Extension of meromorphic maps into Kähler manifolds

By Yum-Tong Siu

In this paper we prove the following result.

Main Theorem. Let $X$ be a complex manifold, $A$ be a subvariety of codimension $\geq 1$ in $X$, and $G$ be an open subset of $X$ which intersects every branch of $A$ of codimension 1. If $M$ is a compact Kähler manifold, then every meromorphic map $f$ (in the sense of Remmert [11, p. 332]) from $(X - A) \cup G$ to $M$ can be extended to a meromorphic map from $X$ to $M$.

The proof depends on the results concerning the Lelong numbers of closed positive currents obtained in [15] and on Bishop's theorem [1, p. 299, Theorem 3] and the techniques of its proof. The methods used in the proof can actually give more general results, for example, results on the extension of a special kind of holomorphic correspondences, and we will discuss such generalizations in this paper. A consequence of such generalizations is that the Main Theorem remains true when $X$ is a normal complex space. There is also a brief discussion of other possible generalizations whose investigation is not yet complete. The Main Theorem (for the case where $X$ is a normal complex space) was announced in [14].

We will apply also the methods used in the proof of the Main Theorem to the problem of extending meromorphic maps across sets of low Hausdorff dimension. By combining these methods with a generalization of Bishop's theorem, we obtain the following result.

Theorem 1. Let $X$ be a complex manifold of dimension $n$ and $M$ be a compact Kähler manifold. Suppose $A$ is a closed subset of $X$ of zero Hausdorff $(2n - 3)$-measure and $f : X - A \to M$ is a meromorphic map in the sense of Remmert. Then $f$ can be extended to a meromorphic map from $X$ to $M$.

The special case of the Main Theorem where $X$ is the open unit ball in $\mathbb{C}^n$ with $n \geq 3$, $f$ is holomorphic, and $A = \{0\}$ was proved by Griffiths [6, p. 29, Theorem 1]. Shiffman [13, p. 333, Theorem 2] relaxed the condition $n \geq 3$ in Griffiths' result to $n \geq 2$ (and proved some slightly more general cases where

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A is compact). The case of the Main Theorem where \( f \) is holomorphic and the codimension of \( A \) is \( \geq 2 \) was conjectured by Griffiths [6, p. 30]. One of the key steps in the proofs of Griffiths' and Shiffman's results is the following lemma. If \( u \) is a plurisubharmonic function on an \( n \)-dimensional complex manifold \( Y \) and is \( C^\infty \) outside a compact subset \( K \) of \( Y \), then
\[
\int_{L-K} (\sqrt{-1} \partial \overline{\partial} u)^n
\]
is finite for every relatively compact open subset \( L \) of \( Y \). Griffiths observed that his conjecture would be confirmed by the arguments of [6] and [13] if one could prove the above lemma for the case where \( K \) is a noncompact subvariety of codimension \( \geq 2 \) in \( Y \) [6, p. 37, Problem 0]. Shiffman and Taylor obtained a simple counterexample showing that for this case the lemma is not true. We give their example in an appendix (according to the wish of Shiffman). This counterexample shows that the proof of a noncompact \( A \) necessarily requires a line of argument different from the case of a compact \( A \).

The Main Theorem was used by Sommese [18] to obtain extension results on reductive group actions on compact Kähler manifolds.

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The three appendices contain material which is referred to in the paper proper but can be regarded as separate entities.

Throughout the paper we use the following notations. The nonnegative integer \( n \) occupies a special position. The coordinates of \( \mathbb{C}^n \) are denoted by \( z = (z_1, \cdots, z_n) \). \(|z|\) means \( (\sum_{i=1}^n |z_i|^2)^{1/2} \). \( B(a, r) \) denotes the open ball in \( \mathbb{C}^n \) with center \( a \) and radius \( r \). \( B(0, r) \) is simply denoted by \( B(r) \), and \( B(1) \) is simply denoted by \( B \). \( \Delta(r) \) denotes the open disc in \( C \) with radius \( r \) and center \( 0 \) and \( \Delta(1) \) is simply denoted by \( \Delta \). Unless the contrary is expressly stated, all complex spaces are reduced. The set of regular points of a complex space \( X \) is denoted by \( \text{Reg } X \). The characteristic function of a set \( A \) is denoted by \( \chi_A \). The topological closure of \( A \) is denoted by \( A^\circ \) or \( \overline{A} \). The boundary of \( A \)
is denoted by $\partial A$. For a subset $E$ of a metric space, $h^p(E)$ denotes the Hausdorff $p$-measure of $E$ [5, p. 171]. In this paper we will be concerned only with the vanishing of $h^p(E)$. For this purpose, two different metrics which dominate each other after multiplication by a constant would give us the same result. So we will not specify the metric used in the calculation of the Hausdorff measure and will use the obviously most convenient one. The pullback of a form $\varphi$ under a map $f$ is denoted by $f^*\varphi$. The pushforward of a current $u$ under a map $g$ is denoted by $g_*u$.

All meromorphic maps in the paper will be in the sense of Remmert [11, p. 367, Definition 15]. In [11], all complex spaces considered there by Remmert are normal (because of Definitions 3, 4, 5 on pp. 337–338 and because of the definition of holomorphic functions in Paragraph 3 of p. 336). We recall here Remmert’s definition of a meromorphic map. Suppose $X$ and $Y$ are normal complex spaces and $X$ is connected. We say that $f: X \to Y$ is a meromorphic map if

(i) to every $x \in X$ is assigned a nonempty compact subset $f(x)$ of $Y$,
(ii) the set of all $(x, y) \in X \times Y$ with $y \in f(x)$ is a connected locally irreducible subvariety of $X \times Y$, and
(iii) there is a dense subset $X^*$ of $X$ such that for every $x \in X^*$ the set $f(x)$ consists of only one point.

One can use the above three conditions to define a meromorphic map of general complex spaces which may not be normal. However, such a definition put us in the awkward situation in which a holomorphic map may fail to be meromorphic because the local irreducibility requirement of condition (ii) in general is not satisfied by a holomorphic map. Moreover, to define a meromorphic map of general complex spaces this way is the same as defining it by first going to the normalizations of the spaces. For the purpose of this paper, such a general situation offers nothing new. Therefore we confine ourselves to Remmert’s definition and consider only meromorphic maps of normal spaces. Note that, if we drop the local irreducibility requirement in condition (ii), we will have difficulty in extending meromorphic maps, because in such a case we have no way to control the part of the graph of a meromorphic map which is over a subvariety of codimension $\geq 1$ in $X$.

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1. Closed positive currents

(1.1) We introduce the following standard notation. Suppose $u$ is an
$l$-current on a $C^\infty$ oriented manifold $M$ of real dimension $m$; that is, $v$ is a continuous linear functional on the LF-space of all $C^\infty(m - l)$-forms $\varphi$ on $M$ with compact supports. We denote the value of $v$ at $\varphi$ by $\int_M v \wedge \varphi$ or simply by $\int v \wedge \varphi$.

A $(k, k)$-current $u$ on an open subset $\Omega$ of $C^*$ is called positive if for every nonnegative $C^\infty$ function $\rho$ on $\Omega$ with compact support and for any linear functions $g_1, \ldots, g_{n-k}$,

$$\int_\Omega u \wedge \rho \prod_{i=1}^{n-k} \frac{\sqrt{-1}}{2} dg_i \wedge d\bar{g}_i$$

is nonnegative. Positivity is a local property and is independent of the coordinate system chosen. If $\varphi$ is a $C^\infty$ positive $(1, 1)$-form on $\Omega$, then $u \wedge \varphi$ is positive on $\Omega$ when $u$ is positive on $\Omega$.

When $u$ is positive on $\Omega$, the coefficients of $u$ are measures. For $A \subset \Omega$, denote by $\chi_A u$ the positive $(k, k)$-current whose coefficients are $\chi_A$ times the coefficients of $u$, where, for a measure $\mu$, $\chi_A \mu$ means the measure defined by

$$(\chi_A \mu)(E) = \mu(A \cap E).$$

Suppose $\widetilde{\Omega}$ an open subset of $C^*$ containing $\Omega$. A positive $(k, k)$-current $\widetilde{u}$ on $\widetilde{\Omega}$ is called the trivial extension of a positive $(k, k)$-current $u$ defined on $\Omega$ if $\chi_{\overline{A}} \widetilde{u} = \widetilde{u}$ and $u = \widetilde{u} | \Omega$. In general, a positive current does not admit a trivial extension to a larger domain.

(1.2) Suppose $u$ is a positive $(k, k)$-current on an open subset $\Omega$ of $C^*$. By the total variation $||u||$ of $u$ we mean the measure

$$u \wedge \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2 \right)^{n-k}.$$ 

For $a \in \Omega$, the Lelong number $n(u, a)$ of $u$ at $a$ is defined as

$$\lim_{r \to 0} \frac{1}{(\pi r^2)^{n-k}} ||u|| (B(a, r))$$

if such a limit exists.

By using smoothing by convolution and Stokes’ theorem, Lelong [10, p. 72] showed that, if $u$ is a closed positive $(k, k)$-current on $\Omega$ and $B(a, r) \subset \Omega$, then

(i) $(1/(\pi r^2)^{n-k}) ||u|| (B(a, r))$ is a nondecreasing function of $r$,

(ii) $n(u, a)$ exists always, and

(iii) $\int_{B(a, r) \setminus a} u \wedge \left( (\sqrt{-1}/2\pi) \partial \bar{\partial} \log |z - a|^2 \right)^{n-k} = (1/(\pi r^2)^{n-k}) ||u|| (B(a, r)) - n(u, a)$. 


An easy consequence of (i) is that \( n(u, a) \) is an upper semicontinuous function of \( a \in \Omega \).

We denote by \( E_c(u) \) or simply by \( E_c \) the set of points \( a \) of \( \Omega \) such that \( n(u, a) \geq c \). Since \( n(u, a) \) is an upper semicontinuous function of \( a \), \( E_c \) is a closed subset of \( \Omega \).

In [15, Main Theorem], it was proved (by using the methods of [2], [3], [9], [17]) that, for \( c > 0 \), \( E_c \) is a subvariety of codimension \( \geq k \) in \( \Omega \). We will not use the full strength of this statement. We need only the following weaker statement.

\[
\bigcup_{c>0} E_c = \bigcup_{i=1}^{\infty} V_i
\]

where \( V_i \) is a subvariety of codimension \( \geq k \) in \( \Omega \) (which may be empty) and

\[
\inf_{x \in V_i} n(u, x) > 0.
\]

The weaker statement follows from the fact that

\[
\bigcup_{c>0} E_c = \bigcup_{i=1}^{\infty} E_{c/2^i}.
\]

We will need this weaker statement only for the case where \( \Omega \) is Stein. So actually for this purpose, the following partial result of Skoda [17, p. 406, Theorem 11.2] suffices: if \( \Omega \) is Stein, then there exists a subvariety \( X_c \) of codimension \( \geq k \) in \( \Omega \) such that \( E_c \subset X_c \subset E_{c/2} \).

To a subvariety \( V \) of pure codimension \( k \) in \( \Omega \), we associate canonically a \((k, k)\)-current \([V]\) defined as follows. For a \( C^\infty(n - k, n - k)\)-form \( \varphi \) on \( \Omega \) with compact support, \( \int [V] \wedge \varphi \) is the integral of \( \varphi \) over \( \text{Reg} \, V \). By locally representing \( V \) as an analytic cover, one can show that \([V]\) is a closed positive current, and, by considering the tangent cone of \( V \) at \( a \), one can show that \( n([V], a) \) is the multiplicity of \( V \) at \( a \) (see [10] and [20]). It follows from (i) that, for \( B(a, r) \subset \Omega \),

\[
\text{Vol}(V \cap B(a, r)) \geq n([V], a) \frac{(\pi r^2)^{n-k}}{(n-k)!}.
\]

We will need the following deep result from [15] on closed positive currents (whose proof requires most of the ingredients in the proof of the analyticity of \( E_c \)). Suppose \( u \) is a closed positive \((k, k)\)-current on an open subset \( \Omega \) of \( \mathbb{C}^n \) and \( V \) is an irreducible subvariety of pure codimension \( k \) in \( \Omega \). Let \( c = \inf_{x \in V} n(u, x) \). Then \( u - c[V] \) is a closed positive \((k, k)\)-current on \( \Omega \) and \( n(u - c[V], x) = 0 \) for almost all \( x \in \text{Reg} \, V \). For, by [15, (12.3)] \( \chi_V u = c[V] \) and hence \( u - c[V] = \chi_{\Omega - V} u \) is positive. Since by [15, (11.5)] the Lelong number of a closed positive \((k, k)\)-current is independent of the local coordinates system used in its definition, it follows from [15, (9.6)] that
\( n(u, x) = c \) for almost all \( x \in \text{Reg } V \); that is, \( n(u - c[V], x) = 0 \) for almost all \( x \in \text{Reg } V \).

We would like to mention a little trivial fact which will be used in this paper. Suppose \( Q \) and \( G \) are complex manifolds, \( W \) is a pure-dimensional subvariety of \( Q \), and \( \pi: Q \to G \) is a holomorphic map such that the restriction of \( \pi \) to \( W \) is proper. Suppose \( Z_1 \) is a subvariety of \( G \) such that \( W \cap \pi^{-1}(Z_1) \) is of codimension \( \geq 1 \) in \( W \). Suppose \( Z_2 \) is a subvariety of \( G - Z_1 \) such that \( W \cap \pi^{-1}(Z_2) \) is of codimension \( \geq 1 \) in \( W - \pi^{-1}(Z_1) \). If \( \omega \) is a \( C^\infty \) form on \( Q \), then

\[ \chi_{Z_1 \cup Z_2 \pi^*([W] \wedge \omega)} = 0. \]

(1.3) We will need two extension results for closed positive currents. The first one is the following.

\[ (*) \] Suppose \( \Omega \) is an open subset of \( C^n \), \( V \) is a subvariety of codimension \( \geq 1 \), and \( G \) is an open subset of \( \Omega \) whose intersection with every branch of \( V \) of codimension 1 is nonempty and irreducible. If \( u \) is a closed positive \((1, 1)\)-current on \( (\Omega - V) \cup G \), then \( u \) can be extended uniquely to a closed positive \((1, 1)\)-current on \( \Omega \). As a consequence, if \( \Omega \) is Stein and \( H^1(\Omega, \mathbb{R}) = 0 \), then there exists a plurisubharmonic function \( h \) on \( \Omega \) such that \( u = \sqrt{-1} \partial \bar{\partial} h \) on \( (\Omega - V) \cup G \).

This result was proved in [15, Theorem 1] and for the last assertion, see for example the proof of [15, (5.4)].

The second extension result is the following result of Harvey [8]: a closed positive \((k, k)\)-current can be trivially extended across a closed subset with zero Hausdorff \((2n - 2k - 1)\)-measure. Actually the following weaker form suffices for our purpose.

\[ (\dagger) \] Suppose \( \bar{u} \) is a closed positive \((k, k)\)-current on an open subset \( \Omega \) of \( C^n \), \( A \) is a closed subset of \( \Omega \) of zero Hausdorff \((2n - 2k - 1)\)-measure, and \( u \) is a closed positive \((k, k)\)-current on \( \Omega - A \) such that \( \bar{u} - u \) is positive on \( \Omega - A \). Then the trivial extension of \( u \) to \( \Omega \) is a closed positive \((k, k)\)-current on \( \Omega \).

This weaker form can be very easily proved by Stokes' theorem and (1.2) (i) (cf. the second half of the proof of [16, (2.23)]).

