A Hartogs Type Extension Theorem for Coherent Analytic Sheaves

Yum-Tong Siu


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A Hartogs type extension theorem
for coherent analytic sheaves

BY YUM-TONG SIU *

For an analytic sheaf $\mathcal{F}$ on a complex space define the $n^{th}$ absolute gap-sheaf $\mathcal{F}^{[n]}$ of $\mathcal{F}$ by the following presheaf

$$U \mapsto \text{ind. lim } \Gamma(U - A, \mathcal{F}),$$

where $A$ runs through all subvarieties of dimension $\leq n$ in $U$ (see [13]). When $\mathcal{F}$ is coherent, $\mathcal{F}^{[n]} = \mathcal{F}$ is equivalent to $\dim S_{k+2}(\mathcal{F}) \leq k$ for $-1 \leq k < n$, where $S_l(\mathcal{F})$ is the subvariety where the homological codimension of $\mathcal{F}$ is $\leq l$ [14, Prop. 13] and [12, Kor. zu Satz III]; cf. [18].

For an analytic subsheaf $\mathcal{G}$ of $\mathcal{F}$ the $n^{th}$ relative gap-sheaf $\mathcal{G}^{[n]}_{\mathcal{F}}$ of $\mathcal{G}$ relative to $\mathcal{F}$ is defined as the subsheaf of $\mathcal{F}$ whose stalks are given as follows: an element $s$ of $\mathcal{F}_x$ belongs to $(\mathcal{G}^{[n]}_{\mathcal{F}})_x$ if and only if for some open neighborhood $U$ of $x$ and some subvariety $A$ of dimension $\leq n$ in $U$, $s$ is induced by some $t \in \Gamma(U, \mathcal{F})$ satisfying $t | U - A \in \Gamma(U - A, \mathcal{G})$ (see [17]).

For $a \in \mathbb{R}^n$ the components of $a$ are denoted by $a_1, \ldots, a_n$. For $a, b \in \mathbb{R}^n$, $a < b$ means $a_i < b_i$ for $1 \leq i \leq n$ and $a \leq b$ means $a_i \leq b_i$ for $1 \leq i \leq n$. $K^* (a)$ denotes

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| < a_i \text{ for } 1 \leq i \leq n\}.$$

$G^* (a, b)$ denotes

$$\{ (z_1, \ldots, z_n) \in K^* (b) \mid |z_i| > a_i \text{ for some } 1 \leq i \leq n\}.$$

Suppose $T$ and $X$ are complex spaces and $t \in T$. If $\mathcal{F}$ is an analytic sheaf on $T \times X$ (respectively $T$), then $\mathcal{F}(t)$ denotes $\mathcal{F}/\mathcal{I}_t$, where $\mathcal{I}$ is the maximum ideal-sheaf on $T \times X$ (respectively $T$) for $\{t\} \times X$ (respectively $\{t\}$). If $\varphi: \mathcal{F} \to \mathcal{G}$ is a sheaf-homomorphism of analytic sheaves on $T \times X$ or $T$, then $\varphi(t): \mathcal{F}(t) \to \mathcal{G}(t)$ is the sheaf-homomorphism naturally induced by $\varphi$.

A subset $A$ of a complex space $X$ is said to be thin in $X$ if $A$ is contained in a countable union of locally closed subvarieties of codimension $\geq 1$ in $X$. $A$ is said to be thick in $X$ if it is not thin in $X$.

An outstanding conjecture in the theory of coherent analytic sheaf extension is the following Hartogs type extension theorem:

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THEOREM 1. Suppose $0 \leq a < b$ in $\mathbb{R}^N$, $D$ is a domain in $\mathbb{C}^n$, and $D'$ is a non-empty open subset of $D$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on $(D \times G^N(a, b)) \cup (D' \times K^N(b))$ satisfying $\mathcal{F}^{[n]} = \mathcal{F}$. Then $\mathcal{F}$ can be uniquely extended to a coherent analytic sheaf $\widetilde{\mathcal{F}}$ on $D \times K^N(b)$ satisfying $\widetilde{\mathcal{F}}^{[n]} = \widetilde{\mathcal{F}}$.

In this paper we solve the above conjecture by proving the following stronger theorem:

MAIN THEOREM. Suppose $0 \leq a < b$ in $\mathbb{R}^N$, $D$ is a domain in $\mathbb{C}^n$, and $\mathcal{F}$ is a coherent analytic sheaf on $D \times G^N(a, b)$ such that $\mathcal{F}^{[n]} = \mathcal{F}$. Suppose $A$ is a thick set in $D$ and, for every $t \in A$, $\mathcal{F}(t)$ can be extended to a coherent analytic sheaf on $\{t\} \times K^N(b)$. Then $\mathcal{F}$ can be uniquely extended to a coherent analytic sheaf $\widetilde{\mathcal{F}}$ on $D \times K^N(b)$ satisfying $\widetilde{\mathcal{F}}^{[n]} = \widetilde{\mathcal{F}}$.

So far the strongest known results in the theory of coherent analytic sheaf extension are the following three theorems proved respectively in [14], [16], and [15].

THEOREM 2. Suppose $0 \leq a < b$ in $\mathbb{R}^N$, $D$ is an open set in $\mathbb{C}^n$, and $\mathcal{F}$ is a coherent analytic sheaf on $D \times G^N(a, b)$ satisfying $\mathcal{F}^{[n+1]} = \mathcal{F}$. Then $\mathcal{F}$ can be uniquely extended to a coherent analytic sheaf $\widetilde{\mathcal{F}}$ on $D \times K^N(b)$ satisfying $\widetilde{\mathcal{F}}^{[n+1]} = \widetilde{\mathcal{F}}$.

THEOREM 3. Suppose $0 \leq a < b$ in $\mathbb{R}^N$, $D$ is a domain in $\mathbb{C}^n$, and $D'$ is a non-empty open subset of $D$. Suppose $\mathcal{G}$ is a coherent analytic sheaf on $D \times K^N(b)$ and $\mathcal{F}$ is a coherent analytic subsheaf of $\mathcal{G}$ on $(D \times G^N(a, b)) \cup (D' \times K^N(b))$ satisfying $\mathcal{F}^{[n]} = \mathcal{F}$. Then $\mathcal{F}$ can be uniquely extended to a coherent analytic subsheaf $\widetilde{\mathcal{F}}$ of $\mathcal{G}$ on $D \times K^N(b)$ satisfying $\widetilde{\mathcal{F}}^{[n]} = \widetilde{\mathcal{F}}$.

THEOREM 4. Suppose $0 \leq a < b$ in $\mathbb{R}^N$, $D$ is an open subset of $\mathbb{C}^n$, and $\mathcal{F}$ is a coherent analytic sheaf on $D \times G^N(a, b)$ satisfying $\mathcal{F}^{[n]} = \mathcal{F}$. Suppose, for every $t \in D$, $\mathcal{F}(t)$ can be extended to a coherent analytic sheaf on $\{t\} \times K^N(b)$. Then $\mathcal{F}$ can be uniquely extended to a coherent analytic sheaf $\widetilde{\mathcal{F}}$ on $D \times K^N(b)$ satisfying $\widetilde{\mathcal{F}}^{[n]} = \widetilde{\mathcal{F}}$.

The above three theorems are clearly consequences of the Main Theorem. However, in the proof of the Main Theorem, Theorem 2 is needed.

The main idea in the proof of the Main Theorem is as follows: First, we reduce the general case to the case where $N = 2$ and $\mathcal{F}$ is locally free. Then it is shown that, to prove this special case, we can assume that $0 < a$, $D$ is Stein, and $\mathcal{F}$ is the restriction of some special locally free analytic sheaf $\mathcal{F}$ defined on $D \times \Omega$, where $\Omega = P_1 \times P_1 - K^2(a)$. Next, we prove that, outside a closed thin set in $D$, the zeroth direct image of $\mathcal{F}$ under the projection
$D \times \Omega \to D$ is locally free. By replacing $\mathcal{F}$ by its tensor product with an appropriate number of copies of the two sheaves coming from the line bundle of the hyperplane section of $\mathbb{P}_*$, we conclude that $\hat{\mathcal{F}}$ has sufficiently many global sections on $D \times \Omega$ minus a closed thin set. Hence, $\mathcal{F}$ has sufficiently many global sections on $D \times G^2(a, b)$ minus a closed thin set. The extendibility of $\mathcal{F}$ follows.

Some propositions in this paper are simple refinements of the corresponding propositions in [15]. We give only brief outlines of the proofs of such propositions, because the proofs of the corresponding propositions are given in detail in [15].

$\mathbb{N}$ denotes the set of all natural numbers and $\mathbb{N}_* = \mathbb{N} \cup \{0\}$. $\mathbb{R}_+$ denotes the set of all positive numbers. $\mathcal{O}$ denotes the structure sheaf of $\mathbb{C}^n$. Complex spaces are not necessarily reduced. If $T$ and $X$ are complex spaces and $t \in T$, then $\{t\} \times X$ is canonically identified with $X$. A coherent analytic sheaf $\mathcal{F}$ is canonically identified with $\mathcal{F}|\text{Supp} \mathcal{F}$, where $\text{Supp} \mathcal{F}$ denotes the support of $\mathcal{F}$. All Banach spaces are over $\mathbb{C}$.

If $X$ is a complex manifold and $E$ is a Banach space, then we denote $X \times E$ by $E_X$. $E_X$ is a trivial bundle with Banach space fibers. For $x \in X$, $\{x\} \times E$ is denoted by $E_x$ and is canonically identified with $E$. If $F$ is another Banach space, a map $v : E_X \to F_X$ is called a homomorphism if $v$ is holomorphic and, for every $x \in X$, $v$ induces a linear map $v_x : E_x \to F_x$. $v$ is called a direct homomorphism if in addition $\text{Ker} \ v_x$ and $\text{Im} \ v_x$ are respectively direct closed subspaces of $E_x$ and $F_x$ for $x \in X$. If $Y$ is an open subset of $X$, then we denote by $v_Y$ the map $E_{Y} \to F_Y$ induced by $v$.

**PROPOSITION 1.** Suppose $X$ is a connected complex manifold and $E, F$ are Banach spaces. Suppose $v : E_X \to F_X$ is a direct homomorphism such that $\dim \text{Ker} \ v_x < \infty$ for every $x \in X$. Then there exists a subvariety $A$ of codimension $\geq 1$ in $X$ such that $\text{Ker} \ v_{x-A}$ is a holomorphic vector bundle over $X-A$ with finite-dimensional fibers.

**PROOF.** By [4, p. 337, Prop. (VI. 2)] there exist $k \in \mathbb{N}_*$ and a subvariety $A$ of codimension $\geq 1$ in $X$ such that $\dim \text{Ker} \ v_x = k$ for $x \in X-A$. Take $x_0 \in X-A$. Let $E'$ and $F'$ be respectively closed subspaces of $E$ and $F$ complementary to $\text{Ker} \ v_{x_0}$ and $\text{Im} \ v_{x_0}$. Define $\tilde{v} : E'_X \oplus F'_X \to F_X$ by $\tilde{v}_x(a \oplus b) = v_x(a) + b$. For some open neighborhood $U$ of $x_0$ in $X-A$, $\tilde{v}_U$ is an isomorphism. Since $\dim \text{Ker} \ v_x = \dim \text{Ker} \ v_{x_0}$ for $x \in U$, by considering the projection $E \to \text{Ker} \ v_x$ parallel to $E'$, we conclude that $E = \text{Ker} \ v_x \oplus E'$ for $x \in U$. Hence $E_U \xrightarrow{v_U} F'_U \xrightarrow{q(\tilde{v}_U)^{-1}} F'_U$ is exact, where $q : E'_U \oplus F'_U \to F'_U$ is the projection onto the second summand. The proposition follows from [2, p. 21, Prop. 3] Q.E.D.
Suppose $X$ is an open subset of a complex space $\tilde{X}$ and $\mathcal{F}$ is a coherent analytic sheaf on $X$. Suppose $\tilde{\mathcal{F}}$ is a coherent analytic sheaf on $\tilde{X}$ extending $\mathcal{F}$ and satisfying a certain condition (C). We say that $\tilde{\mathcal{F}}$ is unique (up to isomorphism) if for any other coherent analytic extension $\mathcal{F}'$ of $\mathcal{F}$ on $\tilde{X}$ satisfying (C) there exists uniquely a sheaf-isomorphism $\tilde{\mathcal{F}} \to \mathcal{F}'$ on $\tilde{X}$ whose restriction to $X$ is the identity map of $\mathcal{F}$.

