrete set of suitable values of \( \arg \varphi^3 \). We discuss below the set of suitable values of \( \arg \varphi^3 \) and compute \( c_t^2/c_s^2 \) for the surface constructed with such a value of \( \arg \varphi^3 \). Two surfaces from the family with different values of \( c_t^2/c_s^2 \) are not biholomorphic.

We let \( p \) take on the values 3, 4, 5, and \( \sigma_\varphi \) be the least positive integer which is \( \geq (1/2 - 1/p) \). The discrete family of surfaces is to be parameterized by a subset of the set of all pairs of numbers of the form \( (p, \sigma) \), where \( p = 3, 4, 5 \) and \( \sigma \) is an integer \( \geq \sigma_\varphi \). For a given pair \( (p, \sigma) \) define

\[ \arg \varphi^3 = \pi \left( \frac{1}{2} - \frac{1}{p} - \frac{2}{\sigma} \right). \]

With these values \( \rho \) and \( \arg \varphi^3 \), we can construct a compact Kähler surface with negative sectional curvature by following the construction given in Sections 4 and 5. In this general case,

\[
\begin{align*}
< p, t_{23}p_{31} &= \pi - \frac{\pi}{p} + \arg \varphi^3 \\
\text{Angle of rotation of } (R_3R_1R_2)^2 &= \frac{3\pi}{p} + \frac{\pi}{2} - 3 \arg \varphi^3, \\
< p, t_{32}p_{31} &= \pi - \frac{\pi}{p} - \arg \varphi^3 \\
\text{Angle of rotation of } (R_2R_1R_3)^2 &= \frac{3\pi}{p} + \frac{\pi}{2} + 3 \arg \varphi^3.
\end{align*}
\]

Let \( \rho \) be the order of \( \bar{\eta}^2 \varphi^3 \) with \( \eta = e^{i \gamma - i \rho \pi} \). For the surface we constructed in Sections 4 and 5 with \( p = 5 \) and \( \arg \varphi^3 = \pi/20 \), the group \( \text{Aut } \Omega \) is of order 3 and consists of the three elements defined by the 3 cyclic permutations \( 1 \rightarrow 2 \rightarrow 3 \) of the coordinates of \( C^3 \). In the general case, the order of \( \text{Aut } \Omega \) is 1 or 3 according as both \( \rho \) and \( \sigma \) are divisible by 3 or not. Let \( m \) be the branching order of the map \( Y \rightarrow B \) along \( E \). Then
\[ \rho \left( \frac{\pi}{2} - \frac{\pi}{p} + \arg \varphi^a \right) = 2m\pi. \]

The computations of Sections 6 and 7 yield in this general case

\[
\frac{c_1^2}{c_2} = \frac{3}{8} \left( \frac{3}{2} - \frac{6}{p} + \frac{6}{p^2} - 2t^2 \right) + \frac{5(m - 1)}{4\rho} \left( \frac{1}{\rho} + \frac{1}{\sigma} \right) + \frac{1}{\rho\sigma},
\]

where \( N = 24(p/(6 - p))^2 \) and \( t = 1/2 - 1/p - 2/\sigma \). This ratio \( c_1^2/c_2 \) is 3 if and only if \( m = 1 \). The only pairs \((p, \sigma)\) that can give \( m = 1 \) are the following:

\[
\begin{align*}
&\begin{cases}
p = 3, & \sigma = 6, 7, 8, 9, 10, 12, 15, 18, 24, 42, \infty \\
p = 4, & \sigma = 4, 5, 6, 8, 12, 20, \infty \\
p = 5, & \sigma = 4, 5, 10, 12.
\end{cases}
\end{align*}
\]

Our discrete family of surfaces is parametrized by \((p, \sigma)\) with \( \sigma \geq \sigma_* \) and \( m \neq 1 \). Two surfaces are biholomorphic (in both the cases \( m = 1 \) and \( m \neq 1 \)) if and only if their parameters \((p, \sigma)\) give the same \(|t|\).

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