2. Proof of the Main Theorem

(2.1) We will prove a statement which is slightly stronger than the Main Theorem. We now formulate that statement. Let us first introduce a definition. Let \( \Omega \) be an open subset of \( C^n \). A subset \( Y \) of \( \Omega \) is said to be thin
of order \( l \) in \( \Omega \) if \( Y \) is contained in a countable union of locally closed subvarieties of dimension \( \leq l \). Suppose \( A \) is a subvariety of codimension \( \geq 1 \) in the open unit ball \( B \) of \( \mathbb{C}^* \), \( S \) is a subvariety of codimension \( \geq 1 \) in \( B - A \), and \( M \) is a compact Kähler manifold of dimension \( m \) with Kähler form \( \omega \). Let
\[
\pi: B \times M \longrightarrow B ,
\sigma: B \times M \longrightarrow M
\]
be the natural projections. Consider the following statement for \( -1 \leq l < n \).

If \( f \) is a holomorphic map from \( B - A - S \) to \( M \) and if the form \( f^* \omega \) on \( B - A - S \) can be extended to a closed positive \((1,1)\)-current \( \varphi \) on \( B \), then there exists a closed subset \( Y_i \) of \( A \cup S \) which is thin of order \( l \) in \( B \) such that \( f \) can be extended to a meromorphic map \( f_i \) from \( B - Y_i \) to \( M \) with graph \( F_i \), and the closed positive \((n - l - 1, n - l - 1)\)-current
\[
\eta_i^* = \left( f^* \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2 \right)^{n-i-1}
\]
on \( B - A - S \) admits a trivial extension to \( B \) which is a closed positive \((n - l - 1, n - l - 1)\)-current on \( B \).

Clearly \((*)_{-1}\) is satisfied. We will prove \((*)_i\), by descending induction on \( l \) and will obtain the Main Theorem from \((*)_{-1}\). If one wishes to invoke Harvey's result quoted in (1.3) in its full strength, then the second part of \((*)_i\), concerning the closed positive current \( \eta_i \) on \( B \) is always automatically satisfied. For,
\[
\eta_i^* = \pi_*(\left[ F_i \right] \wedge \left( \sigma^* \omega + \pi^* \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2 \right)^{n-i-1})
\]
is a closed positive \((n - l - 1, n - l - 1)\)-current on \( B - Y_i \) and hence can be trivially extended to a closed positive \((n - l - 1, n - l - 1)\)-current on \( B \). Moreover, since by the last statement of (1.2),
\[
\chi_{(A \cup S) - Y_i} \eta_i^* = 0 ,
\]
it follows that \( \eta_i^* \) agrees with the trivial extension of \( \eta_i \) on \( B - Y_i \). We have included the second part of \((*)_i\), in the induction process, because we want to use only the weak form of Harvey's result given in (1.3) (†) and its inclusion does not entail any additional effort.

(2.2) We assume that \((*)_i\) is true for some \( 0 \leq l < n \) and want to prove \((*)_{i-1}\). Observe that \((*)_{i-1}\) is a local statement; that is, it suffices to prove
that, for every point $x$ of $B$, we can show that $(\ast)_{i-1}$ is true after we replace $B$ by some open neighborhood of $x$ in $B$.

There exists a plurisubharmonic function $h$ on $B$ which is $C^\infty$ on $B - A - S$ such that

$$\sqrt{-1} \partial \bar{\partial} h = \psi + \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2.$$ 

By subtracting a constant from $h$, we can assume, after replacing $B$ by a slightly smaller open ball, that $h$ is nonpositive on $B$ and we can obtain, by smoothing by convolution, a sequence of $C^\infty$ nonpositive plurisubharmonic functions $h_\nu$ on $B$ so that $h_\nu$ nonincreasingly approaches $h$ pointwise on $B$. We denote also by $\eta_i$ the trivial extension of $\eta_i$ to $B$. Let

$$\xi_\nu = \sqrt{-1} \partial \bar{\partial} h_\nu \wedge \eta_i.$$ 

First we show that a subsequence of $\xi_\nu$ converges weakly to a closed positive current on $B$.

Let $w = (w_1, \cdots, w_n)$ be a unitary coordinate system of $C^\infty$ and let $\alpha$, $\beta$ denote the projections

$$\alpha: x \mapsto (w_1(x), \cdots, w_l(x)), \quad \beta: x \mapsto (w_{l+1}(x), \cdots, w_n(x)).$$

Let $B^r(a, r)$ denote the open ball in $C^\infty$ with center $a$ and radius $r$. Take $x_0 \in Y_i$ and assume that $\alpha^{-1}(\alpha(x_0)) \cap Y_i$ is thin of order 0. There exist $0 < r < s < (1 - |x_0|)/2$ such that $\alpha^{-1}(\alpha(x_0)) \cap Y_i$ is disjoint from

$$H: = \beta^{-1}(B^{s^{-1}}(\beta(x_0), s) - B^{s^{-1}}(\beta(x_0), r)).$$

For some open neighborhood $U$ of $\alpha(x_0)$ in $B^1(\alpha(x_0), (1 - |x_0|)/2)$, $U \times H$ is disjoint from $Y_i$. Let $K$ be a compact open neighborhood of $\alpha(x_0)$ in $U$. Take a nonnegative $C^\infty$ function $\rho_1$ on $U$ with compact support which is identically 1 on an open neighborhood of $K$. Take a nonnegative $C^\infty$ function $\rho_2$ on $B^{s^{-1}}(\beta(x_0), s)$ with compact support which is identically 1 on an open neighborhood of $\beta(x_0)$, $s$. Let $\rho = (\rho_1 \circ \alpha)(\rho_2 \circ \beta)$.

Now

$$\int \rho_\xi \Pi_{i=1}^l \frac{\sqrt{-1}}{2} dw_i \wedge d\bar{w}_i$$

$$= \int_B \rho \sqrt{-1} \partial \bar{\partial} h_\nu \wedge \eta_i \wedge \Pi_{i=1}^l \frac{\sqrt{-1}}{2} dw_i \wedge d\bar{w}_i$$

$$= \int_B \partial \bar{\partial} \rho \wedge \sqrt{-1} h_\nu \eta_i \wedge \Pi_{i=1}^l \frac{\sqrt{-1}}{2} dw_i \wedge d\bar{w}_i.$$
\[ \begin{aligned}
&= \int_B (\rho_1 \circ \alpha) \partial \bar{\partial} (\rho_2 \circ \beta) \wedge \sqrt{-1} h \cdot \eta_1 \wedge \prod_{i=1}^t \frac{\sqrt{-1}}{2} dw_i \wedge d\bar{w}_i \\
&= \int_{\text{Reg} F_t} (\rho_1 \circ \alpha \circ \pi) \partial \bar{\partial} (\rho_2 \circ \beta \circ \pi) \wedge (\sqrt{-1} h \circ \pi) \\
&\quad \cdot \left( \sigma^* \omega + \pi^* \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2 \right)^{n-l-1} \wedge \pi^* \left( \prod_{i=1}^t \frac{\sqrt{-1}}{2} dw_i \wedge d\bar{w}_i \right) \\
&\leq \int_{\text{Reg} F_t} (-h \circ \pi) \left( (\rho_1 \circ \alpha \circ \pi) \partial \bar{\partial} (\rho_2 \circ \beta \circ \pi) \\
&\quad \wedge \left( \sigma^* \omega + \pi^* \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2 \right)^{n-l-1} \wedge \pi^* \left( \prod_{i=1}^t \frac{\sqrt{-1}}{2} dw_i \wedge d\bar{w}_i \right) \right) \\
&< \infty
\end{aligned} \]

where

(i) the second equality follows from the closedness of \( \eta_i \);

(ii) the third equality comes from the fact that \( \rho_1 \circ \alpha \) depends only on \( w_1, \cdots, w_t \) and \( \prod_{i=1}^t (\sqrt{-1}/2) dw_i \wedge d\bar{w}_i \) occurs already in the integrand;

(iii) the fourth equality comes from the definition of \( \eta_i \);

(iv) the finiteness of the last integral is a consequence of the lemma given below in (2.3) in whose statement the meaning of the absolute value signs in the last integrand is explained.

Hence

\[ \sup_{\xi} \int \rho_\xi^* \prod_{i=1}^t \frac{\sqrt{-1}}{2} dw_i \wedge d\bar{w}_i < \infty. \]

From this we conclude that we can select a subsequence of \( \xi_n \) which converges weakly to a closed positive current \( u \) on \( B \), because of the following two observations.

(i) \( \alpha^{-1} \alpha(x_0) \cap Y \) is thin of order 0 for a generic unitary coordinate system \( w \).

(ii) Let \( p = \left( \binom{n}{n-l} \right)^z \). For a generic collection of \( p \) unitary coordinate systems

\[ w^{(p)} = (w_1^{(p)}, \cdots, w_n^{(p)}) \quad (1 \leq \mu \leq p), \]

every coefficient of an \((n-l, n-l)\)-current \( \gamma \) with respect to the coordinate system \( z \) is a linear combination over \( C \) of the coefficients of

\[ \gamma \wedge \prod_{i=1}^t \frac{\sqrt{-1}}{2} dw^{(p)}_i \wedge d\bar{w}^{(p)}_i \]

with respect to the coordinate system \( w^{(p)} \) \((1 \leq \mu \leq p)\). Moreover, the complex numbers which are the coefficients of this linear combination depend only on the coordinate systems \( w^{(p)} \) \((1 \leq \mu \leq p)\) and \( z \) and are independent of \( \gamma \) (see [15, (1.5) and p. 65] or [10, p. 59]).
(2.3) **Lemma.** Suppose $X$ is a subvariety of pure dimension $k$ in an open subset $\Omega$ of $\mathbb{C}^n$ ($1 \leq k < n$), $h$ is a nonpositive plurisubharmonic function on $\text{Reg} X$ and $\varphi$ is a $C^\infty(k, k)$-form on $\Omega$ with compact support. Then
\[
\int_{\text{Reg} X} (-h) |\varphi| < \infty ,
\]
where $|\varphi|$ is the $(k, k)$-form on $\text{Reg} X$ defined as follows: if $\gamma$ is a volume form on $\text{Reg} X$ and the pullback of $\varphi$ to $\text{Reg} X$ equals $f^*\gamma$, then $|\varphi|$ is the form $|f|\gamma$.

This lemma is the generalization to complex spaces of the statement that plurisubharmonic functions on open subsets of $\mathbb{C}^n$ are locally integrable. The lemma follows trivially from a resolution of singularity. However, we want to avoid such complicated machinery. Instead, we will prove it by simple projection techniques.

To prove the lemma, it suffices to show that, for a sufficiently small neighborhood $U$ of an arbitrary point $x_0$ of $X$,
\[
\int_{\text{Reg} U} (-h)(\sqrt{-1}\partial\bar{\partial} |z|^2)^k < \infty .
\]
 Possibly by using a new unitary coordinate system, we can assume without loss of generality that, for each $k$-tuple, $I = (i_1, \ldots, i_k)$, with $1 \leq i_1 < \cdots < i_k \leq n$, there exists an open neighborhood $U_I$ of $x_0$ in $X$ such that the projection
\[
\pi_I: x \mapsto (z_{i_1}(x), \ldots, z_{i_k}(x))
\]
makes $U_I$ an analytic cover over an open subset $G_I$ of $\mathbb{C}^k$ with a subvariety $Z_I$ of $G_I$ as a critical set (in the sense of [7, III. B. 3]). For $y \in G_I - Z_I$, define
\[
h_I(y) = \sum \{h(x) \mid x \in U_I, \pi_I(x) = y\} .
\]
Then $h_I$ is a nonpositive plurisubharmonic function on $G_I - Z_I$ and hence can be extended to a nonpositive plurisubharmonic function $\tilde{h}_I$ on $G_I$. Let $G_I'$ be a relatively compact open neighborhood of $\pi_I(x_0)$ in $G_I$ and let
\[
U = \bigcap_I (U_I \cap \pi_I^{-1}G_I') .
\]
Then
\[
\int_{\text{Reg} U} (-h)(\sqrt{-1}\partial\bar{\partial} |z|^2)^k \leq \sum_I \int_{G_I'} (-\tilde{h}_I)(\sqrt{-1}\partial\bar{\partial} \sum_{s=1}^k |z_s|^2)^k < \infty .
\]

(2.4) We now define a set $\tilde{Y}_{l-1}$, which will be used later to obtain $Y_{l-1}$. Let $B$ be the set of all irreducible subvarieties $V$ of dimension $l$ in $B$ such that
\[
c_v := \inf_{x \in V} n(u, x) > 0 .
\]
We observe that $\mathcal{B}$ is at most countable, because by (1.2) the set of points $x$ of $B$ where $n(u, x) > 0$ is thin of order $l$. Write

$$\mathcal{B} = \{ V_i \}.$$ 

Let

$$Y'_{i-1} = \bigcup_{i=1}^\infty E_{i/\nu}^{\infty}(u) - \bigcup_i V_i.$$ 

We claim that $Y'_{i-1}$ is a subset of $B$ which is thin of order $l - 1$. We know that

$$\bigcup_{i=1}^\infty E_{i/\nu}^{\infty}(u) = \bigcup_{i=1}^\infty W_i$$

where each $W_i$ is an irreducible subvariety of dimension $\leq l$ and satisfies

$$\inf_{x \in W_i} n(u, x) > 0.$$ 

If $\dim W_i = l$, then $W_i \in \mathcal{B}$. Hence $Y'_{i-1}$ is contained in the union of all $W_i$ with $\dim W_i \leq l - 1$ and it is therefore thin of order $l - 1$.

Recall that $u$ is the current obtained at the end of (2.2). We define

$$u_i = u - e_{V_i} \left[ V_i \right].$$ 

Then by (1.2), $u_i$ is a closed positive $(n - l, n - l)$-current on $B$ and $n(u_i, x) = 0$ for almost all $x \in \text{Reg} V_i$. Let

$$Z_i = \bigcup_{i=1}^\infty E_{i/\nu}^{\infty}(u_i) \cap V_i.$$ 

Since $\bigcup_{i=1}^\infty E_{i/\nu}^{\infty}(u_i)$ is a countable union of subvarieties of dimension $\leq l$ in $B$ and since $(\bigcup_{i=1}^\infty E_{i/\nu}^{\infty}(u_i)) \cap \text{Reg} V_i$ is a set of measure zero in $\text{Reg} V_i$, it follows that $Z_i$ is a countable union of subvarieties of dimension $< l$ in $B$. Let

$$\tilde{Y}_{i-1} = Y'_{i-1} \cup \left( \bigcup_{i=1}^\infty Z_i \right).$$ 

Since $u$ is $C^\infty$ on $B - A - S$ and $n(u, x) > 0$ for $x \in V_i$, it follows that $E_{i/\nu}(u)$ and $V_i$ are both contained in $A \cup S$. Hence $\tilde{Y}_{i-1}$ is a subset of $A \cup S$ which is thin of order $l - 1$ in $B$. In general, $\tilde{Y}_{i-1}$ is not a closed subset of $A \cup S$. We will show later that, for $x \in B - \tilde{Y}_{i-1}$, there exists an open neighborhood $U_x$ of $x$ in $B$ such that $f_x | U_x - A - S$ can be extended to a meromorphic map from $U_x$ to $M$. Then $Y_{i-1}$ will be defined as

$$A \cup S - \bigcup \{ U_x \mid x \in B - \tilde{Y}_{i-1} \},$$

which clearly then is a closed subset of $B$ and is thin of order $l - 1$.

(2.5) Fix $x_* \in A \cup S - \tilde{Y}_{i-1}$. Then $x_* \not\in Y_{i-1}$. We have two cases.

Case 1. $n(u, x_*) = 0$.