For open subsets $G \subset \tilde{G}$ of a complex space, the couple $(G, \tilde{G})$ is called an extension pair of order $n$ if for any coherent analytic sheaf $\mathcal{F}$ on $\tilde{G}$ satisfying $\mathcal{F}^{[n]} = \mathcal{F}$ the restriction map $\Gamma(\tilde{G}, \mathcal{F}) \to \Gamma(G, \mathcal{F})$ is bijective.

**Lemma 1.** Suppose $G \subset \tilde{G}$ are open subsets of a complex space $(X, \mathcal{O})$ and $(G, \tilde{G})$ is an extension pair of order $n$. Suppose $\mathcal{F}$ is a coherent analytic sheaf on $G$ satisfying $\mathcal{F}^{[n]} = \mathcal{F}$. Then any coherent analytic extension $\mathcal{F}'$ of $\mathcal{F}$ on $\tilde{G}$ satisfying $\mathcal{F}'^{[n]} = \mathcal{F}'$ is unique.

**Proof.** The lemma follows from the following observation (which can easily be verified by using a two-step local free resolution of $\mathcal{O}$: If $\mathcal{R}$ and $\mathcal{G}$ are coherent analytic sheaves on an open subset of $X$ and $\mathcal{G}^{[n]} = \mathcal{G}$, then $\text{Hom}_0(\mathcal{R}, \mathcal{G})^{[n]} = \text{Hom}_0(\mathcal{R}, \mathcal{G})$. Q.E.D.

Suppose $0 \leq a < b$ in $\mathbb{R}^n$, $D$ is a domain in $\mathbb{C}^n$, and $D'$ is a non-empty open subset of $D$. Let $R = D \times K^n(b)$ and $R' = (D \times G^n(a, b)) \cup (D' \times K^n(b))$. It follows from [13, p. 373, Th. 3] and the exhaustion techniques of [11, § 8] that $(R', R)$ is an extension pair of order $n - 1$. As a consequence, for $a \leq a' < b$ in $\mathbb{R}^n$, $(D \times G^n(a', b), D \times G^n(a, b))$ is an extension pair of order $n$.

**Proposition 2.** Suppose $a \leq a' < b$ in $\mathbb{R}^n$ and $D$ is an open subset of $\mathbb{C}^n$. If $\mathcal{F}$ is a coherent analytic sheaf on $D \times G^n(a', b)$ satisfying $\mathcal{F}^{[n]} = \mathcal{F}$, then any coherent analytic extension $\mathcal{F}'$ of $\mathcal{F}$ on $D \times G^n(a, b)$ satisfying $\mathcal{F}'^{[n]} = \mathcal{F}'$ is unique.

For the rest of this paper we use the following notation: If $A$ is a thick set in a complex space $X$, then $A^t$ denotes the set of $x \in X$ such that, for every open neighborhood $U$ of $x$ in $X$, $U \cap A$ is thick in $X$. It is easily verified that $A^t$ is always thick and closed in $X$ (when $X$ has countable topology).

**Proposition 3.** Suppose $0 \leq a < b$ in $\mathbb{R}^n$, $D$ is a domain in $\mathbb{C}^n$, and $\mathcal{F}$ is a coherent analytic sheaf on $D \times G^n(a, b)$ such that $\mathcal{F}^{[n]} = \mathcal{F}$. Let $A$ be the set of $t \in D$ such that $\mathcal{F}(t)$ can be extended to a coherent analytic sheaf on $\{t\} \times K^n(b)$. Let $T$ be a closed thin set in $D$. Suppose $A$ is thick and suppose for every $t \in A^t - T$ there exist an open neighborhood $U(t)$ of $t$ in $D - T$ and $a \leq a'(t) < b'(t) \leq b$ in $\mathbb{R}^n$ such that $\mathcal{F} | U(t) \times G^n(a'(t), b'(t))$ can be extended
to a coherent analytic sheaf $\tilde{\mathcal{F}}$ on $U(t) \times K^N(b'(t))$. Then $\mathcal{F}$ can be extended to a coherent analytic sheaf on $D \times K^N(b)$.

**Proof.** First we show that $D - T$ is connected. Since $T$ is contained in a countable union of locally closed subvarieties of codimension $\geq 1$, after a linear change of coordinates in $\mathbb{C}^n$, we can assume that, for every $t^0 = (t^0_1, \ldots, t^0_n) \in \mathbb{C}^n$ and every $1 \leq i \leq n$,

$$E_i(t^0) := \{(t_i, \ldots, t_n) \in \mathbb{C}^n | t_j = t^0_j \text{ for } j \neq i\}$$

intersects $T$ in a countable set. Every uniformly bounded holomorphic function $f$ on $D - T$ can be uniquely extended to a holomorphic function on $D$, because $f | E_i(t^0) \cap D - T$ can be uniquely extended to a holomorphic function on $E_i(t^0) \cap D$ for every $t^0 \in \mathbb{C}^n$ and every $1 \leq i \leq n$. If $D - T$ is the union of two disjoint non-empty open subsets $L_1$ and $L_2$, the function on $D - T$ which is identically 0 on $L_1$ and identically 1 on $L_2$ cannot be extended to a holomorphic function on $D$. Hence $D - T$ is connected.

$A^t - T$ is a non-empty closed subset of $D - T$. For $t \in A^t - T$, by replacing $\mathcal{F}$ by $(\tilde{\mathcal{F}}/0_{[n+1, \tilde{\mathcal{F}}]^{[n]}}$, we can assume that $\tilde{\mathcal{F}}^{[n]} = \tilde{\mathcal{F}}$. By Proposition 2, $\tilde{\mathcal{F}}$ can be identified with $\tilde{\mathcal{F}}$ on $U(t) \times K^N(a, b'(t))$. Hence the coherent analytic sheaf $\tilde{\mathcal{F}}$ on $U(t) \times K^N(b)$, which agrees with $\tilde{\mathcal{F}}$ on $U(t) \times K^N(b'(t))$ and agrees with $\mathcal{F}$ on $U(t) \times K^N(a, b)$, extends $\mathcal{F} | U(t) \times K^N(a, b)$. Consequently, for $t \in A^t - T$, $U(t) \subset A^t - T$ is open in $D - T$. Since $D - T$ is connected, $A^t - T = D - T$. U$_{t \in A^t - T} U(t) = D - T$. By Proposition 2, for $t, \tilde{t} \in A^t - T$, $\tilde{\mathcal{F}}$ can be identified with $\tilde{\mathcal{F}}$ on $(U(t) \cap U(\tilde{t})) \times K^N(b)$. Therefore, the coherent analytic sheaf $\mathcal{F}_\mathcal{F}$ on $(D - T) \times K^N(b)$, which agrees with $\tilde{\mathcal{F}}$ on $U(t) \times K^N(b)$ for $t \in D - T$, extends $\mathcal{F} | (D - T) \times K^N(a, b)$.

Take arbitrarily $t^0 \in T$. We can assume that $t^0 = 0$. Since $E_n(t^0) \cap T$ is a countable closed subset of $E_n(t^0)$, we can find $d_1, \ldots, d_{n-1} > 0, d_n > d'_n > 0$ such that $K^*(d_1, \ldots, d_n) \subset D$ and $H := K^*(d_1, \ldots, d_{n-1}) \times G'(d'_n, d_n)$ is disjoint from $T$. Let $\mathcal{F}'$ be the coherent analytic sheaf on $(H \times K^N(b)) \cup (K^*(d_1, \ldots, d_n) \times G'(a, b))$ which agrees with $\mathcal{F}_\mathcal{F}$ on $H \times K^N(b)$ and agrees with $\mathcal{F}$ on $K^*(d_1, \ldots, d_n) \times G'(a, b)$. Since $(\mathcal{F}_\mathcal{F})^{[n]} = \mathcal{F}_\mathcal{F}$, by Theorem 2, $\mathcal{F}'$ can be extended to a coherent analytic sheaf $\mathcal{F}''$ on $K^*(d_1, \ldots, d_n) \times K^N(b)$ satisfying $(\mathcal{F}_\mathcal{F})^{[n]} = \mathcal{F}''$. $\mathcal{F}''$ extends $\mathcal{F} | K^*(d_1, \ldots, d_n) \times G'(a, b)$. Since $t^0$ is an arbitrary point of $T$ and $K^*(d_1, \ldots, d_n)$ is an open neighborhood of $t^0$, by Proposition 2, after piecing together $\mathcal{F}_\mathcal{F}$ and all the $\mathcal{F}''$ obtained by varying $t^0$, we obtain a coherent analytic extension of $\mathcal{F}$ on $D \times K^N(b)$. Q. E. D.

For an open subset $D$ of $\mathbb{C}^n$ an element of $\Gamma(D, \mathcal{O}^n)$ or $\Gamma(D, \mathcal{O}^N)$ is said to be uniformly bounded if all its components admit a common uniform bound.
LEMMA 2. Suppose \( G \subset \subset \tilde{G} \) are Stein domains in \( \mathbb{C}^n \) and \( \varphi : {}_n\mathcal{O}^p \to {}_n\mathcal{O}^N \) is a sheaf-homomorphism on \( \tilde{G} \). Then:

(a) There exist \( r_0 \in \mathbb{N} \) satisfying the following: For every \( r \geq r_0 \) there exists a sheaf-homomorphism \( \theta : {}_n\mathcal{O}^r \to {}_n\mathcal{O}^N \) on \( G \) such that

(i) \( \eta \theta \) is the identity sheaf-isomorphism on \( G \) and

(ii) \( \eta \theta \) is \( \varphi \) on \( G \),

where \( {}_n\mathcal{O}^N \to {}_n\mathcal{O}^r \) is the projection onto the first \( r \) components. Moreover, if \( \varphi \) maps uniformly bounded elements of \( \Gamma(\tilde{G}, {}_n\mathcal{O}^p) \) to uniformly bounded elements of \( \Gamma(\tilde{G}, {}_n\mathcal{O}^N) \), then \( \theta \) can be chosen so that \( \theta \) maps uniformly bounded elements of \( \Gamma(G, {}_n\mathcal{O}^r) \) to uniformly bounded elements of \( \Gamma(G, {}_n\mathcal{O}^N) \).

(b) There exist a sheaf-homomorphism \( \psi : {}_n\mathcal{O}^N \to {}_n\mathcal{O}^p \) on \( G \) and a subvariety \( T \) of codimension \( \geq 1 \) in \( G \) such that

\[
{}_n\mathcal{O}^p(t) \xrightarrow{\psi(t)} {}_n\mathcal{O}^r(t) \xrightarrow{\psi(t)} {}_n\mathcal{O}^N(t)
\]

is exact for \( t \in G - T \).

PROOF. (a) Let \( (\varphi_{\mu})_{\mu \in \mathbb{N}} \in \Gamma(\tilde{G}, {}_n\mathcal{O}^N) \) be the image of \( (0, \ldots, 0, 1, 0, \ldots, 0) \in \Gamma(\tilde{G}, {}_n\mathcal{O}^p) \) under \( \varphi \), where 1 is in the \( \nu \)th position. Let \( \mathcal{M}_\mu \) be the analytic subsheaf of \( {}_n\mathcal{O}^p \) on \( \tilde{G} \) generated by \( \{(\varphi_{2i}, \ldots, \varphi_{2p})\}_{2 \leq \mu} \subset \Gamma(\tilde{G}, {}_n\mathcal{O}^p) \). Since \( \{\mathcal{M}_\mu\}_{\mu \in \mathbb{N}} \) is an increasing sequence of coherent analytic subsheaves of \( {}_n\mathcal{O}^p | \tilde{G} \), there exists \( r_0 \in \mathbb{N} \) such that \( \mathcal{M}_\mu = \mathcal{M}_{r_0} \) on \( G \) for \( \mu \geq r_0 \).

Take arbitrarily \( r \geq r_0 \). Since \( G \) is Stein, it follows from \( \mathcal{M}_\mu = \mathcal{M}_{r_0}(\mu > r) \) that there exist \( \alpha_{\mu_1}, \ldots, \alpha_{\mu_r} \in \Gamma(G, {}_n\mathcal{O}) \) (\( \mu > r \)) such that

\[
(\varphi_{\mu_1}, \ldots, \varphi_{\mu_r}) = \sum_{\lambda=1}^{\mu} \alpha_{\mu_1}(\varphi_{2i}, \ldots, \varphi_{2p}) \quad (\mu > r).
\]

Define \( \theta : {}_n\mathcal{O}^r \to {}_n\mathcal{O}^N \) on \( G \) as follows: For \( x \in G \) and \( (a_1, \ldots, a_r) \in {}_n\mathcal{O}^r_x \),

\[
\theta(a_1, \ldots, a_r) = (b_{\mu})_{\mu \in \mathbb{N}} \in {}_n\mathcal{O}^N_x,
\]

where

\[
\begin{align*}
{_{\mu}}b & = a_{\mu} \quad \text{for } \mu \leq r \\
{_{\mu}}b & = \sum_{\lambda=1}^{r} \alpha_{\mu_1} a_{\lambda} \quad \text{for } \mu > r.
\end{align*}
\]

Then \( r \) and \( \theta \) satisfy the conditions (i) and (ii).

Suppose \( \varphi \) maps uniformly bounded elements of \( \Gamma(\tilde{G}, {}_n\mathcal{O}^p) \) to uniformly bounded elements of \( \Gamma(\tilde{G}, {}_n\mathcal{O}^N) \). Then all \( \varphi_{\mu_\lambda} (\mu \in \mathbb{N}, 1 \leq \nu \leq p) \) admit a common uniform bound on \( \tilde{G} \). We can choose \( \alpha_{\mu_1} \) so that all \( \alpha_{\mu_1}(\mu > r, 1 \leq \lambda \leq r) \) admit a common uniform bound on \( G \), because the map \( \Gamma(\tilde{G}, {}_n\mathcal{O}^p) \to \Gamma(\tilde{G}, {}_n\mathcal{O}^p) \) sending \( (0, \ldots, 0, 1, 0, \ldots, 0) \in \Gamma(\tilde{G}, {}_n\mathcal{O}^r) \) to \( (\varphi_{2i}, \ldots, \varphi_{2p}) \in \Gamma(\tilde{G}, {}_n\mathcal{O}^p) \) (where 1 is in the \( \lambda \)th position) is a continuous linear surjection of Fréchet spaces.
When $\alpha_{\mu}(\mu > r, 1 \leq \lambda \leq r)$ are so chosen, $\theta$ maps uniformly bounded elements of $\Gamma(G, \mathcal{O}^t)$ to uniformly bounded elements of $\Gamma(G, \mathcal{O}^N)$.

(b) Since $\tilde{G}$ is Stein, there exists a sheaf-homomorphism $\psi : \mathcal{O}^t \to \mathcal{O}^p$ on $\tilde{G}$ such that $\text{Im } \psi = \text{Ker } \eta \varphi$ on $G$. Let $T$ be the subvariety of $G$ where $\mathcal{O}^p/\text{Im } \eta \varphi$ is not locally free. Then $\psi$ and $T$ satisfy the conditions. Q.E.D.

Suppose $\pi : X \to Y$ is a holomorphic map of complex spaces. The rank of $\pi$ at $x \in X$ is defined as $\dim_x X - \dim_x \pi^{-1}(\pi(x))$. If $X$ has pure dimension, then the set of $x \in X$ where the rank of $\pi$ is $\leq k$ is a subvariety of $X$ [9, p. 162].

**Proposition 4.** Suppose $0 \leq a < b$ in $\mathbb{R}$, $D$ is a domain in $\mathbb{C}^n$, and $f$ is a meromorphic function on $D \times G^t(a, b)$. Suppose $A$ is a thick set in $D$ and, for $t \in A$, either $\{t\} \times G^t(a, b)$ is contained in the pole set of $f$ or the restriction of $f$ to $\{t\} \times G^t(a, b)$ can be extended to a meromorphic function $g_t$ on $\{t\} \times K^t(b)$. Then $f$ can be extended to a meromorphic function on $D \times K^t(b)$.

**Proof.** By [8, p. 648, Satz 1], we need only prove that, for some non-empty open subset $H$ of $D$ and some $a \leq c < d \leq b$ in $\mathbb{R}$, $f \upharpoonright H \times G^t(c, d)$ can be extended to a meromorphic function on $H \times K^t(d)$.

Let $\pi : D \times G^t(a, b) \to D$ be the natural projection. Let $Z$ be the pole set of $f$. Let $V$ be the subvariety of $Z$ where the rank of $\pi \upharpoonright Z$ is $\leq n - 1$. $\pi(V)$ is thin. For $t \in D - \pi(V)$, $\{t\} \times G^t(a, b)$ is not contained in $Z$.

Take $t^* \in A^t - \pi(V)$. Since $\pi^{-1}(t^*) \cap Z$ is discrete, there exist a connected Stein open neighborhood $U$ of $t^*$ in $D$ and $a \leq a' < b' \leq b$ in $\mathbb{R}$ such that $U \times G^t(a', b')$ is disjoint from $Z$. By replacing $D$ by $U$ and replacing $a$ and $b$ by $a'$ and $b'$, we can assume that $D$ is Stein and $f$ is holomorphic on $D \times G^t(a, b)$.

Let $D'$ be a relatively compact non-empty Stein open subset of $D$ intersecting $A^t$. For $\nu \in \mathbb{N}_*$ let $A_{\nu}$ be the set of all $t \in A \cap D'$ such that $g_t$ has at most $\nu$ poles (with multiplicities counted). For some $p \in \mathbb{N}_*$, $A_{p}$ is thick in $D'$.

Let $f(t, w) = \sum_{\nu=-\infty}^{\infty} f_{\nu}(t)w^\nu$ be the Laurent series expansion in $w$, where $t \in D$ and $w$ is the coordinate of $G^t(a, b)$. Define a sheaf-homomorphism $\varphi : _n\mathcal{O}^{p+1} \to _n\mathcal{O}^N$ on $D$ as follows: For $t \in D$ and $a_0, \ldots, a_p \in \mathcal{O}_t$, $\varphi(a_0, \ldots, a_p) = (b_\nu)_{\nu \in \mathbb{N}} \in _n\mathcal{O}^N_t$ with $b_\nu = \sum_{i=0}^{p} a_i f_{-\nu-i}$. By Lemma 2 there exist $r \in \mathbb{N}$ and a sheaf-homomorphism $\theta : _n\mathcal{O}^r \to _n\mathcal{O}^N$ on $D'$ such that

(i) $\eta \theta = \text{the identity sheaf-isomorphism on } D'$, and

(ii) $\theta \eta \varphi = \varphi$ on $D'$,

where $\eta : _n\mathcal{O}^N \to _n\mathcal{O}^r$ is the projection onto the first $r$ components. Let $\varphi' = \eta \varphi$. 
Then $\text{Ker } \varphi = \text{Ker } \varphi'$ on $D'$ and $\text{Ker } (\varphi(t)) = \text{Ker } (\varphi'(t))$ for $t \in D'$.

Let $T$ be the subvariety of $D'$ where $\mathcal{O}/\text{Im } \varphi'$ is not locally free. There exists $t^0 \in A_p - T$. We have $(\text{Ker } \varphi')(t^0) = \text{Ker } (\varphi'(t^0))$. Since $g_{t^0}$ has at most $p$ poles, there exists a non-zero polynomial $P = \sum_{\nu = 0}^{p} \alpha_{\nu} w^{\nu}$ (where $\alpha_{\nu} \in \mathbb{C}$) such that $Pg_{t^0}$ is holomorphic on $\{t^0\} \times K^1(b)$. From the Laurent series expansion of $g_{t}$ and the power series expansion of $Pg_{t^0}$ we obtain

$$\sum_{\nu = 0}^{p} \alpha_{\nu} f_{-\nu - 1}(t^0) = 0$$

for $\nu \in \mathbb{N}$.

Therefore $0 \neq (\alpha_0, \ldots, \alpha_p) \in \text{Ker } (\varphi(t^0))$. Since $(\text{Ker } \varphi')(t^0) = \text{Ker } (\varphi'(t^0)) = \text{Ker } (\varphi(t^0))$, we have $\text{Ker } \varphi' \neq 0$ on $D'$. Since $D'$ is Stein, and $\text{Ker } \varphi'$ is a coherent analytic sheaf, there exists

$$0 \neq (s_0, \ldots, s_p) \in \Gamma(D', \text{Ker } \varphi').$$

Since $\text{Ker } \varphi = \text{Ker } \varphi'$, we have

$$\sum_{\nu = 0}^{p} s_{\nu} f_{-\nu - 1} = 0 \text{ on } D'$$

for $\nu \in \mathbb{N}$.

$(\sum_{i = 0}^{p} s_i(t)w^i)(\sum_{\nu = -\infty}^{\infty} f_{\nu}(t)w^\nu)$ is a power series which is convergent for $t \in D'$ and $w \in G^1(a, b)$. Hence $f \mid D' \times G^1(a, b)$ can be extended to a meromorphic function on $D' \times K^1(b)$. Q.E.D.

The preceding proof is a modification of E.E. Levi's original proof of his continuation theorem for meromorphic functions [10].

The following three propositions are respectively simple refinements of Proposition (4.5), Theorem (5.3), and Proposition (5.5) in [15]. We present here only brief outlines of their proofs.

**Proposition 5.** Suppose $0 \leq a < b$ in $\mathbb{R}^n$, $D$ is a domain in $\mathbb{C}^n$, and $V$ is a subvariety of pure dimension $n + 1$ in $D \times G^N(a, b)$. Suppose $A$ is a thick set in $D$ and, for every $t \in A$, $\{t\} \times G^N(a, b) \cap V$ can be extended to a subvariety in $\{t\} \times K^N(b)$. Then $V$ can be extended to a subvariety in $D \times K^N(b)$.

**Proposition 6.** Suppose $0 \leq a < b$ in $\mathbb{R}^n$ and $D$ is a domain in $\mathbb{C}^n$. Suppose $\mathcal{G}$ is a coherent analytic sheaf on $D \times K^N(b)$ and $\mathcal{F}$ is a coherent analytic subsheaf of $\mathcal{G}$ on $D \times G^N(a, b)$ such that $\mathcal{F}_{[n]} \mathcal{G} = \mathcal{F}$. Suppose $A$ is a thick set in $D$ and, for every $t \in A$, $\text{Im}(\mathcal{F}(t) \to \mathcal{G}(t))$ can be extended to a coherent analytic subsheaf of $\mathcal{G}(t)$ on $\{t\} \times K^N(b)$. Then $\mathcal{F}$ can be uniquely extended to a coherent analytic subsheaf $\tilde{\mathcal{F}}$ of $\mathcal{G}$ on $D \times K^N(b)$ satisfying $\tilde{\mathcal{F}}_{[n]} \mathcal{G} = \tilde{\mathcal{F}}$.

**Proposition 7.** Suppose $0 \leq a < b$ in $\mathbb{R}^n$, $D$ is a domain in $\mathbb{C}^n$, and $\mathcal{F}$ is a coherent analytic sheaf on $D \times G^N(a, b)$ such that $\mathcal{F}^{[n]} = \mathcal{F}$. Suppose $A$ is a thick set in $D$ and, for every $t \in A$, $\mathcal{F}(t)$ can be extended to a coherent
analytic sheaf on \( \{ t \} \times K^N(b) \). Suppose \( t^0 \in D \) and \( V \) is a subvariety of dimension \( \leq n + 1 \) in \( D \times G^N(a, b) \) such that \( V \) is disjoint from \( \{ t^0 \} \times G^N(a, b) \) and \( \Gamma(D \times G^N(a, b), \mathcal{F}) \) generates \( \mathcal{F} \) on \( D \times G^N(a, b) - V \). Then \( \mathcal{F} \) can be extended to a coherent analytic sheaf on \( D \times K^N(b) \).

**Proof of Proposition 5.** Take \( t \in D \) such that, for some \( a < a^* < b^* < b \) in \( \mathbb{R}^N \), \( V(t) := (\{ t \} \times G^N(a^*, b^*)) \cap V \) has pure dimension 1. (The set of \( t \in D \) not possessing this property is contained in a closed thin set in \( D \).) Suppose \( V(t) \) can be extended to a subvariety \( \tilde{V}(t) \) of pure dimension 1 in \( \{ t \} \times K^N(b^*) \). Take \( a^* < a' < b' < b^* \) in \( \mathbb{R}^N \). By applying [7, p.218, VII.B.3] to \( \tilde{V}(t) \), we can find an open neighborhood \( U \) of \( t \) in \( D \), a holomorphic function \( F \) on \( U \times K^N(b) \), and \( 0 < \alpha < \beta \) in \( \mathbb{R} \) such that \( F(V \cap (U \times G^N(a, a'))) \subseteq K^I(\alpha) \) and the map \( \Phi: U \times K^I(b) \to C^{n+1} \), defined by the coordinate functions of \( C^1 \) and \( F \), makes \( V' := V \cap \Phi^{-1}(U \times G^I(\alpha, \beta)) \cap (U \times K^I(b')) \) an analytic cover over \( U \times G^I(\alpha, \beta) \). Whether \( V' \) can be extended to a subvariety in \( \Phi^{-1}(U \times K^I(\beta)) \cap (U \times K^I(b')) \), which is an analytic cover over \( U \times K^I(\beta) \) under \( \Phi \), is equivalent to whether certain holomorphic functions on \( U \times G^I(\alpha, \beta) \) can be extended to a holomorphic function on \( U \times K^I(\beta) \). On the other hand, by considering Laurent series expansions, we know that every holomorphic function on \( U \times G^I(\alpha, \beta) \), whose restriction to \( \{ t \} \times G^I(\alpha, \beta) \) can be extended to a holomorphic function on \( \{ t \} \times K^I(\beta) \) when \( t \) lies in a thick set in \( U \), can be extended to a holomorphic function on \( U \times K^I(\beta) \). The proposition follows from a subvariety version of Proposition 3.