Case 2. $x_* \in V_{i_*}$ for some $i_*$. In this case, since $x_* \in Z_{i_*}$, $n(u_{i_*}, x_*) = 0$. We define a closed positive $(n - l, n - l)$-current $v$ on $B$ as follows. For Case 1, we define $v = u$ and, for Case 2, we define $v = u_{i_*}$.
Let $F$ be the graph of $f$. For $\bar{x} \in B \times M$ and $r > 0$, let $\bar{B}(\bar{x}, r)$ be the set of all points of $B \times M$ whose distance (with respect to the Euclidean metric of $B$ and the Kähler metric of $M$) from $\bar{x}$ is $< r$. We are going to show that there exist positive numbers $r_*$ and $c_0$ such that, for any $\bar{x}_* \in (\{x_*\} \times M) \cap \bar{F}$ and any $0 < r < r_*$,

\[
(\dagger) \quad \int_{\pi(\bar{B}(\bar{x}_*, r) \cap F)} v \wedge \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |z - x_*|^2 \right)^{i} \geq c_0 r^{2(n-1)}.
\]

Concerning the above inequality we would like to make two observations.

(i) Since $\pi$ maps $F$ biholomorphically onto $B - A - S$, $\pi(\bar{B}(\bar{x}_*, r) \cap F)$ is an open subset of $B - A - S$ and hence of $B$.

(ii) In both Cases 1 and 2, $v$ agrees with $(f^* \omega + (\sqrt{-1}/2) \partial \bar{\partial} |z|^2)^{n-1}$ on $B - A - S$. Clearly $u$ agrees with $(f^* \omega + (\sqrt{-1}/2) \partial \bar{\partial} |z|^2)^{n-1}$ on $B - A - S$. So in Case 1 $v$ agrees with $(f^* \omega + (\sqrt{-1}/2) \partial \bar{\partial} |z|^2)^{n-1}$. In Case 2, as we observed, $V_*$ is contained in $A \cup S$ and as a consequence $u$ agrees with $u_*$ on $B - A - S$. Hence in Case 2 also $v$ agrees with $(f^* \omega + (\sqrt{-1}/2) \partial \bar{\partial} |z|^2)^{n-1}$ on $B - A - S$.

The above inequality can therefore be written as

\[
(*) \quad \int_{\pi(\bar{B}(\bar{x}_*, r) \cap F)} \left( f^* \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2 \right)^{n-1} \wedge \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |z - x_*|^2 \right)^{i} \geq c_0 r^{2(n-1)}.
\]

(2.6) We digress here to recall the following theorem of Bishop [1, p. 299, Theorem 3] which is one of the essential tools used in this paper. If $D$ is an open subset of $C^n$, $A$ is a subvariety of codimension $\geq 1$ in $D$, and $V$ whose volume is finite is a subvariety of pure dimension in $D - A$, then $\bar{V} \cap D$ is a subvariety of $D$. A variation of Bishop's theorem is the following.

If $D$ is an open subset of $C^n$, $A$ is a subvariety of codimension $\geq 1$ in $D$, $S$ is a subvariety of codimension $\geq 1$ in $D - A$, and $V$ whose volume is finite is a subvariety of pure dimension in $D - A - S$, then $\bar{V} \cap D$ is a subvariety of $D$.

This variation is obtained by applying Bishop's theorem twice: first to show that $\bar{V} \cap (D - A)$ is a subvariety of $D - A$ and then to show that $\bar{V} \cap D$ is a subvariety of $D$.

We will also need the following related result.

If $A$ is a subvariety of codimension $\geq 1$ in $B(r)$, $S$ is a subvariety of codimension $\geq 1$ in $B(r) - A$, and $V$ is a subvariety of pure dimension $k$ in $B(r) - A - S$ with $0 \in V$, then the volume of $V$ is at least $cr^{2k}$, where $c$ is a positive constant depending only on $n$ and $k$. 

The case where \( S \) is empty is one of the key steps in the proof of Bishop's theorem. When both \( A \) and \( S \) are empty, this follows from (1.2) \( (\ast) \) with \( c = \pi^k/k! \). In general, if the volume of \( V \) is infinite, then it is clearly \( \geq cr^{2k} \) for any positive number \( c \); and if the volume of \( V \) is finite, then \( \overline{V} \cap B(r) \) is a subvariety of \( B(r) \) and hence it is \( \geq (\pi^k/k!)r^{2k} \).

(2.7) Now we continue with our proof of the inequality (2.5) \( (\ast) \). To prove it, we choose \( r_* > 0 \) such that, for \( \tilde{x} \in \{ x_* \} \times M \), \( \tilde{B}(\tilde{x}, 2r_*) \subset B \times U_{\tilde{z}} \) for some open subset \( U_{\tilde{z}} \) of \( M \) which is biholomorphic to an open subset of \( C^n \). By (2.6) there exists a positive number \( c_* \) such that, for \( \tilde{B}(\tilde{x}, r) \subset \tilde{B}(\tilde{x}, r_*) \) and for any subvariety \( V \) of dimension \( n - l \) in \( \tilde{B}(\tilde{x}, r) - \pi^{-1}(A \cup S) \) with \( \tilde{x} \in V \), we have the inequality \( \text{Vol}(V) \geq c_* r^{(n-l)} \), where \( \text{Vol} \) (as in the rest of this section) is calculated by the Euclidean metric of \( B \) and the Kähler metric of \( M \). (We have used the fact that, in a fixed compact subset of a complex manifold, the volume of a locally closed complex submanifold calculated with respect to a hermitian metric of the manifold is dominated by a constant times the volume calculated with respect to another hermitian metric and the constant is independent of the submanifold.) Let \( c_* = (n - l)! c_* \). Since \( \tilde{x}_* \in \tilde{F} \), there exists a sequence \( \tilde{x}_* \in \tilde{B}(\tilde{x}_*, r_*) \cap F \) which approaches \( \tilde{x}_* \). Let \( x_* = \pi(\tilde{x}_*) \). Let \( d_* \) be the distance between \( \tilde{x}_* \) and \( \tilde{x}_* \) with respect to the Euclidean metric of \( B \) and the Kähler metric of \( M \). Take \( 0 < r < r_* \). Let \( r_* = r - d_* \). We claim that, for \( r_* > 0 \),

\[
(\ast)_* \int_{\tilde{B}(\tilde{x}_*, r_*) \cap F - x_*} \left( f^* \omega + \frac{\sqrt{-1}}{2} \partial \overline{\partial} \log |z|^2 \right)^{n-1} \wedge \left( \frac{\sqrt{-1}}{2\pi} \log |z - x_*|^2 \right)^l \geq c_* (r_*)^{2(n-l)}.
\]

Let \( G \) be the Grassmannian of all \( (n - l) \)-planes in \( C^n \) passing through \( 0 \). Let

\[\mathcal{X} = \{(x, g) \in (C^n - x_*) \times G \mid x - x_* \in g\}\]

and let

\[\alpha: \mathcal{X} \longrightarrow C^n - x_* , \]

\[\beta: \mathcal{X} \longrightarrow G\]

be the natural projections. Let \( \theta \) be the volume form on \( G \) which is invariant under the action of the unitary group \( U(n) \) such that the total volume of \( G \) is 1.

We have

\[\alpha_* \beta^* \theta = \left( \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log |z - x_*|^2 \right)^l\]

(see for example [15, (11.3) \( (\dagger) \)]). Hence
\[
\int_{\pi(\tilde{B}(\tilde{z}_e, r_e) \cap F) - x_v} \left( f^* \omega + \frac{\sqrt{-1}}{2} \partial \overline{\partial} |z|^2 \right)^{n-1} \wedge \left( \frac{\sqrt{-1}}{2 \pi} \partial \overline{\partial} \log |z - x_v|^2 \right) \right) \\
= \int_{\pi(\tilde{B}(\tilde{z}_j, r_j) \cap F) - x_v} \left( f^* \omega + \frac{\sqrt{-1}}{2} \partial \overline{\partial} |z|^2 \right)^{n-1} \wedge \alpha \ast \beta \ast \theta \\
= \int_{\pi(\tilde{B}(\tilde{z}_j, r_j) \cap F) - x_v} \alpha \ast \left( f^* \omega + \frac{\sqrt{-1}}{2} \partial \overline{\partial} |z|^2 \right)^{n-1} \wedge \beta \ast \theta \\
= \int_{g \in \mathcal{G}} \theta \int_{\pi^{-1}(g) \cap \tilde{B}(\tilde{z}_j, r_j) \cap F} \alpha \ast \left( f^* \omega + \frac{\sqrt{-1}}{2} \partial \overline{\partial} |z|^2 \right)^{n-1} \\
(\text{by Fubini's theorem [15, (7.2)]}) \\
= \int_{g \in \mathcal{G}} (n - l) \text{ Vol } (\pi^{-1}(g) \cap \tilde{B}(\tilde{z}_j, r_j) \cap F) \theta \\
\leq \int_{g \in \mathcal{G}} c_0(r_e)^{2(n-1)} \theta \\
= c_0(r_j)^{2(n-1)} .
\]

We cannot simply let \( \nu \to \infty \) and pass to the limit. In general, taking the limit in (\ast \ast), does not yield (2.5) (\ast \ast), as one can easily see in the counter-example

\[ f: C^2 - 0 \longrightarrow P_1 , \]

\[
(z_1, z_2) \longmapsto [z_1, z_2]
\]

and \( x_\ast = 0 \). We have to use \( n(\nu, x_\ast) = 0 \) and the Lebesgue convergence theorem to get (2.5) (\ast \ast).

Take arbitrarily \( \varepsilon > 0 \). Since \( n(\nu, x_\ast) = 0 \), there exists \( 0 < t(\varepsilon) < r \) such that

\[
\frac{1}{(\pi(2t(\varepsilon))^2)} \| v \| (B(x_\ast, 2t(\varepsilon))) < \frac{\varepsilon}{2t} .
\]

There exists \( \nu_0 \) such that \( x_v \in B(x_\ast, t(\varepsilon)) \) for \( \nu \equiv \nu_0 \). Hence, for \( \nu \equiv \nu_0 \),

\[
\frac{1}{(\pi t(\varepsilon))^2} \| v \| (B(x_v, t(\varepsilon))) < \varepsilon .
\]

By (1.2) (iii)

\[
\int_{B(x_v, t(\varepsilon)) - x_v} v \wedge \left( \frac{\sqrt{-1}}{2 \pi} \partial \overline{\partial} \log |z - x_v|^2 \right)^{n-1} < \varepsilon
\]

for \( \nu \equiv \nu_0 \). There exists \( \nu_1 \equiv \nu_0 \), such that \( |x_v - x_\ast| < t(\varepsilon)/2 \) for \( \nu \equiv \nu_1 \). Let \( t_v = t(\varepsilon) - |x_v - x_\ast| \). Then, for \( \nu \equiv \nu_1 \),

\[
\int_{\pi(\tilde{B}(\tilde{z}_j, r_j) \cap F) - B(x_v, t_v)} v \wedge \left( \frac{\sqrt{-1}}{2 \pi} \partial \overline{\partial} \log |z - x_v|^2 \right)^{n-1} \leq c_0(r_v)^{2(n-1)} - \varepsilon ,
\]

that is
\[ \int v \wedge \chi_{x}(\tilde{\xi}_{x}, r) \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |z - x_v|^2 \right)^l \geq c_0(r)^2(n-l) - \varepsilon. \]

Since \( t \geq t(\varepsilon)/2 > 0 \) for all \( \nu \geq \nu_1 \), the coefficients of the \((l, l)\)-form

\[ \chi_n(\tilde{\xi}_{x}, r) \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |z - x_v|^2 \right)^l \]

are bounded uniformly in \( \nu \) for \( \nu \geq \nu_1 \). Moreover, their supports are all contained in the relatively compact subset \( B(x_*, 2r_*) \) of \( B \). Since \( \chi_n(\tilde{\xi}_{x}, r) \)

approaches \( \chi(\tilde{\xi}, r) \) pointwise on \( B \times M \) as \( \nu \to \infty \), it follows that the coefficients of

\[ \chi_n(\tilde{\xi}_{x}, r) \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |z - x_v|^2 \right)^l \]

converge pointwise on \( B \) to the coefficients of

\[ \chi_n(\tilde{\xi}, r) \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |z - x_*|^2 \right)^l. \]

Hence, by the Lebesgue convergence theorem,

\[ \int v \wedge \chi(\tilde{\xi}, r) \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |z - x_*|^2 \right)^l \]

\[ = \lim_{\nu \to \infty} \int v \wedge \chi_n(\tilde{\xi}_{x}, r) \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |z - x_v|^2 \right)^l \]

\[ \geq \lim_{\nu \to \infty} c_0(r)^2(n-l) - \varepsilon \]

\[ = c_0r^2(n-l) - \varepsilon. \]

Since \( \varepsilon \) is an arbitrary positive number, we obtain (2.5) (*).

We want to remark that, though we use the Kähler metric of \( M \) to define \( \tilde{B}(\tilde{z}, r) \), for our purpose actually any hermitian metric of \( M \) can do the job just as well so far as this definition is concerned.

Our next step is to prove that the Hausdorff 2\((n-l)\)-measure \( h^{2(n-l)}(\{x_*\} \times M) \cap \hat{F}^c \) of \( \{x_*\} \times M \) \( \cap \hat{F} \) is zero. It suffices to show that every point \( y \) of \( M \) admits an open neighborhood \( Q \) in \( M \) such that

\[ h^{2(n-l)}(\{x_*\} \times Q) \cap \hat{F} = 0. \]

Some open neighborhood \( \hat{Q} \) of \( y \) in \( M \) is biholomorphic to an open subset \( Q^* \) of \( C^* \) and we identify \( \hat{Q} \) with \( Q^* \). Let \( Q \) be a relatively compact open neighborhood of \( y \) in \( \hat{Q} \).

Define a positive measure \( \mu \) on \( B(x_*, 2r_*) \times \hat{Q} \) as follows: for a subset \( E \) of \( B(x_*, 2r_*) \times \hat{Q} \),

\[ \mu(E) = \int_{\pi(E) \cap \hat{F}} v \wedge \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |z - x_v|^2 \right)^l. \]
Then \( \mu \) is a totally finite measure, because by (1.2) (iii)
\[
\int_{B(x_s, 2r_s) - x_s} v \wedge \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |z - x_s|^2 \right)^t \leq \frac{1}{(\pi(2r_s)^2)^t} \int_{B(x_s, 2r_s)} v \wedge \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2 \right)^t < \infty.
\]
In terms of \( \mu \) the inequality (2.5) (*) becomes
\[
\mu(\overline{B(x_s, r)}) \geq c_0 r^{2(n-1)}
\]
which implies that the \( \mu \)-measure of the Euclidean open ball in \( C^{n+m} \) with center \( \overline{x}_s \) and radius \( r \) is \( \geq c_0 r^{2(n-1)} \) for \( \overline{x}_s \in (\{x_s\} \times Q) \cap \overline{F} \) and \( 0 < r < r_s \), where \( c_0 \) and \( r_s \) are positive numbers depending only on \( Q \). Now (2) follows from the lemma given below with \( U, Z, A, c, r_0, p \) substituted respectively by \( B(x_s, 2r_s) \times \{x_s\} \times Q, F, c_0, r_s, 2(n - l) \).

(2.8) Lemma. Suppose \( U \) is an open subset of \( C^n \), \( Z \) is a closed subset of \( U \), and \( A \) is a subset of \( U - Z \). Suppose \( \mu \) is a totally finite positive measure on \( U, c > 0, r_0 > 0, \) and \( p \geq 0 \) such that \( \mu(A \cap B(a, r)) \geq cr^p \) for \( a \in \overline{A} \cap Z \) and \( 0 < r < r_0 \). Then \( h^q(\overline{A} \cap Z) = 0 \).

Proof. Let \( K \) be an arbitrary compact subset of \( Z \). Let \( \varepsilon_0 \) be the distance of \( K \) from \( \partial U \). Take arbitrarily a positive number \( \varepsilon \) which is less than both \( \varepsilon_0 / 3n \) and \( c / n \). Let \( L \) be the set of all \( \{z_0, \ldots, z_n\} \) such that \( \text{Re } z_j \) and \( \text{Im } z_j \) are integral multiples of \( \varepsilon \) (\( 1 \leq j \leq n \)). We have
\[
\bigcup_{a \in L} B(a, n\varepsilon) = C^n
\]
and, for every \( z \in C^n \), the number of elements in
\[
\{a \in L \mid z \in B(a, 2n\varepsilon)\}
\]
is at most \((4n)^{2n}\). Let \( I \) be the set of all \( a \in L \), such that \( B(a, n\varepsilon) \) intersects \( \overline{A} \cap K \). Let \( G \) be the set of all points of \( U \) with distance from \( K < 3n\varepsilon \). Then
\[
(\ast) \quad \sum_{a \in I} \mu(A \cap B(a, 2n\varepsilon)) \leq (4n)^{2n} \mu(A \cap G).
\]
Since for \( a \in I \),
\[
B(a, n\varepsilon) \cap \overline{A} \cap K \neq \emptyset,
\]
there exists
\[
b_a \in \overline{A} \cap K \cap B(a, n\varepsilon).
\]
Since \( B(b_a, n\varepsilon) \subset B(a, 2n\varepsilon) \), it follows that
\[
(\dagger) \quad \sum_{a \in I} \mu(A \cap B(a, 2n\varepsilon)) \geq \sum_{a \in I} \mu(A \cap B(b_a, n\varepsilon)) \geq \sum_{a \in I} c(n\varepsilon)^p.
\]
We have by (\ast) and (\dagger)
\[ \sum_{\alpha \in I_1} (\text{diameter of } B(\alpha, n\varepsilon))^p \]

\[ = \sum_{\alpha \in I_1} (2n\varepsilon)^p \leq \frac{2^p}{p} (4n)^{2n} \mu(A \cap G_i). \]

Since \( \mu(A) < \infty \) and for any sequence \( \{\varepsilon_j\}_{j=1}^\infty \) of positive numbers strictly decreasing to 0 the set \( A \) contains the disjoint union of \( A \cap G_{i,j} = A \cap G_{i,j+1} \) \( (1 \leq j < \infty) \), it follows that

\[ \lim_{\varepsilon \to 0} \mu(A \cap G_i) = 0. \]

Consequently \( h^p(\bar{A} \cap K) = 0 \) and \( h^p(\bar{A} \cap Z) = 0 \). Q.E.D.

(2.9) Again, take \( x_* \in A \cup S - \bar{Y}_{I-i} \). We want to show that, for some open neighborhood \( D_{x_*} \) of \( x_* \) in \( B \), \( f | D_{x_*} - A - S \) can be extended to a meromorphic map from \( D_{x_*} \) to \( M \). Take \( \bar{x}_* \in ((x_*) \times M) \cap \bar{F} \). We have to show that, for some open neighborhood \( \bar{D} \) of \( \bar{x}_* \) in \( B \times M \), \( \bar{F} \cap \bar{D} \) is a subvariety in \( \bar{D} \). Write \( \bar{x}_* = (x_*, y_*) \). We take an open neighborhood \( \bar{Q} \) of \( y_* \) in \( M \) which is biholomorphic to an open neighborhood \( \bar{Q}^* \) of 0 in \( \mathbb{C}^m \) with \( y_* \) corresponding to 0 and we identify \( \bar{Q} \) with \( \bar{Q}^* \). For \( n - l \leq m \), since

\[ h^2((\{x_*\} \times \bar{Q}) \cap \bar{F}) = 0, \]

by [12, p. 114, Lemma 2] there exists a complex \( (l + m - n) \)-plane \( T \) in \( Q \) (when \( Q \) is given the coordinates of \( Q^* \subset \mathbb{C}^m \)) such that

\[ h^0((\{x_*\} \times (\bar{Q} \cap T - y_*)) \cap \bar{F}) = 0. \]

Let \( g_1, \ldots, g_m \) be a set of linear coordinate functions on \( Q \) vanishing at \( y_* \) such that

\[ T = \{g_1 = \cdots = g_{n-l} = 0\} \]

and, for some \( s_0 > 0 \),

\[ \{\|g_i\| \leq s_0, 1 \leq i \leq m\} \subset Q. \]

For \( n - l > m \), we define \( T \) as the origin and let \( g_i = 0 \) for \( m < i \leq n - l \).

Let \( E \) be the set of points of \( Q \) where

\[ (\sum_{i=n-l+1}^m |g_i|^2)^{1/2} = s_0. \]

Then

\[ (\{x_*\} \times (T \cap E)) \cap \bar{F} = \emptyset. \]

There exists \( 0 < r_0 < \text{Min} (1 - |x_*|, s_0) \) such that

\[ (B(x_*, r_0) \times (P \cap E)) \cap \bar{F} = \emptyset \]

where

\[ P = \{\|g_i\| < r_0, 1 \leq i \leq n - l\}. \]

Regard \( g_1, \ldots, g_m \) naturally as functions on \( B \times Q \). Let
\[ \bar{D} = (B(x_*, r_0) \times Q) \cap \{(\sum_{i=-l+1}^{n} g_i^2)^{1/2} < s_0, g_1 < r_0, \ldots, g_{n-l} < r_0 \} \]

and define

\[ \tilde{\tau}: \bar{D} \longrightarrow B(x_*, r_0) \times \Delta^{n-l}(r_0) \]

by

\[ z_1, \ldots, z_n, g_1, \ldots, g_{n-l}. \]

Let

\[ \tilde{\tau}: \bar{D} \cap \bar{F} \longrightarrow B(x_*, r_0) \times \Delta^{n-l}(r_0), \]

\[ \tau: \bar{D} \cap F \longrightarrow (B(x_*, r_0) - A - S) \times \Delta^{n-l}(r_0) \]

be induced by \( \tilde{\tau} \). By (\( * \)), \( \tilde{\tau} \) is proper. Hence \( \tau \) is also a proper map. Let \( Z = \tau(\bar{D} \cap F) \). \( Z \) is a subvariety of pure dimension \( n \) in \( (B(x_*, r_0) - A - S) \times \Delta^{n-l}(r_0) \). Let

\[ \Phi: (B(x_*, r_0) - A - S) \times \Delta^{n-l}(r_0) \longrightarrow B(x_*, r_0) - A - S \]

be the natural projection. Since \( \pi \) maps \( F \) biholomorphically onto \( B - A - S \), it follows that \( \Phi \) maps \( Z \) biholomorphically onto \( B(x_*, r_0) - A - S \). Let \( \text{Vol} \), be the volume calculated according to the Euclidean metrics of \( B(x_*, r_0) \) and \( \Delta^{n-l}(r_0) \). Let

\[ |g|^2 = \sum_{i=1}^{n-l} |g_i|^2. \]

For notational simplicity we denote \( |z|^2 \circ \tau \) also by \( |z|^2 \). Then

\[ \text{Vol}_{\epsilon}(Z) \leq \frac{1}{n!} \int_{\bar{D} \cap F} \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial}(|z|^2 + |g|^2) \right)^n \]

\[ \leq \frac{2^n}{n!} \int_{\bar{D} \cap F} \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial}(|z|^2 + |g|^2) \right)^{n-l} \wedge \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2 \right)^l \]

\[ \leq \frac{2^n C}{n!} \int_{\bar{D} \cap F} \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial}(|z|^2 + |\sigma^* \omega|^2) \right)^{n-l} \wedge \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2 \right)^l \]

\[ \leq \frac{2^n C}{n!} \int_{B(x_*, r_0)} u \wedge \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2 \right)^l < \infty \]

where

(i) the first inequality follows from \( Z = \tau(\bar{D} \cap F) \);

(ii) the second inequality is obtained by multiplying out both sides, according to the binomial theorem, and by observing that \( (\partial \bar{\partial} |g|^2)^r = 0 \) for \( \nu > n - l \);

(iii) the third inequality is obtained by expressing

\[ \partial \bar{\partial} |g|^2 = \sum_{i,j=1}^{n+m} \lambda_{ij} e_i \wedge \bar{e}_j \]

in terms of a unitary basis \( e_1, \ldots, e_{n+m} \) of \((1, 0)\)-forms with respect to the
Kähler metric of \( B(x_*, r_0) \times Q \) (induced by the Euclidean metric of \( C^n \) and the Kähler metric of \( M \)) and by multiplying out. The constant \( C \) there can be taken to be \((n + m)^{n + m - l} / \max \) times the maximum of the supremums on \( \bar{D} \cap F \) of all minors of order \( \leq n - l \) of the matrix \((\lambda_{ij})_{1 \leq i, j \leq n + m} \).

By Bishop’s theorem,
\[ Z^{-1} \cap (B(x_*, r_0) \times \Delta^{n-l}(r_0)) \]
is a subvariety of \( B(x_*, r_0) \times \Delta^{n-l}(r_0) \). We are going to show that \( \bar{F} \cap \bar{D} \) is a subvariety of \( \bar{D} \). We have
\[ \bar{F}(\bar{D} \cap \bar{F} \cap (A \cup S) \times Q) \subset \bar{Z} \cap ((A \cup S) \times \Delta^{n-l}(r_0)) \).

Take arbitrarily
\[ x' \in \bar{Z} \cap ((A \cup S) \times \Delta^{n-l}(r_0)) \).

We can find an open neighborhood \( W \) of \( x' \) in \( B(x_*, r_0) \times \Delta^{n-l}(r_0) \) and a holomorphic map \( h: W \to \Delta^n \) such that \( h \) makes \( \bar{Z} \cap W \) an analytic cover over \( \Delta^n \). Let \( L \) be the set of points of \( \bar{D} \cap F \cap \tau^{-1}(W) \) where \( h \circ \tau \) is not locally biholomorphic. Let
\[ E = (h \circ \tau)(L) \cup h(\bar{Z} \cap ((A \cup S) \times \Delta^{n-l}(r_0)) \cap W) \).

Then \( E \) is a closed subset of \( \Delta^n \) which is negligible (in the sense of [7, III. B. 2]). The map \( h \circ \tau \) makes \( \bar{D} \cap \bar{F} \cap \tau^{-1}(W) \) an analytic cover over \( \Delta^n \) with critical set \( E \). Since \( \bar{D} \cap F \cap \tau^{-1}(W) - (h \circ \tau)^{-1}(E) \) is a submanifold of \( \tau^{-1}(W - h^{-1}(E)) \), it follows from [7, III. B. 19] that \( \bar{D} \cap \bar{F} \cap \tau^{-1}(W) \) is a subvariety of \( \bar{D} \cap \tau^{-1}(W) \). Hence \( \bar{D} \cap \bar{F} \) is a subvariety of \( \bar{D} \). As a consequence, there exists some open neighborhood \( D_{x_0} \) of \( x_0 \) in \( B \) such that \( f|D_{x_0} - A - S \) can be extended to a meromorphic map from \( D_{x_0} \) to \( M \). Define
\[ Y_{l-1} = (A \cup S) - (\bigcup D_{x_0} | x_0 \in A \cup S - \bar{Y}_{l-1}) \).

Then \( Y_{l-1} \) is a closed subset of \( A \cup S \) which is contained in \( Y_{l-1} \). Hence \( Y_{l-1} \) is thin of order \( l - 1 \) in \( B \). Moreover, \( f \) can be extended to a meromorphic map \( f_{l-1} \) from \( B - Y_{l-1} \) to \( M \). Let \( F_{l-1} \) be the graph of \( f_{l-1} \). Then
\[ \gamma_{l-1} = \pi_* ([F_{l-1}] \wedge (\sigma^* \omega + \pi^* \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2)^{n-l}) \]
is a closed positive \((n - l, n - l)\)-current on \( B - Y_{l-1} \). Since by the last statement of (1.2)
\[ \chi_{(A \cup S) - Y_{l-1}} \gamma_{l-1} = 0 \]
it follows that, on \( B - Y_{l-1} \), \( \gamma_{l-1} \) is equal to the trivial extension of
\[ (f^* \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} |z|^2)^{n-l} \].
That means that the trivial extension of

\[ (f^* \omega + \frac{\sqrt{-1}}{2} \partial \overline{\partial} | z |^2)^{n-1} \]

is a closed positive current on \( B - Y_{l-1} \). Since \( u \) is a closed positive \((n - l, n - l)\)-current on \( B \) and \( u \) agrees with

\[ (f^* \omega + \frac{\sqrt{-1}}{2} \partial \overline{\partial} | z |^2)^{n-1} \]

on \( B - A - S \), it follows that \( u - \eta'_{l-1} \) is positive on \( B \) and, by (1.3) (†), the trivial extension of

\[ (f^* \omega + \frac{\sqrt{-1}}{2} \partial \overline{\partial} | z |^2)^{n-1} \]

is a closed positive current on \( B \). We have thus shown that \((*)\), implies \((*)_{l-1}\) for \( 0 \leq l \leq n - 1 \).

(2.10) Now the Main Theorem follows easily from \((*)_{l-1}\). Without loss of generality we can assume that \( X = B \).

Let \( F' \) be the graph of \( f \). Again, let

\[ \pi: B \times M \longrightarrow B , \]
\[ \sigma: B \times M \longrightarrow M \]

be the natural projections, and let \( \omega \) be the Kähler form of \( M \). Let \( S \) be the set of points of \( B - A \) where \( f \) is not a holomorphic map. \( S \) is a subvariety of codimension \( \geq 2 \) in \( B - A \) (see [11, p. 333]).

\[ \zeta: = \pi_* ([F] \wedge \sigma^* \omega) \]

is a closed positive \((1, 1)\)-current on \( (B - A) \cup G \) whose restriction to \( B - A - S \) agrees with

\[ (f | B - A - S)^* \omega \].

By (1.3) (*), \( \zeta \) can be extended to a closed positive \((1, 1)\)-current on \( B \). Hence the Main Theorem follows from \((*)_{l-1}\).

3. Generalizations of the Main Theorem

(3.1) First we discuss the generalization of the Main Theorem to holomorphic correspondences which generically map points to \( 0 \)-dimensional subvarieties. In order to avoid the task of defining holomorphic correspondences, we do our formulation in terms of the graph.

Suppose \( X \) is a complex manifold, \( A \) is a subvariety of codimension \( \geq 1 \) in \( X \), and \( G \) is an open subset of \( X \) which intersects every branch of \( A \) of
codimension 1. Suppose \( M \) is a compact Kähler manifold, \( Z \) is a subvariety of codimension \( \geq 1 \) in \((X - A) \cup G\), and \( F \) is a subvariety in \(((X - A) \cup G) \times M\) such that \( F \cap (Z \times M) \) is a subvariety of codimension \( \geq 1 \) in \( F \) and \( F - Z \times M \) is locally biholomorphic to \( X \) under the natural projection \( X \times M \to X \). Then \( \bar{F} \) is a subvariety of \( X \times M \).

For the proof it suffices to consider the case where \( X \) is the open unit ball \( B \) of \( \mathbb{C}^n \). Let \( S = Z \cap (B - A) \). Again let \( \omega \) be the Kähler form of \( M \) and

\[
\pi: B \times M \to B, \\
\sigma: B \times M \to M
\]

be the natural projections. By (1.3) (*), the closed positive (1, 1)-current

\[
\pi_*(\lbrack F \rbrack \wedge \sigma^* \omega)
\]

on \((B - A) \cup G\) can be extended to a closed positive (1, 1)-current on \( B \). Since \( F - Z \times M \) is locally biholomorphic to \( B \) under \( \pi \), it follows that

\[
\pi_*(\lbrack F \rbrack \wedge \sigma^* \omega)
\]

is \( C^\infty \) on \((B - A) \cup G - Z\) and, in particular, is \( C^\infty \) on \( B - A - S \). Now the proof of the Main Theorem can be very easily modified to give a proof of the analyticity of \( \bar{F} \cap (B \times M) \). The main thing is to replace the form

\[
\left( f^* \omega + \frac{V - 1}{2} \partial \bar{\partial} |z|^2 \right)^{n-1}
\]

in the proof of the Main Theorem by the form

\[
\lambda_1 := \pi_* \left( \lbrack F \cap \pi^{-1}(B - A - S) \rbrack \wedge \left( \sigma^* \omega + \pi^* \frac{V - 1}{2} \partial \bar{\partial} |z|^2 \right)^{n-1} \right)
\]

and observe that \( F \cap \pi^{-1}(B - A - S) \) is a finite-sheeted topological covering over \( B - A - S \) under \( \pi \). We are not going into every detail, because to do so would be boringly repetitious and anyone who has gone through the proof of the Main Theorem can easily make the obvious modifications.