Q.E.D.

Suppose \( G \subset \tilde{G} \) are domains in \( C^n \) and \( \mathcal{S} \) is a coherent analytic subsheaf of \( _n\mathcal{O}^p | G \) such that \( \mathcal{S}_{(n-1)p} = \mathcal{S} \). \( \mathcal{F} := \mathcal{O}^p/\mathcal{S} \) is a torsion-free coherent analytic sheaf on \( G \). Let \( \lambda: _n\mathcal{O}^p \to \mathcal{F} \) be the quotient map and \( r = \text{rank} \mathcal{F} \). By rearranging the summands of \( _n\mathcal{O}^p \), we can assume that \( \lambda \theta: _n\mathcal{O}^r \to \mathcal{F} \) is injective, where \( \theta: _n\mathcal{O}^r \to _n\mathcal{O}^p \) is the injection into the first \( r \) summands. If \( \mathfrak{M} \) denotes the sheaf of germs of meromorphic functions on \( C^n \), then there is a sheaf-monomorphism \( \psi: \mathcal{F} \to \mathfrak{M}^r \) on \( G \) such that \( \psi \lambda \theta \) is the inclusion map \( _n\mathcal{O}_r \subset \mathfrak{M}_r \). Let \( v_i \) be the image of \((0, \ldots, 0, 1, 0, \ldots, 0) \in \Gamma(G, _n\mathcal{O}^p) \) under \( \psi \lambda \theta \), where 1 is in the \( i^{th} \) position. Then \( \mathcal{S} \) can be extended to a coherent analytic subsheaf of \( _n\mathcal{O}^p | \tilde{G} \) if and only if \( v_1, \ldots, v_p \) can be extended to \( r \)-tuples of meromorphic functions on \( \tilde{G} \).

Suppose \( \mathcal{F} \) is a coherent analytic sheaf on \( \tilde{G} \), \( \eta: _n\mathcal{O}^p \to \mathcal{F} \) is a sheaf-epimorphism on \( \tilde{G} \), and \( \mathcal{R} \) is a coherent analytic subsheaf of \( \mathcal{F} | G \) such that \( \mathcal{R}_{(n-1)p} = \mathcal{R} \). By considering \( \eta^{-1}(\mathcal{R}) \), we conclude that the extendibility of \( \mathcal{R} \) to \( \tilde{G} \) as a coherent analytic subsheaf of \( \mathcal{F} \) is equivalent to the extendibility of certain meromorphic functions from \( G \) to \( \tilde{G} \).
PROOF OF PROPOSITION 6. By the subsheaf version of Theorem 2, $\mathcal{F}_{[n+1]_2} \mathcal{G}$ can be extended to a coherent analytic subsheaf $\tilde{\mathcal{F}}$ of $\mathcal{G}$ on $D \times K^n(b)$ satisfying $\tilde{\mathcal{F}}_{[n+1]_2} = \mathcal{F}$. By replacing $\mathcal{G}$ by $\tilde{\mathcal{F}}$, we can assume without loss of generality that $X := \text{Supp } \mathcal{G}/\mathcal{F}$ has pure dimension $n+1$. By Proposition 5, $X$ can be extended to a subvariety $\tilde{X}$ of pure dimension $n+1$ in $D \times K^n(b)$. By shrinking $G^n(a, b)$, we can assume that $\mathcal{I} \mathcal{G} \subset \mathcal{F}$ for some $k \in \mathbb{N}$, where $\mathcal{I}$ is the maximum ideal-sheaf on $D \times K^n(b)$ for $\tilde{X}$. By replacing $\mathcal{G}$ and $\mathcal{F}$ respectively by $\mathcal{G}/\mathcal{I}^k \mathcal{G}$ and $\mathcal{F}/\mathcal{I}^k \mathcal{G}$, we can assume that $\text{Supp } \mathcal{G} = \tilde{X}$.

Take $t \in D$ such that, for some $a < b^* < b$ in $\mathbb{R}^n$, $\tilde{X}(t) := \{(t) \times K^n(b^*)\} \cap \tilde{X}$ has pure dimension 1. (The set of $t \in D$ not possessing this property is contained in a closed thin set in $D$.) Take $a < a' < b' < b^*$ in $\mathbb{R}^n$. By applying [7, p. 218] to $\tilde{X}(t)$, we can find an open neighborhood $U$ of $t$ in $D$, a holomorphic function $F$ on $U \times K^n(b)$, and $0 < \alpha < \beta$ in $\mathbb{R}$ such that $F(\tilde{X} \cap (U \times K^n(a'))) \subset K^i(\alpha)$ and the map $\Phi: U \times K^n(b) \to C^{n+1}$, defined by the coordinate functions of $C^n$ and $F$, makes $X' := \tilde{X} \cap \Phi^{-1}(U \times G^i(\alpha, \beta)) \cap (U \times K^n(b'))$ an analytic cover over $U \times G^i(\alpha, \beta)$. By considering the zeroth direct images of $\mathcal{F}$ and $\mathcal{G}$ respectively under the maps which are the restrictions of $\Phi$ to $X'$ and $X'': = \tilde{X} \cap \Phi^{-1}(U \times K^i(\beta)) \cap (U \times K^n(b'))$, we conclude, from the observation preceding this proof, that whether $\mathcal{F} \mid X'$ can be extended to a coherent analytic subsheaf of $\mathcal{G} \mid X''$ is equivalent to whether certain meromorphic functions on $U \times G^i(\alpha, \beta)$ can be extended to meromorphic functions on $U \times K^i(\beta)$. The proposition therefore follows from Proposition 4 and the subsheaf version of Proposition 3.

Q.E.D.

PROOF OF PROPOSITION 7. Let $\mathcal{G}$ be the analytic subsheaf of $\mathcal{F}$ generated by $\Gamma(D \times G^N(a, b), \mathcal{F})$. $\mathcal{G}$ can be extended to a coherent analytic sheaf $\tilde{\mathcal{G}}$ on $D \times K^n(b)$. For, if $D^*$ is a relatively compact open subset of $D$ and $a < a^* < b^* < b$ in $\mathbb{R}^n$, then there is a sheaf-epimorphism $\lambda: n + N \mathcal{O}^p \to \mathcal{G}$ on $D^* \times G^N(a^*, b^*)$ and since $(\text{Ker } \lambda)_{[n+1]_n+ N \mathcal{O}^p} = \text{Ker } \lambda$, by the subsheaf version of Theorem 2, Ker $\lambda$ can be extended to a coherent analytic subsheaf of $n + N \mathcal{O}^p$ on $D^* \times K^n(b^*)$.

On $D \times G^N(a, b)$, $\mathcal{R} := (\tilde{\mathcal{G}}/\mathcal{O}_{[n+1]_2})^{[n]} \mathcal{G}$ can be embedded naturally as a subsheaf of $\mathcal{F}$. The subvariety $Z$ in $D \times G^N(a, b)$ where $\mathcal{R}$ disagrees with $\mathcal{F}$ is either empty or of pure dimension $n+1$ in $D \times G^N(a, b)$. Since $Z$ is disjoint from $\{t\} \times G^N(a, b)$, by [11] $Z$ can be extended to a subvariety $\tilde{Z}$ of pure dimension $n+1$ in $D \times K^n(b)$. Suppose $D'' \subset D' \subset D$ are Stein open subsets and $a < a' < b' < b$ in $\mathbb{R}^n$. Since $0_{[n+1]_2} = 0$, we can choose a holomorphic function $f$ on $D' \times K^n(b)$ which vanishes identically on $\tilde{Z}$ and does not vanish identically on any branch of $(D' \times G^N(a, b)) \cap (\text{Supp } 0_{[k]} \mathcal{F})$ for any $k \in \mathbb{N}^*$. 
For some \( l \in \mathbb{N} \), \( f^{i} \mathcal{F} \) is a subsheaf of \( \mathcal{R} \) on \( D'' \times G^{N}(a', b') \). Since \( \mathcal{F} \approx f^{i} \mathcal{F} \) on \( D'' \times G^{N}(a', b') \), if, for some thick set \( B \) in \( D'' \), \( \mathcal{F}(t) \) can be extended to a coherent analytic sheaf on \( \{ t \} \times K^{N}(b) \) for \( t \in B \), then by Proposition 6, \( \mathcal{F} \mid D'' \times G^{N}(a, b) \) can be extended to a coherent analytic sheaf on \( D'' \times K^{N}(b) \). The proposition follows from Proposition 3.

Q.E.D.

For a ring \( R \) and \( l \in \mathbb{N} \), let \( \mathcal{O}[Z; l] \) be the module of all polynomials in \( Z \) with coefficients in \( R \) and having degrees \( \leq l \). For \( r \in \mathbb{N} \), \( \mathcal{O}[Z; l] \) denotes the direct sum of \( r \) copies of \( \mathcal{O}[Z; l] \). For a sheaf \( \mathcal{R} \) of rings, \( \mathcal{O}[Z; l] \) denotes the sheaf defined by the presheaf \( U \mapsto \Gamma(U, \mathcal{R})^{r}[Z; l] \).

We make the following two natural identifications. Suppose \( z_{1}, \ldots, z_{n+1} \) are the coordinates of \( \mathbb{C}^{n+1}, t \in \mathbb{C}^{n} \), and \( f_{0}, \ldots, f_{l} \in \mathcal{O}_{t} \).

(i) \( \mathcal{O}^{r}[z_{n+1}; l] \) is identified with a subsheaf of \( \mathcal{O}^{r} \mid G \times 0 \times \mathbb{C}^{n+1} \) by regarding naturally \( \sum_{t=0}^{l} f_{i} z^{i}_{n+1} \in \mathcal{O}_{t} \) as an element of \( \mathcal{O}^{r}(t, 0) \).

(ii) \( \mathcal{O}^{r}[z_{n+1}; l] \) is identified with \( \mathcal{O}^{r}[t] \) by identifying \( \sum_{t=0}^{l} f_{i} z^{i}_{n+1} \in \mathcal{O}_{t}(t, 0) \) with \( (f_{0}, \ldots, f_{l}) \in \mathcal{O}_{t}(t, 0) \).

**PROPOSITION 8.** Suppose \( l \in \mathbb{N} \), \( G \subset \subset \mathbb{C}^{n+1} \) are Stein open subsets of \( \mathbb{C}^{n} \), and \( \mathcal{O}^{r}: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1} \) is a sheaf-homomorphism on \( \mathcal{G} \times 0 \subset \mathbb{C}^{n+1} \). Then there exist a sheaf-homomorphism \( \psi: \mathcal{O}[Z; l] \rightarrow \mathcal{O}^{r}[Z_{n+1}; l] \) on \( G \) and a subvariety \( T \) of codimension \( \geq 1 \) in \( G \) such that, for \( t \in G - T \),

\[
\text{Im} \psi(t) = (\mathcal{O}[Z_{n+1}; l])^{0} \cap \mathcal{P}(t)^{-1}(\mathcal{O}[Z_{n+1}; l])^{0}.
\]

**PROOF.** Let \( (\mathcal{P}_{\nu})_{\nu \in \mathbb{N}} \in \Gamma(\mathcal{G} \times 0, \mathcal{O}^{r}) \) be the image of \( (0, \ldots, 0, 1, 0, \ldots, 0) \)

\( \in \Gamma(\mathcal{G} \times 0, \mathcal{O}^{r}) \) under \( \mathcal{O}^{r} \), where \( 1 \) is in the \( \nu^{1} \)th position. Let \( \mathcal{P}_{\nu} = \sum_{i=0}^{\infty} \mathcal{P}_{\nu} z^{i}_{n+1} \) be the power series expansion in \( Z_{n+1} \) where \( \mathcal{P}_{\nu} \in \Gamma(\mathcal{G}, \mathcal{O}) \). Define a sheaf-homomorphism \( \mathcal{P}: \mathcal{O}[Z_{n+1}; l] \rightarrow \mathcal{O}^{r} \) on \( G \) as follows: For \( t \in G \) and \( a_{\nu} \in \mathcal{O}_{t} \), \( 1 \leq \nu \leq p, 0 \leq i \leq l \), the image of \( (a_{\nu})_{\nu \leq p, 0 \leq i \leq l} \in \mathcal{O}^{r}[Z_{n+1}; l] \) under \( \mathcal{P} \) is \( (b_{\nu})_{\nu \in \mathbb{N}, \nu > 1} \in \mathcal{O}^{r} \), where

\[
b_{\nu} = \sum_{t=1}^{p} \sum_{i=0}^{\infty} \mathcal{P}_{\nu} z^{i}_{n+1} a_{\nu}.
\]

By Lemma 2(b) there exist a sheaf-homomorphism \( \psi': \mathcal{O}[Z_{n+1}; l] \rightarrow \mathcal{O}^{r}[Z_{n+1}; l] \) on \( G \) and a subvariety \( T \) of codimension \( \geq 1 \) in \( G \) such that

\[
\mathcal{O}[Z_{n+1}; l] \xrightarrow{\psi'(t)} \mathcal{O}^{r}[Z_{n+1}; l] \xrightarrow{\mathcal{P}(t)} \mathcal{O}^{r}(t)
\]

is exact for \( t \in G - T \). Through the identification of \( (\mathcal{O}[Z_{n+1}; l])^{0} \) and \( \mathcal{O}^{r}[Z_{n+1}; l] \), \( \psi' \) corresponds to a sheaf-homomorphism \( \psi: \mathcal{O}[Z_{n+1}; l] \rightarrow (\mathcal{O}[Z_{n+1}; l])^{0} \) on \( G \). Then \( \psi \) and \( T \) satisfy the requirements.

Q.E.D.

Suppose \( G \subset \subset \mathbb{C}^{n} \) are Stein open subsets of \( \mathbb{C}^{n} \). Suppose \( U_{i} \subset \subset \mathbb{C}^{n+1} \) are open disc neighborhoods of \( 0 \) in \( \mathbb{C}^{n} \) with centers at \( 0 \). Suppose \( r \in \mathbb{N} \) and
$M$ is a non-singular $r \times r$ matrix of holomorphic functions on $\tilde{G} \times \tilde{U}_1 \times \tilde{U}_2$. For $t \in \tilde{G}$ let $M_{(t)}$ be the $r \times r$ matrix of holomorphic functions on $\tilde{U}_1 \times \tilde{U}_2$ obtained by restricting $M$ to $\{t\} \times \tilde{U}_1 \times \tilde{U}_2$ and identifying $\tilde{U}_1 \times \tilde{U}_2$ with $\{t\} \times \tilde{U}_1 \times \tilde{U}_2$.

Let $E$ and $F$ be respectively the Hilbert spaces of all $r$-tuples of square integrable holomorphic functions on $U_2$ and $U_1 \times U_2$. Let $w_1$, $w_2$ be the coordinate functions of $\tilde{U}_1 \times \tilde{U}_2 \subset \mathbb{C}^2$. Suppose $m = (m_1, m_2) \in \mathbb{N}_+^*$. Let $E^{m_1+1}$ be the direct sum of $m_1 + 1$ copies of $E$ and let $F''$ be the subset of $F$ consisting of elements whose power series expansions in $w_2$ are polynomials of degree $\leq m_2$. Define a homomorphism $u : \tilde{G} \times E^{m_1+1} \to \tilde{G} \times F$ of trivial bundles with Hilbert space fibers as follows: For $t \in \tilde{G}$ and $f_0, \ldots, f_{m_1} \in E$, $u$ maps $(t, f_0, \ldots, f_{m_1}) \in \tilde{G} \times E^{m_1+1}$ to $(t, g) \in \tilde{G} \times F$, where

$$g(w_1, w_2) = M_{(t)} \left( \sum_{i=0}^{m_1} f_i(w_2) w_i^i \right).$$

**Proposition 9.** There exist a subvariety $T$ of codimension $\geq 1$ in $G$ and a subvariety $T'$ of codimension $\geq 1$ in $G - T$ such that $u^{-1}((G - T - T') \times F')$ is a holomorphic vector bundle with finite-dimensional fibers.

**Proof.** We can assume without loss of generality that the radius of $U_1$ is $< 1$ and the radius of $\tilde{U}_1$ is $> 1$. Choose Stein open subsets $G \subset \subset G'' \subset \subset G' \subset \subset \tilde{G}$ of $\mathbb{C}^r$ and choose open discs $U_2 \subset \subset U_2'' \subset \subset U_2' \subset \subset \tilde{U}_2$ in $\mathbb{C}$ with centers at 0.

Let $M = \sum_{i=0}^m M_i w_i$ be the power series expansion in $w_1$, where $M_i$ is an $r \times r$ matrix of holomorphic functions on $\tilde{G} \times \tilde{U}_2$. Define a sheaf-homomorphism

$$\alpha : (a+1)\mathcal{O}^{r(m_1+1)} \to (a+1)\mathcal{O}^{N*}_{a+1}$$
on $G' \times U_2'$ as follows: For $x \in G' \times U_2'$ and $\sigma_0, \ldots, \sigma_{m_1} \in (a+1)\mathcal{O}_{a+1}^r$, $\alpha(\sigma_0, \ldots, \sigma_{m_1}) = (\tau_\mu)_{\mu \in \mathbb{N}_+^*} \in (a+1)\mathcal{O}^{N*}_{a+1}$, where

$$\tau_\mu = \sum_{\nu=0}^{\min(\nu, m_1)} M_{\nu-\nu} \sigma_\nu \quad (\mu \in \mathbb{N}_+^*).$$

$\alpha$ maps uniformly bounded elements of $\Gamma(G' \times U_2', (a+1)\mathcal{O}^{r(m_1+1)})$ to uniformly bounded elements of $\Gamma(G' \times U_2', (a+1)\mathcal{O}^{N*})$.

By Lemma 2 (a) there exist $p \in \mathbb{N}$ and a sheaf-homomorphism $\theta : (a+1)\mathcal{O}^p \to (a+1)\mathcal{O}^p$ on $G'' \times U_2''$ such that

(i) $\eta \theta$ is the identity sheaf-isomorphism on $G'' \times U_2''$,

(ii) $\theta \eta \alpha = \alpha$ on $G'' \times U_2''$,

where $\eta : (a+1)\mathcal{O}^{N*} \to (a+1)\mathcal{O}^p$ is the projection onto the first $p(a+1)\mathcal{O}$-components, and
(iii) $\theta$ maps uniformly bounded elements of $\Gamma(G'' \times U'_2, (\pi_+^{\mathcal{O}})^p)$ to uniformly bounded elements of $\Gamma(G'' \times U'_2, (\pi_+^{\mathcal{O}})^{N*}).$

By Proposition 8 there exist a sheaf-homomorphism $\psi_2: \pi_+^{\mathcal{O}} \to (\pi_+^{\mathcal{O}})[w_2; m_2]^p$ on $G$ and a subvariety $T_1$ of codimension $\geq 1$ in $G$ such that, for $t \in G - T_1$,

$$\text{Im } \psi_2(t) = (\pi_+^{\mathcal{O}})[w_2; m_2]^p \cap \theta(t)^{-1}( (\pi_+^{\mathcal{O}})[w_2; m_2])^{N*}.$$ 

For $t \in G$ define a linear map $\beta_t: C^* \to F$ as follows: Suppose $a \in C^*$. Through the canonical isomorphism $C^* \cong (\pi_+^{\mathcal{O}})[w_2; m_2]^p$ we can regard $a$ as an element of $\pi_+^{\mathcal{O}}(t)$. $\psi_2(t)(a) \in (\pi_+^{\mathcal{O}})[w_2; m_2]^p$. We can regard $\psi_2(t)a$ as an element of $\Gamma((t) \times U'_2, (\pi_+^{\mathcal{O}})[w_2; m_2]^p)$.

$$\theta(t) \psi_2(t)(a) \in \Gamma((t) \times U'_2, (\pi_+^{\mathcal{O}})[w_2; m_2])^{N*} \cong \Gamma(U'_2, (\pi_+^{\mathcal{O}})[w_2; m_2])^{N*}.$$ 

Let $\theta(t) \psi_2(t)(a) = (f_{\beta})_{\beta \in N*}$, where $f_{\beta} \in \Gamma(U'_2, (\pi_+^{\mathcal{O}})[w_2; m_2])$. Define $\beta_\iota(a)$ to be the square integrable holomorphic function $\sum_{\pi \in N*} f_{\beta} u_\iota^\pi$ on $U_1 \times U_2$.

For $t \in G$ define $v_1: E^{m_1+1} \oplus C^* \to F$ by $v_1(a \oplus b) = u_\iota(a) + \beta_\iota(b)$. Since, by virtue of the non-singularity of $M$, $u_\iota$ maps $E^{m_1+1}$ homeomorphically onto a closed subspace of $F$ for $t \in G$, it follows that $\text{Im } v_1$ is a closed subspace of $F$ and $\dim \text{Ker } v_1 < \infty$ for $t \in G$. Define $v: G \times (E^{m_1+1} \oplus C^*) \to G \times F$ by $v(t, a \oplus b) = v_1(a \oplus b)$. Then $v$ is a direct homomorphism of trivial bundles with Hilbert space fibers. Since $\dim \text{Ker } v_1 < \infty$ for $t \in G$, by Proposition 1, there exists a subvariety $T_2$ of codimension $\geq 1$ in $G$ such that $\text{Ker } v_{G-T_2}$ is a holomorphic vector bundle over $G - T_2$ with finite-dimensional fibers. Set $T = T_1 \cup T_2$.

Let $\omega: (G - T) \times (E^{m_1+1} \oplus C^*) \to (G - T) \times E^{m_1+1}$ be the natural projection. Let $\omega': \text{Ker } v_{G-T} \to (G - T) \times E^{m_1+1}$ be the restriction of $\omega$ to $\text{Ker } v_{G-T}$. It is easily verified that $\text{Im } \omega' = u_\iota^{-1}((G - T) \times F')$.

For every $t \in G - T$, both $\text{Ker } \omega'_t$ and $\text{Im } \omega'_t$ are finite-dimensional. $\omega'$ is a direct homomorphism of trivial bundles with Hilbert space fibers. By Proposition 1, there exists a subvariety $T'$ of codimension $\geq 1$ in $G - T$ such that $\text{Ker } \omega'_{G-T-T'}$ is a holomorphic vector bundle with finite-dimensional fibers.

$$\text{Ker } \omega'_{G-T-T'} \subset \text{Ker } v_{G-T-T'}, \overset{\omega'_{G-T-T'}}{\longrightarrow} (G - T - T') \times E^{m_1+1}$$

is exact. Hence, by [2, p. 21, Prop. 3], $u_\iota^{-1}((G - T - T') \times F') = \text{Im } \omega'_{G-T-T'}$ is a holomorphic vector bundle with finite-dimensional fibers. Q.E.D.

Suppose $\mathcal{L}$ be the sheaf associated to the line bundle of a hyperplane section of $P_i$. Let $\mathcal{L}_i$ be the inverse image of $\mathcal{L}$ under the projection $C^* \times P_1 \times P_1 \to P_1$ onto the $i^{th}$ $P_1$-factor ($i = 1, 2$).

Let $z_1, z_2$ be the inhomogeneous coordinates of $P_1 \times P_1$, where $z_i$ may take on the value $\infty$. Suppose $a \in \mathbb{R}_2^+$ and for $i = 1, 2$, let
\[ W_i = \{ (z_1, z_2) \in P_1 \times P_1 \mid |z_i| > a_i \} . \]

Let \( \Omega = W_1 \cup W_2 \). Suppose \( \mathcal{B} \) is a locally free analytic sheaf of rank \( r \) on \( \tilde{G} \times \Omega \) and suppose there exists a sheaf-isomorphism \( \varphi_i : \tilde{\mathcal{O}}^r \rightarrow \mathcal{B} \) on \( \tilde{G} \times W_i \) (\( i = 1, 2 \)), where \( \tilde{\mathcal{O}} \) is the structure sheaf of \( C^* \times P_1 \times P_1 \). For \( m = (m_1, m_2) \in N_+^2 \), let \( \mathcal{B}^{(m)} = \mathcal{B} \otimes \mathcal{L}^{(m_1)}_1 \otimes \mathcal{L}^{(m_2)}_2 \), where \( \mathcal{L}^{(m_i)} \) is the tensor product of \( m_i \) copies of \( \mathcal{L} \).