Now we obtain the following generalization of the Main Theorem.

The Main Theorem remains true when \( X \) is assumed to be a normal complex space (instead of a complex manifold).

The reason is as follows. The problem is local in nature. We can assume that \( \tau: X \to B \) is an analytic cover such that, for some subvariety \( A' \) of codimension \( \geq 1 \) in \( A \), \( \tau \) maps \( A - A' \) biholomorphically onto \( \tau(A - A') \). The problem of extending \( f \) can be interpreted as extending the holomorphic correspondence from \((B - \tau(A)) \cup \tau(G \cap (A - A'))\) to \( M \) defined by \( f \) and \( \tau \).
And, to extend the holomorphic correspondence, we can use the result we have just proved.

(3.2) We consider another kind of generalization of the Main Theorem. The existence of $G$ means that the restriction of $f$ to $X - A$ can be extended across some point of every branch of $A$ of codimension 1. Since $(\ast)$, of (2.1) is much stronger than the Main Theorem, it is possible to replace the assumption of the existence of $G$ by a weaker assumption and still get the same conclusion. Our result is as follows.

$(\ast)$ Suppose $M$ is a compact Kähler manifold with Kähler form $\omega$ and $f$ is a meromorphic map from $\Delta^* \times (\Delta - 0)$ to $M$. Let $S$ be the subvariety of codimension $\geq 2$ in $\Delta^* \times (\Delta - 0)$ consisting of points where $f$ is not a holomorphic map. Suppose $Z$ is a subset of $\Delta^*$ which is not of Lebesgue measure zero in $\Delta^*$ such that, for $z \in Z$, $f \restriction (z) \times (\Delta - 0) - S$ can be extended to a holomorphic map from $(z) \times \Delta$ to $M$. Then $f$ can be extended to a meromorphic map from $\Delta^* \times \Delta$ to $M$.

This implies that for meromorphic maps into compact Kähler manifolds the definition of Remmert agrees with that of stoll (see [6, p. 30]).

In view of $(\ast)$, of (2.1) it suffices to prove that

$$\eta: = (f \restriction \Delta^* \times (\Delta - 0) - S)^* \omega$$

can be extended to a closed positive $(1, 1)$-current on $\Delta^* \times (\Delta - 0)$. Since, for $z \in Z$, $f \restriction (z) \times (\Delta - 0) - S$ can be extended to a holomorphic map from $(z) \times \Delta$ to $M$, it follows that, for $z \in Z$, the pullback of $\eta$ to $(z) \times (\Delta - 0) - S$ can be extended to a $C^\infty$ closed positive $(1, 1)$-form on $(z) \times \Delta$. The desired extendibility of $\eta$ is a consequence of the following statement which will be proved in Appendix 2 by using Hörmander’s $L^2$ estimates of $\bar{\partial}$ [9].

(†) Suppose $T$ is a subvariety of codimension $\geq 2$ in $\Delta^* \times (\Delta - 0)$ and $A$ is a subset of $\Delta^*$ which is not of Lebesgue measure zero in $\Delta^*$. If $u$ is a $C^\infty$ closed positive $(1, 1)$-form on $\Delta^* \times (\Delta - 0) - T$ such that, for $z \in A$, the pullback of $u$ to $(z) \times (\Delta - 0) - T$ can be extended to a closed positive $(1, 1)$-current on $(z) \times \Delta$, then $u$ can be extended to a closed positive $(1, 1)$-current on $\Delta^* \times \Delta$.

R. Harvey wrote me that he and J. Polking proved the following by a method not involving Hörmander’s $L^2$ estimates of $\bar{\partial}$ [9].

(#) If $\varphi$ is a plurisubharmonic function on $\Delta^* \times (\Delta - 0)$ and $E$ is a subset of $\Delta^*$ with positive harmonic capacity such that, for $z \in E$,

$$\sqrt{-1} \bar{\partial} \bar{\partial} (\varphi \restriction (z) \times (\Delta - 0))$$
can be extended to a closed positive \((1, 1)\)-current on \(\{z\} \times \Delta\), then \(\sqrt{-1} \partial \bar{\partial} \varphi\) can be extended to a closed positive \((1, 1)\)-current on \(\Delta^a \times \Delta\).

Since every closed positive \((1, 1)\)-current on \(\Delta^a \times (\Delta - 0)\) can be written as \(\sqrt{-1} \partial \bar{\partial} \varphi\) for some plurisubharmonic function \(\varphi\) on \(\Delta^a \times (\Delta - 0)\), it follows that \((\#)\) implies \((\dagger)\). By using \((\#)\), we can further generalize \((\ast)\) so that the set \(Z\) in \((\ast)\) is required only to have positive harmonic capacity instead of being not of Lebesgue measure zero. However, this still is not the conjectured sharpest result which should require only that \(Z\) is not thin of order \(n - 1\). So far this conjecture has not been proved.

(3.3) We conclude this section with some remarks about another kind of generalization of the Main Theorem whose investigation is still far from complete. In this kind of generalization we seek to relax the Kähler condition on the range manifold.

First we introduce the definition of a strictly positive \((l, l)\)-form. Suppose \(u\) is an \((l, l)\)-covector of \(C^*\). It is said that \(u\) is strictly positive if

\[
\left\langle u, \frac{2}{\sqrt{-1}} v_1 \wedge \bar{v}_1 \wedge \cdots \wedge \frac{2}{\sqrt{-1}} v_i \wedge \bar{v}_i \right\rangle
\]

is positive for every linearly independent set of vectors \(v_i, \ldots, v_i\), where \(\langle \cdot, \cdot \rangle\) is the natural pairing between \((l, l)\)-covectors and \((l, l)\)-vectors. An \((l, l)\)-form on an open subset of \(C^*\) is said to be strictly positive if it is strictly positive at every point as an \((l, l)\)-covector.

Suppose \(M\) is a compact complex manifold and \(\omega\) is a \(C^\infty\) closed strictly positive \((k + 1, k + 1)\)-form on \(M\). Suppose \(X\) is a connected \(n\)-dimensional complex manifold, \(A\) is a subvariety of codimension \(\geq 1\) in \(X\), and \(G\) is an open subset of \(X\) which intersects every branch of \(A\) of codimension 1. Suppose \(F\) is a subvariety of pure dimension \(n + k\) in \(((X - A) \cup G) \times M\) such that the map \(p: F \rightarrow (X - A) \cup G\) induced by the natural projection \(X \times M \rightarrow X\) is flat. Assume further that for some \(x_0 \in (X - A) \cup G\), \(p^{-1}(x_0) \subset \text{Reg}(F)\) and the map \(p\) has Jacobian rank \(n\) at every point of \(p^{-1}(x_0)\). The problem is to prove that \(F\) is a subvariety of \(X \times M\). An alternative formulation of the problem is in terms of the extendibility of certain meromorphic correspondences. One way to attack the problem is to use Douady's result [4].

Let \(D\) be the Douady space of all (not necessarily reduced) complex subspaces of \(M\). Let \(\bar{D}\) be the normalization of the reduction of \(D\). Since \(p\) is flat, \(F\) defines a holomorphic map \(f\) from \((X - A) \cup G\) to \(\bar{D}\). Let \(Y\) be the branch of \(\bar{D}\) containing the image of \(f\). We have a complex subspace \(V\) of \(Y \times M\) such that the map \(q: V \rightarrow Y\) induced by the natural projection
$Y \times M \to Y$ is flat and, for every $y \in Y$, $q^{-1}(y)$ is the complex subspace of $M$ corresponding to $y$. Consider first the special case where the following three conditions are satisfied.

(i) $V$ and $Y$ are both complex manifolds.

(ii) $Y$ is compact.

(iii) $q$ has Jacobian rank equal to $\dim Y$ everywhere.

Let $\tau: V \to M$ be the map induced by the natural projection $Y \times M \to M$. It is easy to verify that $q_*(\tau^*\omega)$ is a Kähler form on $Y$. By the Main Theorem $f$ can be extended to a meromorphic map from $X$ to $Y$ and hence $\bar{F}$ is a subvariety of $X \times M$. In general, of course, the three conditions are not satisfied. However, we can get modified forms of (i) and (iii). By the assumptions imposed on the point $x_0$, it can be shown that there exists a subvariety $Y_i$ of codimension $\geq 1$ in $Y$ with $\text{Im } f \subset Y_i$ such that both $Y - Y_i$ and $V - q^{-1}(Y_i)$ are regular and $q$ has Jacobian rank equal to $\dim Y$ at every point of $V - q^{-1}(Y_i)$. It should not be difficult to strengthen the Main Theorem so that the conditions (i) and (ii) can be replaced by the above modified forms, provided that $Y$ is compact. Unfortunately it is still not known which branch of the Douady space of all complex subspaces of a compact complex manifold is compact. Hironaka told me that he has an example of a compact complex manifold whose Douady space has a noncompact branch, but so far there is no example of a compact Kähler manifold whose Douady space has a noncompact branch. So, in our case, to increase the chances that $Y$ is compact, we may have to add the additional assumption that $M$ carries also a $C^\omega$ strictly positive $(k, k)$-form.

Another question is the following. Suppose $M$ is a compact complex manifold and $\omega$ is a closed strictly positive $(k, k)$-form on $M$. Suppose $X$ is a complex manifold, $A$ is a subvariety of codimension $\geq k + 1$ in $X$, and $f$ is a meromorphic map from $X - A$ to $M$. Can $f$ be extended to a meromorphic map from $X$ to $M$?

Consider the special case where $X = \mathbf{P}_s$ and $f$ is a local embedding. Let $G$ be the Grassmannian of all $(k - 1)$-dimensional planes in $\mathbf{P}_s$ and let $\tilde{A} \subset G$ be the set of $(k - 1)$-dimensional planes in $\mathbf{P}_s$ which intersect $A$. Then $\tilde{A}$ is a subvariety of codimension $\geq 2$ in $G$. Define a holomorphic correspondence $F$ from $G - \tilde{A}$ to $M$ which assigns to $T \in G - \tilde{A}$ the set $f(T)$. If $F$ can be extended to a holomorphic correspondence $\bar{F}$ from $G$ to $M$, then the meromorphic map from $\mathbf{P}_s$ to $M$ defined by

$$x \mapsto \bigcap \{\bar{F}(T) \mid x \in T \in G\}$$

extends $f$. The extendibility of $F$ is precisely the problem we discussed above.
4. Proof of Theorem 1

(4.1) For the proof of Theorem 1, we need the following generalization of the theorem of Bishop.

**Theorem 2.** Let $U$ be an open subset of $C^n$ and $\varphi$ be a plurisubharmonic function on $U$. Suppose the set $E$ of all points $x$ of $U$ with $\varphi(x) = -\infty$ is closed. Let $A$ be a subvariety of $U - E$ of pure dimension. If the volume of $A$ is finite, then the topological closure $A^- \cap U$ of $A$ in $U$ is a subvariety of $U$.

This generalization is obtained from the version of the proof of Bishop's theorem given in [16, (2.14)] by refining the result [10, p. 54, Proposition 1] that a plurisubharmonic function on a Stein domain is the upper envelope of functions of the form $a \log |f|$, where $a > 0$ and $f$ is holomorphic. Details will be given in Appendix 3.

We would like to remark that $E$ in general is not closed. For example, let $\{a_v\}$ be a countable dense subset of $\Delta - 0$, where $\Delta$ is the open unit 1-disc, and let $c_v$ be positive numbers such that

$$\sum_{v=1}^{\infty} c_v \log \frac{|a_v|}{2} > -\infty.$$  

Then the function

$$\varphi(z) = \sum_{v=1}^{\infty} c_v \log \frac{|z - a_v|}{2}$$

is a subharmonic function on $\Delta$, because it is a series of nonpositive subharmonic functions and it is not $-\infty$ at 0. The set of points of $\Delta$ where $\varphi$ is $-\infty$ is dense in $\Delta$ but does not contain 0. Therefore it is not closed.

Next, we would like to remark that Theorem 2 is not true if we only assume that $E$ is a closed subset of

$$Z: = \{x \in U \mid \varphi(x) = -\infty\}$$

instead of all of $Z$. A simple counter-example is the following.

$$U = C^2;$$

$$\varphi(z_1, z_2) = \log |z_1|;$$

$$E = \{z_1 = 0, |z_2| = 1\};$$

$$A = \{z_1 = 0, |z_2| < 1\}.$$  

We have the following corollary to Theorem 2.

**Corollary.** Suppose $U$ is an open subset of $C^n$ and $\{E_v\}$ is a countable set of subvarieties of $U$ such that $E_v = \bigcup_{v=1}^{\infty} E_v$ is closed in $U$. Let $A$ be a
subvariety of $U - E$ of pure dimension. If the volume of $A$ is finite, then $A^{-} \cap U$ is a subvariety of $U$.

The corollary is derived from Theorem 2 by constructing a plurisubharmonic function in the following way. Since the corollary is local in nature, we can assume without loss of generality that each $E_{v}$ is irreducible and is the set of common zeros of a finite number of holomorphic functions $g_{i}^{(v)} (1 \leq i \leq k_{v})$ on $U$ with

$$\sup_{x \in U} |g_{i}^{(v)}(x)| \leq 1.$$  

Let

$$\varphi_{v} = \log \left( \frac{1}{k_{v}} \sum_{i=1}^{k_{v}} |g_{i}^{(v)}|^2 \right)$$

and let $\{K_{v}\}$ be a sequence of compact subsets of $U - E$ such that $K_{v}$ is contained in the interior of $K_{v+1}$ and

$$U - E = \bigcup_{v=1}^{\infty} K_{v}.$$  

Choose $c_{v} > 0$ such that

$$\sup_{x \in K_{v}} c_{v} \varphi_{v}(x) < \frac{1}{2}.$$  

Let

$$\varphi = \sum_{v=1}^{\infty} c_{v} \varphi_{v}.$$  

Then $\varphi$ is a plurisubharmonic function on $U$ such that

$$E = \{x \in U \mid \varphi(x) = -\infty\}.$$  

In our proof of Theorem 1, it suffices to use the Corollary of Theorem 2 instead of Theorem 2 itself. A direct proof of the Corollary of Theorem 2 can be derived from Bishop's theorem itself. However, Theorem 2 is of interest by itself and therefore we prove it in Appendix 3 instead of simply proving its corollary.

(4.2) For the proof of Theorem 1, first we observe that, by using Theorem 2 instead of Bishop's theorem in the proof of $(\ast)_{-1}$ of (2.1) we obtain the following stronger statement.

$(\dagger)$ Suppose $\varphi$ is a plurisubharmonic function on the open unit ball $B$ of $C^{n}$ such that the $-\infty$ set $E$ of $\varphi$ is closed in $B$. Suppose $S$ is a subvariety of codimension $\geq 1$ in $B - E$ and $M$ is a compact Kähler manifold with Kähler form $\omega$. If $f$ is a holomorphic map from $B - E - S$ to $M$ and if the form $f^{*}\omega$ on $B - A - S$ can be extended to a closed positive $(1, 1)$-current on $B$, then $f$ can be extended to a meromorphic map from $B$ to $M$. 
To prove Theorem 1, we let $X$, $M$, $A$, $f$ be as in the statement of Theorem 1. Since the problem is local in nature, we can assume without loss of generality that $X = B$. Let $S$ be the set of points of $B - A$ where $f$ is not a holomorphic map. $S$ is a subvariety of codimension $\geq 2$ in $B - A$ (see [11, p. 333]). Since the Hausdorff $(2n - 3)$-measure of $S$ is zero and $A \cup S$ is a closed subset of $B$, by replacing $A$ by $A \cup S$, we can assume without loss of generality that $f$ is a holomorphic map at every point of $B - A$.