**Proposition 10.** For \( m \in N_+^2 \), there exist a subvariety \( T_m \) of codimension \( \geq 1 \) in \( G \) and a subvariety \( T_m' \) of codimension \( \geq 1 \) in \( G - T_m \) such that the restriction map

\[ \alpha_i : \Gamma((G - T_m - T_m') \times \Omega, \mathcal{B}^{(m)} \rightarrow \Gamma([t] \times \Omega, \mathcal{B}^{(m)}(t)) \]

is surjective.

**Proof.** Let \( M \) be the non-singular \( r \times r \) matrix of holomorphic functions on \( \tilde{G} \times (W_1 \cap W_2) \) representing the sheaf-isomorphism \( \varphi_1^{-1} \varphi_2 : \tilde{\mathcal{O}}^r \rightarrow \tilde{\mathcal{O}}^r \) on \( \tilde{G} \times (W_1 \cap W_2) \). Choose \( a < b \) in \( R_+^2 \). Let \( U_i = \{ z \in P_1 \mid |z| > b_i \} (i = 1, 2) \). Let \( w_i = 1/z_i \) (\( i = 1, 2 \)). Define \( E, F, u, \) and \( F' \) exactly as in the paragraph preceding Proposition 9. By Proposition 9, there exist a subvariety \( T_m \) of codimension \( \geq 1 \) in \( G \) and a subvariety \( T_m' \) of codimension \( \geq 1 \) in \( G - T_m \) such that \( u^{-1}((G - T_m - T_m') \times F') \) is a holomorphic vector bundle with finite-dimensional fibers. We can assume without loss of generality that \( T_m \) and \( T_m' \) respectively have pure codimension 1 in \( G \) and \( G - T_m \). By Cartan’s Theorem A, for \( t \in G - T_m - T_m' \), the restriction map \( \beta_i : \Gamma(G - T_m - T_m', u^{-1}((G - T_m - T_m') \times F')) \rightarrow u_i^{-1}(F') \) is surjective.

Let \( \Omega' = (U \times P_1) \cup (P_1 \times U_2) \). Take \( t \in G - T - T' \). Consider the following commutative diagram:

\[
\begin{array}{ccc}
\Gamma((G - T_m - T_m') \times \Omega', \mathcal{B}^{(m)})) & \xrightarrow{\alpha_i} & \Gamma([t] \times \Omega', \mathcal{B}^{(m)}(t)) \\
\sigma \downarrow & \downarrow \tau & \\
\Gamma((G - T_m - T_m') \times \Omega, \mathcal{B}^{(m)}) & \xrightarrow{\beta_i} & u_i^{-1}(F')
\end{array}
\]

where \( \sigma \) and \( \tau \) are defined as follows. Suppose \( f_0, \cdots, f_m \) are holomorphic sections of \( (G - T_m - T_m') \times E \) over \( G - T_m - T_m' \) such that \( u(f_0, \cdots, f_m) \) is a holomorphic section of \( (G - T_m - T_m') \times F' \) over \( G' - T_m - T_m' \). Since the tensor product of \( m_i \) copies of \( \mathcal{L} \) is isomorphic to the sheaf of germs of meromorphic functions on \( P_1 \), whose only possible poles are poles of order \( \leq m_i \) at \( \infty \), it follows that \( \sum_{i=0}^{m_i} f_i w_i \) and \( u(f_0, \cdots, f_m) \) can be naturally regarded respectively as elements of

\[ \Gamma((G' - T_m - T_m') \times P_1 \times U_2, \tilde{\mathcal{O}} \otimes \mathcal{L}^{(m_1)}_1 \otimes \mathcal{L}^{(m_2)}_2) \]
and
\[ \Gamma \left( (G' - T_m - T'_m) \times U_1 \times P_1, \mathcal{O}^{(m)}_{1} \otimes \mathcal{O}^{(m_2)}_{2} \right). \]

Define \( \sigma(f_0, \cdots, f_{m_1}) \) to be the section of \( \mathcal{B}^{(m)} \) which agrees with \( \varphi_i' \left( \sum_{i=0}^{m_1} f_i w_i \right) \) on \( (G' - T_m - T'_m) \times P_1 \times U_2 \) and agrees with \( \varphi_i'(u(f_0, \cdots, f_{m_1})) \) on \( (G' - T_m - T'_m) \times U_1 \times P_1 \), where \( \varphi_i' : \mathcal{O}^{(m_1)}_1 \otimes \mathcal{O}^{(m_2)}_2 \to \mathcal{B}^{(m)} \) is induced by \( \varphi_i \) \((i = 1, 2)\). \( \tau \) is defined in a manner entirely analogous to the definition of \( \sigma \).

Since the following two restriction maps are bijective:
\[ \Gamma \left( (G - T_m - T'_m) \times \Omega, \mathcal{B}^{(m)} \right) \to \Gamma \left( (G - T_m - T'_m) \times \Omega', \mathcal{B}^{(m')} \right), \]
\[ \Gamma \left( \{ t \} \times \Omega, \mathcal{B}^{(m)}(t) \right) \to \Gamma \left( \{ t \} \times \Omega', \mathcal{B}^{(m')}(t) \right), \]
it is easily verified that \( \sigma \) and \( \tau \) are isomorphisms. The proposition follows.

**Q.E.D.**

**Proposition 11.** Suppose \( A \) is a thick set in \( G \) and for every \( t \in A \), \( \mathcal{B}(t) \) can be extended to a coherent analytic sheaf on \( \{ t \} \times P_1 \times P_1 \). Then \( \mathcal{B} | G \times \Omega \) can be extended to a coherent analytic sheaf on \( G \times P_1 \times P_1 \).

**Proof.** By Proposition 10, for \( m \in \mathbb{N}^\times \) we can find a subvariety \( T_m \) of codimension \( \geq 1 \) in \( G \) and a subvariety \( T'_m \) of codimension \( \geq 1 \) in \( G - T_m \) such that for \( t \in G - T_m - T'_m \) the restriction map
\[ \Gamma \left( (G - T_m - T'_m) \times \Omega, \mathcal{B}^{(m)} \right) \to \Gamma \left( \{ t \} \times \Omega, \mathcal{B}^{(m)}(t) \right) \]
is surjective.

Since \( A \) is thick, there exists \( t^* \in A - \bigcup_{m \in \mathbb{N}^\times} (T_m \cup T'_m) \). Since \( \mathcal{B}(t^*) \) can be extended to a coherent analytic sheaf on \( \{ t^* \} \times P_1 \times P_1 \), we can find \( m^* \in \mathbb{N}^\times \) such that \( \Gamma \left( \{ t^* \} \times \Omega, \mathcal{B}^{(m^*)}(t^*) \right) \) generates \( \mathcal{B}^{(m^*)}(t^*) \) on \( \{ t^* \} \times \Omega \). Since the restriction map
\[ \Gamma \left( (G - T_m - T'_m) \times \Omega, \mathcal{B}^{(m^*)} \right) \to \Gamma \left( \{ t^* \} \times \Omega, \mathcal{B}^{(m^*)}(t^*) \right) \]
is surjective, by Nakayama's lemma, \( \Gamma \left( (G - T_m - T'_m) \times \Omega, \mathcal{B}^{(m^*)} \right) \) generates \( \mathcal{B}^{(m^*)} \) on \( \{ t^* \} \times \Omega \). By Proposition 7, \( \mathcal{B}^{(m^*)} | (G - T_m - T'_m) \times \Omega \) can be extended to a coherent analytic sheaf on \( (G - T_m - T'_m) \times P_1 \times P_1 \). By Proposition 3, \( \mathcal{B}^{(m^*)} | G \times \Omega \) can be extended to a coherent analytic sheaf on \( G \times P_1 \times P_1 \).

**Q.E.D.**

Suppose \( \alpha < \alpha' < \beta' < \beta \) in \( \mathbb{R}_+ \). Let
\[ U_1 = K^1(\beta), \quad U_2 = \{ z \in P_1 \mid |z| > \alpha \} \]
\[ U'_1 = K^1(\beta'), \quad U'_2 = \{ z \in P_1 \mid |z| > \alpha' \}. \]

Suppose \( I_r \) is the \( r \times r \) identity matrix. The following lemma is proved in [6, p. 427, Heftungs lemma] by some Laurent series and infinite product arguments.
LEMMMA 3. There exists $\varepsilon = \varepsilon(\alpha', \beta') \in \mathbb{R}$ satisfying the following: If $H$ is an open subset of $C^n$ and $M$ is an $r \times r$ matrix of holomorphic functions on $H \times (U_1 \cap U_2)$ such that the uniform bound on $H \times (U_1 \cap U_2)$ of every entry of $M - I_r$ is $< \varepsilon$, then there exists a non-singular $r \times r$ matrix $P_i$ of holomorphic functions on $H \times U_i'$ $(i = 1, 2)$ such that $M = P_1^{-1}P_2$ on $H \times (U_1' \cap U_2')$.

Suppose $G_1$, $G_2$ are open subsets of a complex manifold $(X, \mathcal{O})$ and $M$ is a non-singular $r \times r$ matrix of holomorphic functions defined on an open subset of $X$ containing $G_1 \cap G_2$. Then $\mathcal{L}(G_1, G_2)$ denotes the locally free analytic sheaf of rank $r$ on $G_1 \cup G_2$ characterized as follows. There exists a sheaf-isomorphism $\varphi_i: \mathcal{O} \to \mathcal{L}_G(G_1, G_2)$ on $G_i$ $(i = 1, 2)$ such that the sheaf-isomorphism $\varphi_1^{-1}\varphi_2: \mathcal{O} \to \mathcal{O}$ on $G_1 \cap G_2$ is represented by $M$ (when elements of $\mathcal{O}$ are written as column vectors).

PROPOSITION 12. Suppose $D$ is a domain in $C^n$ and $\mathcal{F}$ is a locally free analytic sheaf of rank $r$ on $D \times P_1$. Let $A$ be the subset of $D$ consisting of all $t$ such that $\mathcal{F}(t) \simeq \mathcal{O}$ on $(t) \times P_1$, where $\mathcal{O}$ is the structure sheaf of $P_1$. Then there exists a subvariety $T$ in $D$ such that

(i) $A \cap T = \emptyset$,

(ii) for every Stein open subset $W$ of $D - T$ which either is contractible or has the same homotopy type as the 1-sphere, $\mathcal{F}$ is isomorphic to $\hat{\mathcal{O}}$ on $W \times P_1$, where $\hat{\mathcal{O}}$ is the structure sheaf of $C^n \times P_1$.

PROOF. When $A = \emptyset$, we can set $T = D$. So we assume that $A \neq \emptyset$.

Let $\mathcal{R}$ be the zero direct image of $\mathcal{F}$ under the natural projection $\pi: D \times P_1 \to D$. $\mathcal{R}$ is a coherent analytic sheaf on $D$. Let $T_1$ be the subvariety of $D$ where $\mathcal{R}$ is not locally free.

Let $\mathcal{G}$ be the analytic subsheaf of $\mathcal{F}$ generated by $\mathcal{R}$, i.e. $s \in \mathcal{G}$ if and only if there exist $v_1, \ldots, v_p \in \Gamma(U \times P_1, \mathcal{F})$ for some open neighborhood $U$ of $\pi(x)$ such that $s \in (\sum_{i=1}^p \hat{\mathcal{O}} v_i)_x$. $\mathcal{G}$ is coherent. Let $T_2$ be the subvariety of $D \times P_1$ where $\mathcal{G}$ disagrees with $\mathcal{F}$.

Let $T = T_1 \cup \pi(T_2)$. We claim that $T$ satisfies the requirement. Take $t^0 \in A$. Let $G$ be an open polydisc neighborhood of $t^0$ in $D$. By [5, p. 270, Satz 6] there exists a non-singular $r \times r$ matrix $M$ of holomorphic functions on $G \times (U_1 \cap U_2)$ such that $\mathcal{F} | G \times P_1 \simeq \mathcal{L}(G \times U_1, G \times U_2)$. Since $\mathcal{F}(t^0) \simeq \mathcal{O}$ on $\{t^0\} \times P_1$, there exists a non-singular $r \times r$ matrix $M_i$ of holomorphic functions on $\{t^0\} \times U_i$ $(i = 1, 2)$ such that $M = M_1^{-1}M_2$ on $\{t^0\} \times (U_1 \cap U_2)$. By replacing $M$ by $(M_1 \circ \sigma)(M_1 \circ \sigma)^{-1}$ (where $\sigma: G \times (U_1 \cap U_2) \to \{t^0\} \times (U_1 \cap U_2)$ is induced by the map $G \to \{t^0\}$), we can assume that $M = I_r$ on $\{t^0\} \times (U_1 \cap U_2)$.