Let $\omega$ be the Kähler form of $M$. $f^* \omega$ is a $C^\infty$ closed positive $(1, 1)$-form on $B - A$. By Harvey's result quoted in (1.3) $f^* \omega$ can be trivially extended to a closed positive $(1, 1)$-current $\psi$ on $B$ (this can also be obtained by a simple argument using Stokes' theorem as in [16, (2.23)]).

Let $E$ be the set of all points $x$ of $B$ such that the Lelong number $n(\psi, x)$ of $\psi$ at $x$ is positive. By (1.2), $E$ is the union of a countable number of subvarieties of codimension $\geq 1$ in $B$.

Now Theorem 1 will follow from (†) if we can prove the following two statements:

(I) Every $x_0 \in A - E$ admits an open neighborhood $U$ in $B$ such that $f$ can be extended to a holomorphic map from $(B - A) \cup U$ to $M$.

(II) $E$ is closed in $B$.

For, when (II) is satisfied, according to the argument following the statement of the Corollary to Theorem 2, for every $x \in E$ we can find an open neighborhood $W$ of $x$ in $B$ and construct a plurisubharmonic function on $W$ whose $-\infty$ set is $E \cap W$.

(4.3) Let $F \subset (B - A) \times M$ be the graph of $f$. Before we prove (I), we prove first the following intermediate result.

(•) For every $x_0 \in A - E$ the Hausdorff 2-measure of $([x_0] \times M) \cap \bar{F}$ is zero.

Fix $x_0 \in A - E$. It suffices to show that every point $y$ of $M$ admits an open neighborhood $Q$ in $M$ such that

$$ h^2(([x_0] \times Q) \cap \bar{F}) = 0. $$

This can be obtained by modifying slightly the proof of (2.7) (‡) for the case $l = n - 1$. The only modification needed is the following. In the proof of (2.7) (•), the fact that the $A$ there is a subvariety is used when we invoke the related result of Bishop's theorem quoted in (2.6) to get

(†)

$$ \text{Vol} (\pi^{-1}(y) \cap \bar{B}(\vec{z}_0, r_0) \cap F) \geq c_0(r_0)^{2(n-1)}. $$

Now the $A$ here is not a subvariety in general. However, since the Hausdorff
(2n − 3)-measure of A is zero, for a fixed $x_0$, almost all complex lines $g$ in $\mathbb{C}^n$ containing $x_0$ do not intersect $A$ [12, p. 114, Lemma 2] and hence, for these $g$, (†) holds because of (1.2) (*).

(4.4) To prove (I), fix arbitrarily a point $x_0$ of $A − E$. Since $h^2_{2n−3}(A) = 0$, by [12, p. 114, Lemma 2] there exists a complex line $T$ containing $x_0$ such that $T$ is disjoint from $A − x_0$. Without loss of generality we can assume that $x_0 = 0$ and

$$T = \{z_1 = \cdots = z_{n−1} = 0\}.$$

Since by (4.3) (*), $h^i(\{[x_0] \times M\} \cap \bar{F}) = 0$ and since

$$h^i((T − x_0) \times M) \cap \bar{F}) = h^i((T − x_0) \times M) \cap F) = 0,$$

we have

$$h^i((T \times M) \cap \bar{F}) = 0.$$

Fix arbitrarily $\bar{x}_0 \in ([x_0] \times M) \cap \bar{F}$. We want to prove that $\bar{F}$ is a subvariety at $\bar{x}_0$. Let $\bar{x}_0 = (x_0, y)$ with $y \in M$. Some open neighborhood $Q$ of $y$ in $M$ is biholomorphic to an open subset $Q^\ast$ of $\mathbb{C}^n$. We identify $Q$ with $Q^\ast$. Without loss of generality we can assume that $\bar{x}_0 = 0$. Since $h^i((T \times Q) \cap \bar{F}) = 0$, by [12, p. 114, Lemma 2] there exists a linear function $g$ on $B \times Q$ such that $g, z_1, \cdots, z_{n−1}$, are linearly independent and

(∗) $$h^i((T \times Q) \cap \bar{F} \cap \{g = 0\}) = 0.$$

Choose linear functions $w_1, \cdots, w_m$ on $B \times Q$ such that $w_1 = g$ and $z_1, \cdots, z_n, w_1, \cdots, w_m$ form a coordinate system of $B \times Q$. It follows from (∗) that we can find $0 < r < 1/2$ such that

$$\{\sum_{i=1}^{n} |z_i|^2 + \sum_{j=1}^{m} |w_j|^2 < r^2\}$$

is relatively compact in $B \times Q$ and

$$(T \times Q) \cap \bar{F} \cap \{g = 0\}$$

is disjoint from

$$\{\sum_{i=1}^{n} |z_i|^2 + \sum_{j=1}^{m} |w_j|^2 = r^2\}.$$

In other words, $\bar{F}$ is disjoint from

$$\{z_1 = \cdots = z_{n−1} = w_1 = 0, |z_n|^2 + \sum_{j=2}^{m} |w_j|^2 = r^2\}.$$

Hence there exists $0 < s < 1/2$ such that

$$\{\sum_{i=1}^{n−1} |z_i|^2 + |w_1|^2 < s^2, |z_n|^2 + \sum_{j=2}^{m} |w_j|^2 < r^2\}$$

is relatively compact in $B \times Q$ and $\bar{F}$ is disjoint from

$$\{\sum_{i=1}^{n−1} |z_i|^2 + |w_1|^2 < s^2, |z_n|^2 + \sum_{j=2}^{m} |w_j|^2 = r^2\}.$$
By [7, p. 167, V.D. 3] we can choose an entire function \( h \) on \( C^n \times C^n \) which is so close to \( w \), on an open neighborhood of \((B \times Q)^-\) that

(i) \( z_1, \ldots, z_n, h, w_2, \ldots, w_n \) form a coordinate system of \( B \times Q \);

(ii) \[ \Omega = \left\{ \sum_{i=1}^{n-1} |z_i|^2 + |h|^2 < s^2, \ |z_n|^2 + \sum_{j=2}^{n} |w_j|^2 < r^2 \right\} \]

is relatively compact in \( B \times Q \);

(iii) \( \vec{F} \) is disjoint from

\[ \left\{ \sum_{i=1}^{n-1} |z_i|^2 + |h|^2 \leq s^2, \ |z_n|^2 + \sum_{j=2}^{n} |w_j|^2 = r^2 \right\} \]

(iv) the map \((B \times Q) \cap F \rightarrow C^n\) defined by \( z_1, \ldots, z_{n-1}, h \) has only 0-dimensional fibers.

Let

\[ \tau: \Omega \cap \vec{F} \rightarrow B(s) \]

be defined by \( z_1, \ldots, z_{n-1}, h \). Then \( \tau \) is proper. Let

\[ Z = \tau(\Omega \cap (A \times Q) \cap \vec{F}) \]

Then \( \tau \) makes \( \tau^{-1}(B(s) - Z) \) an analytic cover over \( B(s) - Z \). We want to prove that

\[ \tau^{-1}(B(s) - Z)^- \cap \Omega \]

is an analytic cover over \( B(s) \). By the usual method of considering elementary symmetric polynomials (cf. [16, (2.8)]), it suffices to show that every holomorphic function on \( B(s) - Z \) which is locally bounded on \( B(s) \) can be extended to a holomorphic function on \( B(s) \). For this, we need only prove that \( h^{2n-1}(Z) = 0 \) (see [12, Lemma 3] or [16, (2.16)]). Consider the following commutative diagram

\[ \begin{array}{ccc} B(s) \times \Delta(r) & \overset{p}{\longrightarrow} & B \\
\alpha \downarrow & & \downarrow \\
\Omega & \overset{\beta}{\longrightarrow} & B(s) \\
\downarrow \vec{\tau} & & \\
B(s) & & 
\end{array} \]

where

(i) \( \alpha \) is defined by \( z_1, \ldots, z_{n-1}, h, z_n \);

(ii) \( \beta \) is the natural projection;

(iii) \( \vec{\tau} \) is defined by \( z_1, \ldots, z_{n-1}, h \);

(iv) \( p \) is the projection onto the first \( n - 1 \) coordinates of \( B(s) \) and the coordinate of \( \Delta(r) \).

We have
\[ Z \subset \tilde{\tau}(\Omega \cap (A \times Q)) = \beta(\alpha(\Omega \cap (A \times Q))) \subset \beta(p^{-1}(A)) . \]

Since \( h^{x_n-3}(A) = 0 \), it follows that \( h^{x_n-1}(p^{-1}(A)) = 0 \) (cf. [16, (2.8)]) and \( h^{x_n-1}(\beta(p^{-1}(A))) = 0 \). Hence \( h^{x_n-1}(Z) = 0 \). To finish the proof that \( \tilde{F} \) is a subvariety at \( \tilde{x}_0 \), it remains to show that
\[ \tau^{-1}(B(s) - Z)^- \cap \Omega = F^- \cap \Omega . \]

Clearly
\[ \tau^{-1}(B(s) - Z)^- \cap \Omega \subset F^- \cap \Omega . \]

Suppose \( \tau^{-1}(B(s) - Z)^- \cap \Omega \) is different from \( F^- \cap \Omega \). Then
\[ D: = F \cap \Omega - \tau^{-1}(B(s) - Z)^- \cap \Omega \]
is a nonempty open subset of \( F \cap \Omega \). Since \( \tau \mid F \cap \Omega \) has only 0-dimensional fibers, it follows that \( \tau(D) \) is an open subset of \( B(s) \), contradicting \( \tau(D) \subset Z \).

So \( \tilde{F} \) is a subvariety at \( \tilde{x}_0 \). Since \( \tilde{x}_0 \) is an arbitrary point of \( \{ x_0 \} \times M \), there exists an open neighborhood \( U \) of \( x_0 \) in \( B \) such that \( \tilde{F} \cap (U \times M) \) is a subvariety of \( U \times M \). Hence \( f \mid U - A \) can be extended to a meromorphic map \( \tilde{f} \) from \( U \) to \( M \). To finish the proof of (I), it remains to show that \( \tilde{f} \) is a holomorphic map at \( x_0 \). Let \( S \) be the set of points of \( U \) where \( \tilde{f} \) is not a holomorphic map. Then by [11, p. 333] \( S \) is a subvariety of \( U \) of codimension \( \geq 2 \). We claim that \( S = E \cap U \). In particular, this claim would imply that \( \tilde{f} \) is a holomorphic map at \( x_0 \) and would finish the proof of (I).

Take \( x \in U - E \). We want to show that \( x \in S \). We can assume without loss of generality that \( x \in A \). Since \( x \in E \), it follows from (4.3) (*) that \( h^x((x) \times M) \cap \tilde{F}) = 0 \). So \( \dim((x) \times M) \cap \tilde{F}) = 0 \). Take \( \tilde{x} \in ((x) \times M) \cap \tilde{F} \). Then there exists a connected open neighborhood \( W \times Q \) of \( \tilde{x} \) in \( U \times M \) with \( W \subset U \) and \( Q \subset M \) such that \( \pi \) makes \( (W \times Q) \cap \tilde{F} \) an analytic cover over \( W \). Since \( ((W - A) \times Q) \cap F \) is single-sheeted over \( W - A \) under \( \pi \) and since \( W \) is connected, it follows that \( (W \times Q) \cap \tilde{F} \) is single-sheeted over \( W \) under \( \pi \) and \( (W \times Q) \cap \tilde{F} \) is the graph of a holomorphic map from \( W \) to \( Q \). Hence \( \tilde{f} \) is a holomorphic map at \( x \) and \( x \in S \).

Now take \( x \in U - S \). Then \( (\tilde{f} \mid U - S)^* \omega \) is a \( C^\infty \) closed positive (1, 1)-form on \( U - S \) and it agrees with \( f^* \omega \) on \( U - S - A \). Since \( \psi \) is the trivial extension of \( f^* \omega \), it follows that \( (\tilde{f} \mid U - S)^* \omega \) agrees with \( \psi \) on \( U - S \). Hence \( \psi \) is \( C^\infty \) on \( U - S \). As a consequence, \( n(\psi, x) = 0 \) and \( x \in E \). We have thus proved \( S = E \cap U \) and (I) is proved.

(II) follows very easily. Let \( x_0 \) vary in \( A - E \) and let \( H \) be the union of all the open neighborhoods \( U \). Then \( H \) is an open subset of \( B \) and \( f \mid H - A \) can be extended to a meromorphic map \( f^* \) from \( H \) to \( M \). We have just proved that \( E \cap H \) is precisely the set of points where \( f^* \) is not holomor-
phic. Hence $E \cap H$ is a closed subset of $H$. Since $E = (E \cap H \cap A) \cup (A - H)$, it follows that $E$ is closed in $A$ and is closed.

**Remark.** By using the method of (3.1), one can easily show that *Theorem 1 remains true when $X$ is assumed to be a normal complex space (instead of a complex manifold).*

**Appendix 1. The example of B. Shiffman and B. A. Taylor**

Let $n > 1$ and $Z = \{z_2 = \cdots = z_n = 0\}$. The example is a plurisubharmonic function $u$ on the open unit ball $B$ of $\mathbb{C}^n$ which is $C^\infty$ on $B - Z$ such that $\int_{B(r) - Z} (\sqrt{-1} \partial \bar{\partial} u)^n$ is infinite for $0 < r < 1$.

Before we give the example of Shiffman and Taylor, we first evaluate a simple integral. Take a positive integer $k$ and a positive number $A$. Let

$$f_{k, A}(z) = \left| \frac{z_1^k}{A} \right|^2 + \sum_{i=2}^n |z_i|^2.$$

Define a coordinate system $w = (w_1, \cdots, w_n)$ by

$$\begin{cases}
  w_1 = \frac{z_1^k}{A} \\
  w_i = z_i \quad (2 \leq i \leq n).
\end{cases}$$

Then $f_{k, A}(z) = |w|^2$. For $r > 0$,

$$1 = \frac{1}{(\pi r^2)^n} \int_{|w| < r} \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} |w|^2 \right)^n = \frac{1}{(\pi r^2)^n} \int_{|w| = r} \left( \frac{\sqrt{-1}}{2} \bar{\partial} |w|^2 \right)^n \wedge \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} |w|^2 \right)^{n-1}$$

$$= \int_{|w| = r} \frac{\sqrt{-1}}{2\pi} \bar{\partial} \log |w|^2 \wedge \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |w|^2 \right)^{n-1} = \frac{1}{k} \int_{f_{k, A}(z) = r^2} \frac{\sqrt{-1}}{2\pi} \bar{\partial} \log f_{k, A} \wedge \left( \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log f_{k, A} \right)^{n-1}$$

where

(i) the first equality results from the volume of $B(r)$ being $(\pi r^2)^n/n!$;

(ii) the second equality is by Stokes’ theorem;

(iii) the third equality comes from directly expanding the integrand involving $\log |w|^2$;

(iv) the fourth equality is a consequence of the fact that the map $z \mapsto w(z)$ is $k$ to 1 outside $\{z_1 = 0\}$;

(v) the fifth equality holds because the last integrand, being the pull-
back of a closed form, is closed on $C^s = 0$ and because \( \{ f_{k, A}(z) = r^2 \} \) is homologous to \( \{ |z| = r \} \) in $C^s = 0$.