Choose a relatively compact neighborhood $Q$ of $(U_1' \cap U_2')$ in $U_1 \cap U_2$. 

By continuity arguments, there exists an open neighborhood $H$ of $t^*$ in $G$ such that the uniform bound on $H \times Q$ of every entry of $M-I$, is $< \varepsilon(\alpha', \beta')$, where $\varepsilon(\alpha', \beta')$ comes from Lemma 3. There exists a non-singular $r \times r$ matrix $P_i$ of holomorphic functions on $H \times U'_i (i = 1, 2)$ such that $M = P_i^{-1}P_i$ on $H \times \left(U'_1 \cup U'_2\right)$. Hence $F | H \times P_i \approx \hat{\Theta}^r$ on $H \times P_i$. $H \cap T = \emptyset$ and $\mathcal{R} \approx \hat{\Theta}^r$ on $H$.

Suppose $W$ is a Stein open subset of $D-T$ which either is contractible or has the same homotopy type as the 1-sphere. Since $\mathcal{R}$ is a locally free analytic sheaf of rank $r$ on $D-T$, by [5, p. 270, Satz 6], $\mathcal{R}$ is isomorphic to $\hat{\Theta}^r$ on $W$. $\mathcal{R}$ is generated on $W$ by $r$ elements of $\Gamma(W, \mathcal{R})$. $F$ is generated on $W \times P_i$ by $r$ elements of $\Gamma(W \times P_i, F)$. Hence $F \approx \hat{\Theta}^r$ on $W \times P_i$. Q.E.D.

For $\alpha \in \mathbb{R}_+$ let

$$L(\alpha) = \{z \in P_i | |z| > \alpha\}.$$

**Proposition 13.** Suppose $a < b$ in $\mathbb{R}_+$, $D$ is a contractible Stein domain in $\mathbb{C}^n$, and $F$ is a locally free analytic sheaf of rank $r$ on $D \times G^2(a, b)$. Suppose $t^* \in D$ and $F(t^*)$ can be extended to a coherent analytic sheaf on $[t^*] \times K^2(b)$. Then there exists a closed thin set $T$ in $D$ satisfying the following:

For $t \in D-T$ there exist

(i) a relatively compact open neighborhood $U(t)$ of $t$ in $D-T$,

(ii) $a_1 < a'_i(t) < b'_i(t) < b_i$ in $\mathbb{R}$,

(iii) $a'_i(t) > a_2$ in $\mathbb{R}$,

(iv) a non-singular $r \times r$ matrix $P_i$ of holomorphic functions on $U(t) \times G^i(a'_i(t), b'_i(t)) \times L(a_2)$,

(v) a non-singular $r \times r$ matrix $M_i$ of holomorphic functions on $U(t) \times L(a'_i(t)) \times L(a'_i(t))$,

such that

(a) on $U(t) \times G^2((a'_i(t), a_2), (b'_i(t), b_2))$ $F$ is isomorphic to

$$\mathcal{L}_{P_i} = \mathcal{L}_{P_i} \left( U(t) \times G^i(a'_i(t), b'_i(t)) \times P_i, U(t) \times K^i(b'_i(t)) \times L(a_2) \right),$$

(b) on $U(t) \times ((G^i(a'_i(t), b'_i(t)) \times P_i) \cup (K^i(b'_i(t)) \times L(a'_i(t))))$ $\mathcal{L}_{P_i}$ is isomorphic to

$$\mathcal{L}_{M_i} = \mathcal{L}_{M_i} \left( U(t) \times L(a'_i(t)) \times P_i, U(t) \times P_i \times L(a'_i(t)) \right).$$

Consequently,

($\alpha$) for $t' \in U(t)$, $F(t')$ can be extended to a coherent analytic sheaf on $\{t'\} \times K^2(b)$ if and only if $\mathcal{L}_{M_i}(t')$ can be extended to a coherent analytic sheaf on $\{t'\} \times P_i \times P_i$,

($\beta$) $F | U(t) \times G^2(a, b)$ can be extended to a coherent analytic sheaf on $U(t) \times K^2(b)$ if and only if $\mathcal{L}_{M_i}$ can be extended to a coherent analytic sheaf on $U(t) \times P_i \times P_i$. 
PROOF. By [5, p. 270, Satz 6] there exists a non-singular \( r \times r \) matrix \( S \) of holomorphic functions on \( D \times G^i(a_i, b_i) \times G^i(a_2, b_2) \) such that

\[
\mathcal{F} \cong \mathcal{O}_S \left( D \times G^i(a_i, b_i) \times K^i(b_2), D \times K^i(b_1) \times G^i(a_2, b_2) \right).
\]

Since \( \mathcal{F}(t^0) \) can be extended to a coherent analytic sheaf on \( \{t^0\} \times K^2(b) \), \( \mathcal{F}(t^0) \) is isomorphic to \( \mathcal{O}^r \) on \( \{t^0\} \times G^2(a, b) \). Hence we can assume that \( S = I_r \) on \( \{t^0\} \times G^i(a_i, b_i) \times G^i(a_2, b_2) \).

Consider the following locally free analytic sheaf of rank \( r \) on \( D \times G^i(a_i, b_i) \times P_1 \):

\[
\mathcal{O}_S := \mathcal{O}_S \left( D \times G^i(a_i, b_i) \times K^i(b_2), D \times G^i(a_i, b_i) \times L(a_2) \right).
\]

Since \( S = I_r \) on \( \{t^0\} \times G^i(a_i, b_i) \times G^i(a_2, b_2) \), by Proposition 12, there exists a subvariety \( Z \) of codimension \( \geq 1 \) in \( D \times G^i(a_i, b_i) \) such that, for every Stein open subset \( W \) of \( D \times G^i(a_i, b_i) - Z \) which has the same homotopy type as the 1-sphere, \( \mathcal{O}_S \) is isomorphic to \( \mathcal{O}^r \) on \( W \times P_1 \), where \( \mathcal{O} \) is the structure sheaf of \( C^* \times P_1 \times P_1 \). We can assume without loss of generality that \( Z \) has pure dimension 1 in \( D \times G^i(a_i, b_i) \).

Let \( \pi: D \times G^i(a_i, b_i) \to D \) be the natural projection. Let \( A \) be the subvariety of \( Z \) where the rank of \( \pi|_Z \) is \( \leq n - 1 \). Take \( a_i < a_i^* < b_i^* < b_i \) in \( \mathbb{R} \).

Let \( T = \pi(A \cap \{D \times G^i(a_i^*, b_i)^\}) \). We claim that \( T \) satisfies the requirement.

Take \( t \in D - T \). \( \{t\} \times G^i(a_i^*, b_i^*) \cap Z \) is at most discrete. We can find \( a_i^* < a_i'(t) < b_i'(t) < b_i^* \) in \( \mathbb{R} \) such that \( \{t\} \times G^i(a_i'(t), b_i'(t)) \) is disjoint from \( Z \). By continuity arguments, we can find an open polydisc neighborhood \( \hat{U}(t) \) of \( t \) in \( D - T \) such that \( \hat{U}(t) \times G^i(a_i'(t), b_i'(t)) \) is disjoint from \( Z \). \( \mathcal{O}_S \) is therefore isomorphic to \( \mathcal{O}^r \) on \( \hat{U}(t) \times G^i(a_i'(t), b_i'(t)) \times P_1 \). Hence there exist non-singular \( r \times r \) matrices of holomorphic functions \( P \) and \( P' \) on \( \hat{U}(t) \times G^i(a_i'(t), b_i'(t)) \times L(a_2) \) and \( \hat{U}(t) \times G^i(a_i'(t), b_i'(t)) \times K^i(b_2) \) respectively such that \( S = (P')^{-1}P \) on \( \hat{U}(t) \times G^i(a_i'(t), b_i'(t)) \times G^i(a_2, b_2) \).

Let \( \hat{P} = P \circ \sigma \), where

\[
\sigma: \hat{U}(t) \times G^i(a_i'(t), b_i'(t)) \times P_1 \to \hat{U}(t) \times G^i(a_i'(t), b_i'(t)) \times \{\infty\}
\]

is induced by \( P_1 \to \{\infty\} \). By replacing \( P \) and \( P' \) respectively by \( \hat{P}^{-1}P \) and \( \hat{P}^{-1}P' \), we can assume without loss of generality that \( P = I_r \) on \( \hat{U}(t) \times G^i(a_i'(t), b_i'(t)) \times \{\infty\} \).

Consider the following locally free analytic sheaf of rank \( r \) on \( \hat{U}(t) \times P_1 \times L(a_2) \):

\[
\mathcal{O}_S := \mathcal{O}_S \left( \hat{U}(t) \times L(a_i'(t)) \times L(a_2), \hat{U}(t) \times K^i(b_i'(t)) \times L(a_2) \right).
\]

Let \( p: \hat{U}(t) \times P_1 \times L(a_2) \to \hat{U}(t) \times L(a_2) \) be the natural projection. For \( s \in \hat{U}(t) \times \{\infty\} \), \( \mathcal{O}_S \approx \mathcal{O}^r \) on \( p^{-1}(s) \), where \( \mathcal{O} \) is the structure sheaf of \( P_1 \). By Proposition 12, there exists an open neighborhood \( B \) of \( \hat{U}(t) \times \{\infty\} \) in \( \hat{U}(t) \times L(a_2) \).
such that, for any contractible Stein open subset $B'$ of $B$, $\mathcal{O}$ is isomorphic to $\tilde{\mathcal{O}}^r$ on $p^{-1}(B')$.

Choose $a'_s(t) > a_s$ in $\mathbb{R}$ and a relatively compact open polydisc neighborhood $U(t)$ of $t$ in $\tilde{U}(t)$ such that $U(t) \times L(a'_s(t)) \subset B$. $\mathcal{O}$ is isomorphic to $\tilde{\mathcal{O}}^r$ on $U(t) \times P_1 \times L(a'_s(t))$. It follows that there exist non-singular $r \times r$ matrices of holomorphic functions $M_t$ and $M'$ on $U(t) \times L(a'_s(t)) \times L(a'_s(t))$ and $U(t) \times K^1(b'(t)) \times L(a'_s(t))$ respectively such that $P = M_t(M')^{-1}$ on $U(t) \times G^1(a'_s(t), b'_s(t)) \times L(a'_s(t))$.

Set $P_t$ to be the restriction of $P$ to $U(t) \times G^1(a'_s(t), b'_s(t)) \times L(a_s)$. It is easy to verify that $P_t$ and $M_t$ satisfy the conditions (a) and (b), and (a) and (b) follow from Proposition 2.

Q.E.D.

PROPOSITION 14. Suppose $a < b$ in $\mathbb{R}^n$, $D$ is a domain in $\mathbb{C}^n$, and $\mathcal{F}$ is a locally free analytic sheaf of rank $r$ on $D \times G^a(b)$. Suppose $A$ is a thick set in $D$ and, for every $t \in A$, $\mathcal{F}(t)$ can be extended to a coherent analytic sheaf on $\{t\} \times K^a(b)$. Then $\mathcal{F}$ can be extended to a coherent analytic sheaf on $D \times K^a(b)$.

PROOF. By Proposition 3, we can assume that $D$ is contractible and Stein. We can also assume without loss of generality that $A$ is the set of all $t \in D$ such that $\mathcal{F}(t)$ can be extended to a coherent analytic sheaf on $\{t\} \times K^a(b)$. We use the same notation as in the statement of Proposition 13.

Take $t \in A^t - T$. $U(t) \cap A$ is thick in $U(t)$. By Proposition 11, $\mathcal{L}_{\mathcal{M}_t}$ can be extended to a coherent analytic sheaf on $U(t) \times P_1 \times P_1$. $\mathcal{F} \mid U(t) \times G^a(b)$ can be extended to a coherent analytic sheaf on $U(t) \times K^a(b)$. The proposition follows from Proposition 3.

Q.E.D.

Proposition 14 is the locally free case of the Main Theorem. To prove the Main Theorem, we need only reduce the general case to the locally free case. This reduction is a simple refinement of the corresponding reduction in [15]. Therefore we present here only brief outlines.