Now we are ready to give the example. For $\nu \geq 1$ let
\[
D_\nu = B \left( 1 - \frac{1}{\nu} \right) \setminus \left\{ \sum_{i=1}^n |z_i|^2 > \frac{1}{\nu^2} \right\}.
\]
The key point in their example is the following observation. For $\mu, \nu \geq 1$ there exists a positive number $C_{\mu, \nu}$ satisfying the following. For $k \geq 1$ there exists a positive number $A(k)$ such that every partial derivative of $\log f_{k, A(k)}$ up to order $\mu$ is bounded by $C_{\mu, \nu}$ on $D_\nu$. This is clear, because, when one differentiates directly, $f_{k, A(k)}$ which occurs in the denominator is not smaller than $1/\nu^2$ and because, whenever $k$ occurs not as an exponent, it is in a term containing $1/A(k)$ as a factor.

Fix $0 < r < 1$. For $\nu \geq 2$, let $k_\nu = (2^C C_{\nu, \nu})^\nu$. Since $\log f_{k_{\nu, A(k_{\nu})}}$ is a plurisubharmonic function on $C^s$ which is $C^\infty$ on $C^s = 0$, it follows that, smoothing by convolution, we can obtain a $C^\infty$ plurisubharmonic function $g_\nu$ on $C^s$ such that every partial derivative of $g_\nu$ up to order $\nu$ is bounded by $2C_{\nu, \nu}$ on $D_{\nu-1}$ and
\[
\int_{|z| = r} \frac{(\nabla^2 - 1)}{2\pi} \partial \bar{\partial} g_\nu \wedge \left( \frac{(\nabla^2 - 1)}{2\pi} \partial \bar{\partial} g_\nu \right)^{n-1} \geq \frac{1}{2} k_\nu.
\]
By applying Stokes’ theorem to the last integral, we obtain
\[
\int_{|z| < r} \left( \frac{(\nabla^2 - 1)}{2\pi} \partial \bar{\partial} g_\nu \right)^n \geq \frac{1}{2} k_\nu.
\]
Define
\[
\sum_{\nu=2}^\infty \frac{1}{2^C C_{\nu, \nu}} g_\nu.
\]
Then $u$ is clearly $C^\infty$ on $B - Z$. On $B$, $u$ is plurisubharmonic, because, on $B(1 - 1/\nu_0)$, $g_\nu \leq 2C_{\nu, \nu}$ for $\nu \geq \nu_1$ (for some $\nu_1$ depending on $\nu_0$) by the maximum principle on $\{z_i = \text{const.}\}$ and
\[
\sum_{\nu=1}^\infty \frac{1}{2^C C_{\nu, \nu}} (g_\nu - 2C_{\nu, \nu})
\]
is the limit of a nonincreasing sequence of plurisubharmonic functions which, being finite on $D_{\nu_0}$, is not identically $-\infty$ on $B(1 - 1/\nu_0)$ for $\nu_0 > 1$. The integral
\[
\int_{B(1) - Z} (\nabla^2 - 1)^n \partial \bar{\partial} u)
\]
is infinite, because
\[
\int_{B(r) - Z} (\sqrt{-1} \partial \bar{\partial} u)^n \geq \sum_{s=1}^{\infty} \frac{1}{(2^s C_{s\nu})^n} \int_{B(r) - Z} (\sqrt{-1} \partial \bar{\partial} g_s)^n \\
\geq \sum_{s=1}^{\infty} \frac{(2\pi)^n}{(2^s C_{s\nu})^n} \frac{1}{2} k_s ,
\]
where for the first inequality we have used the fact that a product of \(C^\infty\) positive \((1, 1)\)-forms is positive.

**Appendix 2. Extension of closed positive \((1, 1)\)-currents**

We prove the following in this appendix.

Suppose \(S\) is a subvariety of codimension \(\geq 2\) in \(\Delta^n \times (\Delta - 0)\) and \(\omega\) is a \(C^\infty\) closed positive \((1, 1)\)-form on \(\Delta^n \times (\Delta - 0) - S\). If \(A\) is a subset of \(\Delta^n\) which is not of Lebesgue measure zero in \(\Delta^n\) such that, for \(z \in A\), the pull-back of \(\omega\) to \(\{z\} \times (\Delta - 0) - S\) can be extended to a closed positive \((1, 1)\)-current on \(\{z\} \times (\Delta - 0)\), then \(\omega\) can be extended to a closed positive \((1, 1)\)-current on \(\Delta \times (\Delta - 0)\).

We use \((z_1, \cdots, z_n, w)\) as the coordinates of \(C^{n+1}\) and, as usual, denote \((z_1, \cdots, z_n)\) by \(z\). Since \(\dim S \leq n - 1\), we can assume without loss of generality that \(A \times (\Delta - 0)\) is disjoint from \(S\). First we show that there exists a subset \(Z\) of \(A\) which is not of Lebesgue measure zero in \(\Delta^n\) and there exists a plurisubharmonic function \(\varphi\) on \(\Delta^n \times (\Delta - 0)\) such that \(\sqrt{-1} \partial \bar{\partial} \varphi = \omega\) on \(\Delta^n \times (\Delta - 0) - S\) and, for \(z \in Z\), \(\varphi \mid \{z\} \times (\Delta - 0)\) can be extended to a subharmonic function on \(\{z\} \times \Delta\). \(\omega\) can be extended to a closed positive \((1, 1)\)-current \(\eta\) on \(\Delta^n \times (\Delta - 0)\) by (1.3) (+) or Harvey’s result given in (1.3) (or in this case even by [13, p. 333, Main Lemma]).

Since
\[
\Delta^n \times (\Delta - 0) \text{ is Stein and} \\
H^\ell(\Delta^n \times (\Delta - 0), C) = 0 ,
\]
there exists a plurisubharmonic function \(\bar{\varphi}\) on \(\Delta^n \times (\Delta - 0)\) such that
\[
\eta = \sqrt{-1} \partial \bar{\partial} \bar{\varphi}
\]
on \(\Delta^n \times (\Delta - 0)\). Since for \(z \in A\) the pullback \(\eta_z\) of \(\eta\) to \(\{z\} \times (\Delta - 0)\) can be extended to a closed positive \((1, 1)\)-current on \(\{z\} \times (\Delta - 0)\), it follows that for \(z \in A\) there exists a subharmonic function \(\bar{\psi}_z\) on \(\{z\} \times \Delta\) such that
\[
\eta_z = \sqrt{-1} \partial \bar{\partial} \bar{\psi}_z
\]
on \(\{z\} \times \Delta\). Hence, for \(z \in A\), \(\bar{\varphi}(z, w) - \bar{\psi}_z(w)\) is a harmonic function of \(w\) for \(w \in \Delta - 0\). For \(z \in A\), there exist \(c_z \in \mathbb{R}\) and a holomorphic function \(g_z\) on \(\Delta - 0\) such that
for $w \in \Delta - 0$. By a consequence of Hörmander’s ([9]) $L^2$ estimates of $\overline{\partial}$ [15, (5.2) and (5.3)] (compare [2, p. 275, Existance Theorem] and [3]) there exists a holomorphic function $f(z, w) \neq 0$ on $\Delta_n \times (\Delta - 0)$ such that

$$
\int_{\Delta_n \times (\Delta - 0)} \exp \left( -\overline{\varphi}(z, w) \right) \left| f(z, w) \right|^2 \left( \prod_{i=1}^n \frac{\sqrt{-1}}{2} dz_i \wedge d\bar{z}_i \right) \wedge \frac{\sqrt{-1}}{2} dw \wedge d\bar{w} < \infty .
$$

Hence there exists a subset $A'$ of $A$ which is not a set of Lebesgue measure zero in $\Delta^*$, and $A'$ is such that, for $z \in A'$,

$$
\int_{\{z\} \times (\Delta - 0)} \exp \left( -\overline{\varphi}(z, w) \right) \left| f(z, w) \right|^2 \frac{\sqrt{-1}}{2} dw \wedge d\bar{w} < \infty
$$

and $f(z, w)$ as a function of $w$ is not identically zero. Let $m_z$ be an integer $\leq c_z/2$. Then

$$-2m_z \log |w| \leq -c_z \log |w| \quad (0 < |w| < 1) .$$

Hence, for $z \in A'$ and $0 < |w| < 1$,

$$-\psi_z(w) - 2m_z \log |w| - \Re g_z(w) \leq -\psi_z(w) - c_z \log |w| - \Re g_z(w) = -\overline{\varphi}(z, w) .$$

For $z \in A'$ let

$$h_z(w) = f(z, w)w^{-m_z} \exp \left( -\frac{1}{2} g_z(w) \right) .$$

Then

$$
\int_{\{z\} \times (\Delta - 0)} \exp \left( -\psi_z(w) \right) \left| h_z(w) \right|^2 \frac{\sqrt{-1}}{2} dw \wedge d\bar{w} = \int_{\{z\} \times (\Delta - 0)} \exp \left( -\psi_z(w) - 2m_z \log |w| - \Re g_z(w) \right) \cdot \left| f(z, w) \right|^2 \frac{\sqrt{-1}}{2} dw \wedge d\bar{w} \leq \int_{\{z\} \times (\Delta - 0)} \exp \left( -\overline{\varphi}(z, w) \right) \left| f(z, w) \right|^2 \frac{\sqrt{-1}}{2} dw \wedge d\bar{w} < \infty .
$$

Since $\psi_z$ is subharmonic on $\{z\} \times \Delta$ for $z \in A$, it follows that

$$\liminf_{w \to 0} \exp \left( -\psi_z(w) \right) \geq \exp \left( -\psi_z(0) \right) > 0$$

and, for $z \in A'$, $h_z(w)$ as a function of $w$ is locally square integrable at $w = 0$. 
We conclude that, for \( z \in A' \), \( h_z(w) \) as a function of \( w \) can be extended to a holomorphic function \( \tilde{h}_z(w) \) on \( \Delta \). Let \( k_z \) be the vanishing order of \( \tilde{h}_z(w) \) at \( w = 0 \). Let

\[
A_k = \{ z \in A' \mid k_z = k \}.
\]

Since \( A' \) is the union of all \( A_k \) \( (k \geq 0) \), there exists one \( A_k \) which is not a set of Lebesgue measure zero in \( \Delta^n \). Let \( V \) be the zero-set of \( f(z, w) \). By the definition of \( h_z \), for \( z \in A' \), the zero-set of \( h_z \) equals \((\{z\} \times (\Delta - 0)) \cap V \). For \( z \in A' \), the zero-set of \( h_z(w) \) is a subset of the zero-set of \( \tilde{h}_z \), which is a subvariety of \( \Delta \). By Stoll’s theorem [19, p. 169, Satz 2], \( \tilde{V} \cap (\Delta^n \times \Delta) \) is a subvariety of \( \Delta^n \times \Delta \). Let \( \tilde{f}(z, w) \) be a holomorphic function on \( \Delta^n \times \Delta \) whose zero-set is \( \tilde{V} \cap (\Delta^n \times \Delta) \). For \( z \in A_k \),

\[
\begin{align*}
\phi(z, w) &= 2 \log |f(z, w)| + 2k \log |w| \\
&= \psi_z(w) + c_z \log |w| + \Re g_z(w) - 2 \log |f(z, w)| + 2k \log |w| \\
&= \psi_z(w) + (c_z - 2m_z) \log |w| - 2 \log \left| \frac{h_z(w)}{w^k} \right|
\end{align*}
\]
is locally bounded from above at \( w = 0 \). For \( (z, w) \in \Delta^n \times (\Delta - 0) \), define

\[
\varphi(z, w) = \phi(z, w) - 2 \log \left| \frac{f(z, w)}{\tilde{f}(z, w)} \right| + 2k \log |w|.
\]

Then

\[
\sqrt{-1} \partial \bar{\partial} \varphi = \sqrt{-1} \partial \bar{\partial} \bar{\phi} = \eta
\]
on \( \Delta^n \times (\Delta - 0) \) and, for \( z \in A_k \),

\[
\varphi(z, w) = 2 \log |\tilde{f}(z, w)| + (\phi(z, w) - 2 \log |f(z, w)| + 2k \log |w|)
\]
is locally bounded from above at \( w = 0 \) (because the term in parentheses has just been proved to be locally bounded from above). This function \( \varphi \) satisfies all the conditions we imposed.

What remains is to show that \( \varphi \) can be extended to a plurisubharmonic function on \( \Delta^n \times \Delta \). Observe that, since \( \eta \) is \( C^\infty \) on \( \Delta^n \times (\Delta - 0) - S \), \( \varphi \) is also \( C^\infty \) on \( \Delta^n \times (\Delta - 0) - S \). For \( z \in \Delta^n \) and \( w \in \Delta - 0 \), define

\[
u(z, w) = \sup_{0 \leq \theta \leq \pi} \varphi(z, e^{i\theta} w).
\]

Fix \( 0 < r_0 < 1 \) and an arbitrary positive integer \( m \). For every positive integer \( k \), let \( C_k \) be the set of all \( z \in \Delta^n \) such that

\[
\int_{\{|z| \times (\Delta(r_0) - 0)\}} \exp \left(-m\nu(z, w) - 2 \log |w|\right) \frac{1}{2} dw \wedge \bar{d}w \leq k.
\]

Since \( \nu \) is upper semicontinuous on \( \Delta^n \times (\Delta - 0) \), it follows from Fatou’s lemma that each \( C_k \) is closed in \( \Delta^n \). Let \( C = \bigcup_{k=1}^{\infty} C_k \). We claim that \( C \) has
no interior. Suppose the contrary. Then $C$ contains some nonempty open subset $U$ of $\Delta^*$. By the Baire category theorem, some $C_k$ contains a nonempty open subset $W$ of $U$. It follows that

$$
\int_{W \times (\Delta(r_0) - 0)} \exp \left( -m u(z, w) \right) \frac{1}{w} z^2 \left( \prod_{i=1}^{n} \frac{\sqrt{-1}}{2} d z_i \wedge d \bar{z}_i \right) \wedge \frac{\sqrt{-1}}{2} d w \wedge d \bar{w}
$$

is finite. Take $z^0 \in W$. Since $u$ is plurisubharmonic on $\Delta^* \times (\Delta(r_0) - 0)$, from a consequence of Hörmander's ([9]) estimates of $\tilde{\alpha}$ (cf. [15, (5.2)]) it follows that there exists a holomorphic function $F$ on $\Delta^* \times (\Delta(r_0) - 0)$ which is square integrable on $\Delta^* \times (\Delta(r_0) - 0)$ with respect to the weight function $e^{-m u}$ and which agrees with $1/w$ on $\{z^0\} \times (\Delta(r_0) - 0)$. There exists a subset $L$ of $\Delta^*$ with Lebesgue measure zero in $\Delta^*$ such that, for $z \in \Delta^* - L$,

$$
\int_{\{z\} \times (\Delta(r_0) - 0)} \exp \left( -m u(z, w) \right) | F(z, w) |^2 \frac{\sqrt{-1}}{2} d w \wedge d \bar{w} < \infty .
$$

For $z \in A_k - L$, $u$ is bounded from above on $\{z\} \times (\Delta(r_0) - 0)$ and hence $F|\{z\} \times (\Delta(r_0) - 0)$ is square integrable with respect to the Lebesgue measure. It follows that, for $z \in A_k - L$, $F|\{z\} \times (\Delta(r_0) - 0)$ can be extended to a holomorphic function on $\{z\} \times \Delta(r_0)$. Since $A_k - L$ is not a set of Lebesgue measure zero in $\Delta^*$ and hence not contained in a subvariety of codimension $\geq 1$ in $\Delta^*$, by using the Laurent series expansion of $F$ in $w$, we conclude that $F$ can be extended to a holomorphic function on $\Delta^* \times \Delta(r_0)$, which contradicts the fact that $F$ agrees with $1/w$ on $\{z^0\} \times (\Delta(r_0) - 0)$. So we have proved that $C$ has no interior and $\Delta^* - C$ is dense in $\Delta^*$.

Since, for fixed $z$, $m u(z, w) + 2 \log |w|$ is a convex function of $\log |w|$, either $m u(z, w) + 2 \log |w|$ decreases to an element of $R \cup \{-\infty\}$ or increases to $\infty$ as $|w| \to 0$ (for $|w|$ sufficiently small). When $z \in \Delta^* - C$, by definition of $C$ the second case cannot occur. Hence, for $z \in \Delta^* - C$, $m \varphi(z, w) + 2 \log |w|$ as a function of $w$ can be extended to a subharmonic function on $\Delta$. It follows that

$$
(m \varphi(z, w) + 2 \log |w|) \leq m \sup_{|w|=1/2} \varphi(z, w) + 2 \log \frac{1}{2}
$$

for $z \in \Delta^* - C$ and $0 < |w| < 1/2$. Since $\Delta^* - C$ is dense in $\Delta^*$ and $\varphi$ is continuous on $\Delta^* \times (\Delta - 0) - S$, it follows that $(*)$ holds for

$$(z, w) \in \Delta^* \times \left( \Delta \left( \frac{1}{2} \right) - 0 \right) - S$$

and hence also for

$$(z, w) \in \Delta^* \times \left( \Delta \left( \frac{1}{2} \right) - 0 \right).$$
By dividing \((\ast)\) by \(m\) and letting \(m \to \infty\), we conclude that 
\[
\varphi(z, w) \leq \sup_{|w| = 1/2} \varphi(z, w)
\]
for
\[
(z, w) \in \Delta^a \times \left(\Delta \left(\frac{1}{2}\right) - 0\right).
\]
So \(\varphi\) can be extended to a plurisubharmonic function on \(\Delta^a \times \Delta\).

A consequence of the extension result we have just proved is the following. If \(\omega\) is a \(C^\infty\) closed positive \((1, 1)\)-form on \(\Delta^a \times (\Delta - 0)\) and \(||\omega||\) is a totally finite measure on \(\Delta^a \times (\Delta - 0)\), then the trivial extension of \(\omega\) is a closed positive \((1, 1)\)-current on \(\Delta^a \times \Delta\). The reason is the following. Let \(z_{n+1} = w\) and let
\[
w = \sum_{i, j = 1}^{n+1} \omega_{ij} \frac{\sqrt{-1}}{2} dz_i \wedge d\bar{z}_j.
\]
Then
\[
||\omega|| = (n - 1)! \left(\sum_{i=1}^{n+1} \omega_{ii}\right) \prod_{i=1}^{n+1} \frac{\sqrt{-1}}{2} \left|dz_i \wedge d\bar{z}_i\right|.
\]
It follows from the total finiteness of \(||\omega||\) that, for almost all
\[
z = (z_1, \ldots, z_n) \in \Delta^a,
\]
we have
\[
\int_{\{z\} \times (\Delta - 0)} \left(\sum_{i=1}^{n+1} \omega_{ii}\right) \frac{\sqrt{-1}}{2} dz_{n+1} \wedge d\bar{z}_{n+1} < \infty.
\]
Therefore, for almost all \(z \in \Delta^a\), the pullback of \(\omega\) to \(\{z\} \times (\Delta - 0)\), which equals
\[
\omega_{n+1, n+1} \frac{\sqrt{-1}}{2} dz_{n+1} \wedge d\bar{z}_{n+1},
\]
can be trivially extended to a current on \(\{z\} \times \Delta\) which is automatically a closed positive \((1, 1)\)-current. Hence \(\omega\) can be extended to a closed positive \((1, 1)\)-current \(\tilde{\omega}\) on \(\Delta^a \times \Delta\). By [15, (6.3)], \(\chi_{\Delta^a \times (\Delta - 0)} \tilde{\omega}\) is closed on \(\Delta^a \times \Delta\), but it is precisely the trivial extension of \(\omega\) to \(\Delta^a \times \Delta\).

By using the subgroup of elements of \(GL(n + 1, \mathbb{C})\) which leave every point of \(\Delta^a \times 0\) fixed, we can approximate weakly a closed positive \((1, 1)\)-current on \(\Delta^a \times (\Delta - 0)\) by \(C^\infty\) closed positive \((1, 1)\)-forms (cf. [15, (9.2)]) to obtain the following. If \(\omega\) is a closed positive \((1, 1)\)-current on \(\Delta^a \times (\Delta - 0)\) and \(||\omega||\) is a totally finite measure on \(\Delta^a \times (\Delta - 0)\), then the trivial extension of \(\omega\) is a closed positive \((1, 1)\)-current on \(\Delta^a \times \Delta\).
By using the projection techniques of [15, (13.7)-(13.9)], we can obtain from the previous statement the following.

Suppose \( \Omega \) is an open subset of \( \mathbb{C}^n \) and \( A \) is a subvariety of codimension \( \geq k \) in \( \Omega \). If \( \omega \) is a closed positive \((k, k)\)-current on \( \Omega - A \) such that, for every compact subset \( K \) of \( \Omega \), \( \| \omega \| (K - A) \) is finite, then the trivial extension of \( \omega \) is a closed positive current on \( \Omega \).

We are not providing the details here, because the situation is very much analogous to that of [15, (13.7)-(13.9)] so that anyone reading [15] can easily provide the details himself.

We conclude this appendix by giving a few conjectures related to the material presented here.

Conjecture 1. Suppose \( \Omega \) is an open subset of \( \mathbb{C}^n \) and \( A \) is a subvariety of codimension \( \geq 1 \) in \( \Omega \). If \( \omega \) is a closed positive \((k, k)\)-current on \( \Omega - A \) such that, for every compact subset \( K \) of \( \Omega \), \( \| \omega \| (K - A) \) is finite, then \( \omega \) can be extended to a closed positive \((k, k)\)-current on \( \Omega \).

This conjecture can be regarded as Bishop's theorem for closed positive currents and the statement we have just proved is the equi-dimensional case which is much shallower. In the conjecture we do not say that the trivial extension of \( \omega \) is a closed positive \((k, k)\)-current on \( \Omega \), because, if \( \tilde{\omega} \) is a closed positive extension of \( \omega \) to \( \Omega \), it is still not known whether \( \chi_{\delta} \tilde{\omega} \) is closed. If it is the case, then by smoothing as in [15, (9.2)] we can assume in addition that \( \omega \) is \( C^\infty \) on \( \Omega - A \).

The following conjecture is the closed positive current case of Bishop's lemma [1, p. 290, Lemma 3] which is related to Conjecture 1 in a way somewhat analogous to the subvariety case (cf. (2.6)).

Conjecture 2. Suppose \( A \) is a subvariety of \( B(r) \) not containing 0. If \( \omega \) is a closed positive \((k, k)\)-current on \( B(r) - A \), then,

\[
\| \omega \| (B(r) - A) \geq (\pi r^2)^{n-k}n(u, 0).
\]

Finally we give as a conjecture the sharper \((k, k)\)-current case of the result stated at the beginning of this appendix.

Conjecture 3. Suppose \( \omega \) is a closed positive \((k, k)\)-current on \( \Delta^* \times (\Delta^k - 0) \) and \( A \) is a subset of \( \Delta^* \) which is not of Lebesgue measure zero. If for every \( z \in A \) the slice \( \omega \mid \{z\} \times (\Delta^k - 0) \) (for definition see for example [15, (10.3)]) can be extended to a closed positive \((k, k)\)-current on \( \{z\} \times \Delta^k \), then the trivial extension of \( \omega \) is a closed positive \((k, k)\)-current on \( \Delta^* \times \Delta^k \).

Note that the result of Harvey and Polking, which Harvey told me,
implies this conjecture for \( k = 1 \). The case \( k > 1 \) cannot be obtained simply by projection, because the projection technique introduced in [15] makes use of smoothing first which would obliterate all the information concerning the extendibility of the slices. Some other projection technique without prior smoothing should be devised.

Appendix 3. A generalization of Bishop’s Theorem

We prove Theorem 2 in this appendix. Since the proof is analogous to the version of the proof of Bishop’s theorem given in [16, (2.14)] we will only present the modifications that need to be made in the proof of [16, (2.14)] in order to obtain Theorem 2.

Our first step is to prove [16, (2.11)] with \( P \) as a closed subset of \( U \) which is the \( -\infty \) set of a plurisubharmonic function \( \varphi \) on \( U \) (instead of being a subvariety). Everything there carries over directly except the equation \( \mu(\sigma \cap P \cap \bar{B}) = 0 \). To obtain it, it suffices to show that

\[
\varphi(0) \leq \int \varphi \, d\mu.
\]

Since

\[
\log |f(0)| \leq \int \log |f| \, d\mu
\]

for every holomorphic function \( f \) on \( U \), to prove (\#), by using Fatou’s lemma, it suffices to prove the following.

(\dagger) \( \varphi \) is the lim sup of a sequence of plurisubharmonic functions which are locally uniformly bounded from above and are of the form \( a \log |f| \), where \( a \) is a positive number and \( f \) is a holomorphic function on \( U \).

Let

\[
\Omega = \{(z, w) \in U \times \mathbb{C} \mid \log |w| + \varphi(z) < 0\}.
\]

Then \( \Omega \) is an open neighborhood of \( U \times 0 \) and \( \Omega \) is Stein. We can find a finite number of holomorphic functions \( g^{(1)}, \ldots, g^{(k)} \) on \( \Omega \) such that the map \( G: \Omega \to \mathbb{C}^k \) defined by these functions is proper. In particular, for every \( z \in U \), the restriction of \( G \) to \( (\{z\} \times \mathbb{C}) \cap \Omega \) is a proper map from \( (\{z\} \times \mathbb{C}) \cap \Omega \) to \( \mathbb{C}^k \). It follows that

(\#) if \( z \in U \) and \( w^{(\nu)} \in \mathbb{C} \) satisfy

\[
|w^{(\nu)}| < e^{-\varphi(z)}
\]

and

\[
\lim_{\nu \to \infty} |w^{(\nu)}| = e^{-\varphi(z)},
\]

then
\[
\lim_{\nu \to \infty} \sum_{i=1}^{k} | g^{(i)}(z, w^{(\nu)}) |^2 = \infty.
\]

Now for \(1 \leq i \leq k\) we let
\[
g^{(i)}(z, w) = \sum_{\nu=0}^{\infty} g^{(i)}_{\nu}(z) w^\nu
\]
be the power series expansion of \(g^{(i)}(z, w)\) in \(w\). For \(z \in U\),
\[
r_i(z) = \liminf_{\nu \to \infty} | g^{(i)}_{\nu}(z) |^{-1/\nu}
\]
is the radius of convergence of the power series expansion of \(g^{(i)}(z, w)\) in \(w\).
Because of \((\mathcal{Z})\), for \(z \in D\),
\[
e^{-\varphi(z)} = \min_{1 \leq i \leq k} r_i(z).
\]
This means that, for \(z \in D\),
\[
\varphi(z) = \max_{1 \leq i \leq k} \limsup_{\nu \to \infty} \frac{1}{\nu} \log | g^{(i)}_{\nu}(z) |.
\]
Define positive numbers \(a_{\nu}\) and holomorphic functions \(f_{\nu}\) \((0 \leq \nu < \infty)\) as follows.
\[
a_{\nu+1} = \frac{1}{\nu+1},
\]
\[
f_{\nu+1} = g^{(i)}_{\nu+1}.
\]
Then, for \(z \in D\),
\[
\varphi(z) = \limsup_{\nu \to \infty} a_{\nu} \log | f_{\nu}(z) |.
\]

We claim that the sequence of functions \(a_{\nu} \log | f_{\nu} |\) is locally uniformly bounded from above on \(D\). It suffices to show that, for each \(1 \leq i \leq k\), the sequence of functions \((1/\nu) \log | g^{(i)}_{\nu} |\) \((\nu \geq 1)\) is locally uniformly bounded from above on \(D\).

Take \(1 \leq i \leq k\). There exist a positive number \(\eta\) and a compact neighborhood \(K\) of \(z^0\) in \(D\) such that the series
\[
\sum_{\nu=0}^{\infty} g^{(i)}_{\nu}(z) w^\nu
\]
converges uniformly on \(K \times \{|w| \leq \eta\}\). Hence all the terms in the series are uniformly bounded on \(K \times \{|w| = \eta\}\) by some number \(M \geq 1\). It follows that, for \(\nu \geq 1\) and \(z \in K\),
\[
\frac{1}{\nu} \log | g^{(i)}_{\nu}(z) | \leq \frac{1}{\nu} \log M - \log \eta
\]
\[
\leq \log M - \log \eta.
\]

From this new form of \([16, (2.11)]\), we conclude as in \([16, (2.13)]\) holds with \(P\) a closed subset of \(U\) and the \(-\infty\) set of a plurisubharmonic function \(\varphi\) on \(U\).
Our next step is to modify the proof of [16, (2.14)]. Use the notations there, except that $E$ is a now a closed subset of $U$ and is the $-\infty$ set of a plurisubharmonic function $\varphi$ on $U$. ($\pi$ of [16, p. 62, line 6] should read $\bar{z}.$) Now we do not have $h$. Instead, we do the following. Let $Z$ be the critical set of the analytic cover $\pi^{-1}(G) \to G$. For $x \in G - Z$ define

$$\psi_0(x) = \sum_{y \in \pi^{-1}(x)} \varphi(y) .$$

$\psi_0$ can be extended to a plurisubharmonic function $\psi$ on $G$. Instead of proving that $f$ is continuous, we prove that $\psi(x) \to -\infty$ as $x$ approaches a point of $W \cap \partial G$. To do that, we have to replace inequalities involving $|f|$ by new inequalities obtained by substituting $\psi$ for $\log |f|$. We have also to substitute $\varphi$ for $\log |h|$. [16, (2.4)] and [16, (2.3)] are used and we have to modify them. The modification of [16, (2.3)] is easy. All that is needed is to observe that, if a holomorphic function defined on a domain minus the $-\infty$ set of a plurisubharmonic function is locally bounded on the domain, then it can be extended to the whole domain. For the modification of [16, (2.4)], we have to prove it with $E$ as a closed subset of $D \times G$ which is the $-\infty$ set of a plurisubharmonic function $\varphi$ on $D \times G$. As in the proof of [16, (2.4)], we assume that $\dim A \geq k$ and want to derive a contradiction. We can assume without loss of generality that $k = 1$, $D = \Delta$, $D' = \Delta(\varepsilon)$ for some $0 < \varepsilon < 1$ and $A$ is irreducible. We can also assume that $\varphi$ is strongly plurisubharmonic on $D \times G$ and is bounded from above on $D \times G$ by some positive number $M$. Take $x_0 \in A$. Let $w$ be the coordinate of $D$. Take $|w(x_0)| < \eta < 1$. There exists a positive integer $m_0$ such that, for $m \geq m_0$

$$(*) \quad \varphi(x_0) - m \log |w(x_0)| > M - m \log \eta .$$

Fix arbitrarily $m \geq m_0$. Let

$$\varphi_m(x) = \varphi(x) - m \log |w(x)|$$

and

$$A_m = \{x \in A \mid \varphi_m(x) \geq \varphi_m(x_0)\} .$$

Then $A_m$ is compact, because by $(*)$ every $x$ of $A_m$ satisfies $|w(x)| \leq \eta$ and hence it is the intersection of $A$ with the compact subset

$$\{x \in D \times K \mid \varepsilon \leq |w(x)| \leq \eta, \varphi_m(x) \geq \varphi_m(x_0)\}$$

of $D \times K - E$. Since $\varphi_m$ is upper semicontinuous on $(D - 0) \times G$, the supremum of $\varphi_m$ on $A_m$ is assumed by some point $x_m$ of $A_m$. It follows from the definition of $A_m$ that $\varphi_m(x_m)$ is also the supremum of $\varphi_m$ on $A$. Since $\varphi_m$ is strongly plurisubharmonic on $(D - 0) \times G$, this is a contradiction.

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References


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