PROPOSITION 15. Suppose $0 \leq a < b$ in $\mathbb{R}^n$, $D$ is a domain in $\mathbb{C}^n$, and $\mathcal{F}$ is a coherent analytic sheaf on $D \times G^a(b)$ such that $\mathcal{F}^{[n]} = \mathcal{F}$ and $0^{[n+2]} = \mathcal{F}$. Suppose $A$ is a thick set in $D$ and, for $t \in A$, $\mathcal{F}(t)$ can be extended to a coherent analytic sheaf on $\{t\} \times K^a(b)$. Then $\mathcal{F}$ can be extended to a coherent analytic sheaf on $D \times K^a(b)$.

PROOF. We can assume that $\mathcal{F} \neq 0$. Supp $\mathcal{F}$ has pure dimension $n + 2$ and hence by [11] can be extended to a subvariety $X$ of pure dimension $n + 2$ in $D \times K^a(b)$.

Take $t \in D$ such that, for some $a < b^* < b$ in $\mathbb{R}^n$, $X(t) = \{(t) \times K^a(b^*)\} \cap X$
has pure dimension 2. (The set of $t \in D$ not possessing this property is contained in a closed thin set.) Take $a < a' < b' < b^*$ in $\mathbb{R}^N$. By applying [7, VII. B. 3, p. 218] to $X(t)$, we can find an open neighborhood $U$ of $t$ in $D$, $0 < \alpha < \beta$ in $\mathbb{R}^3$, and a holomorphic map $F' : U \times K^N(b) \to \mathbb{C}^2$ such that $F'(X \cap (U \times K^N(a'))) \subset K^2(\alpha)$ and the map $\Phi : U \times K^N(b) \to C^{*+2}$, defined by the coordinate functions of $C^*$ and $F'$, makes $X \cap \Phi^{-1}(U \times K^2(\beta)) \cap (U \times K^N(b'))$ an analytic cover over $U \times K^2(\beta)$.

The proposition follows from Propositions 3 and 14 and from considering the zeroth direct image of $\mathcal{F}$ under the map which is the restriction of $\Phi$ to $X \cap \Phi^{-1}(U \times G^2(\alpha, \beta)) \cap (U \times K^N(b'))$. Q.E.D.

Suppose $\pi : X \to (Y, \mathcal{O})$ is a holomorphic map and $\mathcal{F}$ is a coherent analytic sheaf on $X$. $\mathcal{F}$ is said to be $\pi$-flat at $x \in X$ if $\mathcal{F}_x$, when naturally regarded through $\pi$ as a $\mathcal{O}_{(x)}$-module, is a flat $\mathcal{O}_{(x)}$-module. The subset $Z$ of $X$ where $\mathcal{F}$ is not $\pi$-flat is a subvariety in $X$ and, when $Y$ is reduced, $\pi(Z)$ is thin in $Y$ [3]. We need the above statement only for the case where $Y$ is a manifold and this case can be proved by very elementary means [15, Lem. (5.1)].

**Proposition 16.** Suppose $0 \leq a < b$ in $\mathbb{R}^N$, $D$ is a contractible Stein open subset of $C^*$, and $\mathcal{F}$ is a $p$-flat coherent analytic sheaf on $D \times K^N(b)$, where $p : D \times K^N(b) \to D$ is the natural projection. Suppose $1 \leq \nu \leq N-1$ and

$$0 \to \mathcal{O}^{\nu}_{\mathbb{R}^N} \to \cdots \to \mathcal{O}^{\nu}_0 \to \mathcal{F} \to 0$$

is an exact sequence of sheaf-homomorphisms on $D \times K^N(b)$. Suppose $\xi \in H^\nu(D \times G^N(a, b), \mathcal{F})$. For $t \in D$, let $\xi_t \in H^\nu(|t| \times G^N(a, b), \mathcal{F}(t))$ be the image of $\xi$ under the map

$$H^\nu(D \times G^N(a, b), \mathcal{F}) \to H^\nu(|t| \times G^N(a, b), \mathcal{F}(t))$$

induced by the natural map $\mathcal{F} \to \mathcal{F}(t)$. Suppose $A$ is a thick set in $D$ and for $t \in A$ there exists a holomorphic function $f_t$ on $K^1(b)$ such that $f_t$ is nowhere zero on $G^i(a, b)$ and $(f_t \circ \pi) \xi_t = 0$, where $\pi : C^N \to C$ is defined by $\pi(z_1, \ldots, z_N) = z_i$. Then there exists a holomorphic function $F$ on $D \times K^1(b)$ such that $F$ is nowhere zero on $D \times G^i(a, b)$ and $(F \circ \Pi) \xi = 0$, where $\Pi : D \times C^N \to D \times C$ is defined by $\Pi(t, z) = (t, \pi(z))$.

**Proof.** We need only prove the case where $\mathcal{F} = \mathcal{O}_{\mathbb{R}^N}$ and $\nu = N-1$, because $H^\mu(D \times G^N(a, b), \mathcal{O}_{\mathbb{R}^N}) = 0$ for $1 \leq \mu \leq N-2$. Let $H = G^i(a, b) \times \cdots \times G^i(a_N, b_N)$. $\xi$ can be represented by a holomorphic function $\Theta$ on $D \times H$ whose Laurent series expansion in $z_2, \ldots, z_N$ has the form

$$\Theta = \sum_{\nu_2, \ldots, \nu_N} \Theta_{\nu_2, \ldots, \nu_N} z_2^{\nu_2} \cdots z_N^{\nu_N},$$

where $z_1, \ldots, z_N$ are the coordinates of $C^N$. For $t \in D$, let $\Theta_t$ be the restriction
of $\Theta$ to $\{t\} \times H$. $(f_t \circ \pi)\xi_t = 0$ means that $(f_t \circ \pi)\Theta_t$ can be extended to a holomorphic function on
\[ \tilde{H} := K^1(b_1) \times G^1(a_2, b_2) \times \cdots \times G^1(a_n, b_n) . \]
$(F \circ \Pi)\xi = 0$ means that $(F \circ \Pi)\Theta$ can be extended to a holomorphic function on $D \times \tilde{H}$. The proposition follows from Proposition 4. Q. E. D.

**Proof of the Main Theorem.** Let $\mathcal{G} = 0_{[n+2]}$ and $\mathcal{R} = \mathcal{F}/\mathcal{G}$. Let $\pi : D \times K^N(b) \to D$ be the natural projection. By Proposition 3, we can assume without loss of generality that $\mathcal{F}$ is $\pi$-flat and $\mathcal{R}^{[n]} = \mathcal{R}$. By Proposition 15, $\mathcal{G}$ can be extended to a coherent analytic sheaf $\tilde{\mathcal{G}}$ on $D \times K^N(b)$ satisfying $\tilde{\mathcal{G}}^{[n]} = \tilde{\mathcal{G}}$. By Theorem 2, $\mathcal{R}^{[n+1]}$ can be extended to a coherent analytic sheaf $\tilde{\mathcal{R}}$ on $D \times K^N(b)$ satisfying $\tilde{\mathcal{R}}^{[n+1]} = \tilde{\mathcal{R}}$. By Proposition 6, being a subsheaf of $\tilde{\mathcal{R}} | D \times G^N(a, b)$, $\mathcal{R}$ can be extended to a coherent analytic sheaf $\tilde{\mathcal{R}}$ on $D \times K(b)$ satisfying $\tilde{\mathcal{R}}^{[n]} = \tilde{\mathcal{R}}$.

Take $t_0 \in D$ such that, for some contractible Stein open neighborhood $U$ of $t_0$ in $D$ and some $a < b' < b$ in $\mathbb{R}^N$, $\mathcal{G}$ and $\tilde{\mathcal{R}}$ are $\pi$-flat on $U \times K^N(b')$ and there exists an exact sequence of sheaf-homomorphisms
\[ 0 \longrightarrow \mathcal{O}^{\mathbb{N}}_{\mathbb{N}-2} \longrightarrow \cdots \longrightarrow \mathcal{O}^{\mathbb{N}}_{0} \longrightarrow \mathcal{G} \longrightarrow 0 \]
on $U \times K^N(b')$. (The set of $t_0 \in D$ not possessing this property is contained in a closed thin set in $D$.)

Suppose for some thick set $B$ in $U$, $\mathcal{F}(t)$ can be extended to a coherent analytic sheaf $\tilde{\mathcal{F}}(t)$ on $\{t\} \times K^N(b)$ for $t \in B$. We can assume that $\tilde{\mathcal{F}}(t)^{[0]} = \tilde{\mathcal{F}}(t)$ for $t \in B$. For $t \in B$, we have the following commutative diagram:
\[
\begin{array}{c}
\Gamma \left( U \times G^N(a, b'), \mathcal{F} \right) \longrightarrow \Gamma \left( U \times G^N(a, b'), \mathcal{R} \right) \longrightarrow H^1 \left( U \times G^N(a, b'), \mathcal{G} \right) \\
\downarrow \parallel \vert \downarrow \parallel \\
\Gamma \left( \{t\} \times G^N(a, b'), \mathcal{F}(t) \right) \longrightarrow \Gamma \left( \{t\} \times G^N(a, b'), \mathcal{R}(t) \right) \longrightarrow H^1 \left( \{t\} \times G^N(a, b'), \mathcal{G}(t) \right)
\end{array}
\]
where $\lambda^*(t)$ is induced by $\lambda(t) : \tilde{\mathcal{F}}(t) \to \tilde{\mathcal{R}}(t)$ which is the unique extension of $\mathcal{F}(t) \to \mathcal{R}(t)$. Since $\operatorname{Supp} \operatorname{Coker} \lambda(t)$ is a finite subset of $K^N(b') - G^N(a, b')$, from the above diagram and Proposition 16, we conclude that $\Gamma \left( U \times G^N(a, b'), \mathcal{F} \right)$ generates $\mathcal{F}$ on $U \times G^N(a, b')$. The Main Theorem follows from Propositions 3 and 7. Q. E. D.

By using the techniques of [1], we obtain from Theorem 1 the following two theorems concerning the extension of coherent analytic sheaves across pseudoconcave boundaries.
A real-valued function $\varphi$ on a complex space $X$ is said to be strongly $n$-pseudoconvex at $x \in X$ if there exist a biholomorphic map $\tau$ from a neighborhood $U$ of $x$ in $X$ onto a complex subspace of an open subset $D$ of $\mathbb{C}^n$ and a real-valued infinitely differentiable function $\bar{\varphi}$ on $D$ such that $\varphi = \bar{\varphi} \tau$ on $U$ and at every point in $D$ the Hermitian matrix $(\partial^2 \bar{\varphi}/\partial z_i \partial \bar{z}_j)_{1 \leq i,j \leq n}$ has at least $N - n + 1$ positive eigenvalues.

An open subset $D$ of a complex space $X$ is said to be strongly $n$-pseudoconcave at $x \in X$ if there is a strongly $n$-pseudoconvex function $\varphi$ on some open neighborhood $U$ of $x$ in $X$ such that $D \cap U = \{y \in U : \varphi(y) > \varphi(x)\}$.

**Theorem 5.** Suppose $X$ is a complex space, $x \in X$, and $D$ is an open subset of $X$ which is strongly $n$-pseudoconcave at $x$. Then there exists an open neighborhood $U$ of $x$ in $X$ such that every coherent analytic sheaf $\mathcal{F}$ on $U \cap D$ satisfying $\mathcal{F}^{[n]} = \mathcal{F}$ can be uniquely extended to a coherent analytic sheaf $\tilde{\mathcal{F}}$ on $U$ satisfying $\tilde{\mathcal{F}}^{[n]} = \tilde{\mathcal{F}}$.

**Theorem 6.** Suppose $\varphi$ is a strongly $n$-pseudoconvex function on a complex space $X$ such that $\varphi : X \to (a, b)$ is a proper map, where $a \in \{-\infty\} \cup \mathbb{R}$ and $b \in \mathbb{R} \cup \{\infty\}$. Then for every $c \in (a, b)$ every coherent analytic sheaf $\mathcal{F}$ on $\{x \in X : \varphi > c\}$ satisfying $\mathcal{F}^{[n]} = \mathcal{F}$ can be uniquely extended to a coherent analytic sheaf $\tilde{\mathcal{F}}$ on $X$ satisfying $\tilde{\mathcal{F}}^{[n]} = \tilde{\mathcal{F}}$.

Instead of strong pseudoconvexity, the above two theorems can also be formulated with the weaker condition of $*$-strong convexity [13, p. 370, Defs. 4 and 5].

**University of Notre Dame**

**References**


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