Curvature Approach to Second Main Theorem of Nevanlinna

Yum-Tong Siu

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• One crucial ingredient in our discussion about the curvature approach to the second main theorem of Nevanlinna is the construction of a metric on the Riemann sphere \( \mathbb{P}_1 \) based on a given set of \( q \) distinct points \( a_1, \ldots, a_q \) of \( \mathbb{P}_1 \).

• The construction is motivated by the Poincaré metric \( ds^2_{D^*} \) of Gaussian curvature 1 on the puncture open unit disk \( D^* = \mathbb{D} - \{0\} \). We now discuss \( ds^2_{D^*} \).

• We have seen the use of the Poincaré metric

\[
ds^2_{D_R} = \frac{4R^2|dz|^2}{(R^2 - z\bar{z})^2}
\]

on \( D_R \) which is of constant curvature \(-1\).
• The Poincaré metric $ds^2_{\mathbb{D}^*}$ of the punctured open unit disk $\mathbb{D}^* = \mathbb{D} - \{0\}$ is induced from the Poincaré metric on its universal cover.

• The universal cover of $\mathbb{D}^*$ can be described by the map $w \mapsto z = e^w$ with the condition $0 < |z| < 1$ translated to $\text{Re} \ w = \log |z|$ belonging to the negative real axis $(-\infty, 0)$.

• In other words, the universal cover of $\mathbb{D}^*$ is the open left half plane with the exponential map as the covering map.

• We can do a rotation of $\frac{\pi}{2}$ radians in the $w$-space so that we can use the open upper half plane $\mathbb{H}$ as the universal cover of $\mathbb{D}^*$ with $w \mapsto z = e^{iw}$ as the covering map.
• We can use the biholomorphic map $w \mapsto \zeta = \frac{w-i}{w+i}$ from $\mathbb{H}$ onto $\mathbb{D}$ to pull back the Poincaré metric

$$ds^2_\mathbb{D} = \frac{4|d\zeta|^2}{(1-|\zeta|^2)^2}$$

of $\mathbb{D}$ (with coordinate $\zeta$)

• to get the Poincaré metric

$$ds^2_\mathbb{H} = \frac{4 \left| d \left( \frac{w-i}{w+i} \right) \right|^2}{\left( 1 - \left| \frac{w-i}{w+i} \right|^2 \right)^2}$$

on $\mathbb{H}$. 
Since
\[ \frac{d}{dw} \left( \frac{w - i}{w + i} \right) = \frac{2i}{(w + i)^2} \]

and
\[
|w + i|^2 - |w - i|^2 = |w|^2 - iw + i\bar{w} + 1 - (|w|^2 + iw - i\bar{w} + 1) = -2iw + 2i\bar{w} = \frac{2}{i}(w - \bar{w}) = 4 \text{ Im } w,
\]
it follows that
\[
ds_{\mathbb{H}}^2 = \frac{4 \left| d \left( \frac{w - i}{w + i} \right) \right|^2}{\left( 1 - \left| \frac{w - i}{w + i} \right|^2 \right)^2} = \frac{4 \left| \frac{2i}{(w + i)^2} \right|^2 |dw|^2}{\left( 1 - \left| \frac{w - i}{w + i} \right|^2 \right)^2} = \frac{16 |dw|^2}{\left( |w + i|^2 - |w - i|^2 \right)^2} = \frac{|dw|^2}{|\text{Im } w|^2}.
\]
• We now use the map \( w \mapsto z = e^{iw} \) to push forward the Poincaré metric of \( ds_{\mathbb{H}}^2 \) to get the Poincaré metric of \( \mathbb{D}^* \).

\[
ds_{\mathbb{D}^*}^2 = \frac{|d (-i \log z)|^2}{|\text{Im} (-i \log z)|^2} = \frac{|dz|^2}{|z|^2 (\log |z|)^2}.
\]

• The curvature of the metric \( ds_{\mathbb{D}^*}^2 \) is \(-1\).

• We will model the construction of a (possibly singular) metric of \( \mathbb{P}_1 \) on this particular Poincaré metric

\[
ds_{\mathbb{D}^*}^2 = \frac{|dz|^2}{|z|^2 (\log |z|)^2}
\]
by modifying the following computation which directly checks that its curvature is $-1$. We have

$$\partial \bar{z} \log \frac{1}{|z|^2 (\log |z|)^2} = -\partial \bar{z} (\log z + \log \bar{z} - 2 \log (\log |z|))$$

$$= -\frac{1}{z} - 2 \frac{\partial z \log |z|}{\log |z|} = -\frac{1}{z} - \frac{\partial z (\log z + \log \bar{z})}{\log |z|} = -\frac{1}{z} - \frac{1}{\bar{z} \log |z|}$$

and

$$\partial z \partial \bar{z} \log \frac{1}{|z|^2 (\log |z|)^2} = \partial z \left( -\frac{1}{\bar{z}} - \frac{1}{\bar{z} \log |z|} \right)$$

$$= \frac{\partial z \log |z|}{\bar{z} (\log |z|)^2} = \frac{\partial z (\log z + \log \bar{z})}{2 \bar{z} (\log |z|)^2} = \frac{1}{2 |z|^2 (\log |z|)^2}. $$
From

$$\Delta \log \frac{1}{|z|^2 (\log |z|)^2} = 4 \partial_z \partial_{\bar{z}} \log \frac{1}{|z|^2 (\log |z|)^2} = \frac{2}{|z|^2 (\log |z|)^2}$$

it follows that the curvature

$$- \frac{\Delta \log \frac{1}{|z|^2 (\log |z|)^2}}{2 \frac{1}{|z|^2 (\log |z|)^2}} \equiv -1.$$

In this computation the crucial step is the identity

$$\partial_z \partial_{\bar{z}} \log \frac{1}{|z|^2 (\log |z|)^2} = \frac{1}{2 |z|^2 (\log |z|)^2}.$$

The right-hand side is the (coefficient of the) metric of $\mathbb{D}^*$ (with the appropriate adjustment of a constant factor because of the use of $\partial_z \partial_{\bar{z}}$).
Construction of Metric From $q$ Distinct Points of $\mathbb{C}$

- We will actually use the left-hand side for the purpose of constructing a metric of $\mathbb{P}_1$.
- In order to model the construction of a (possibly singular) metric of $\mathbb{P}_1$ on the Poincaré metric of $\mathbb{D}^*$, we take $q$ distinct points $a_1, \ldots, a_q \in \mathbb{C}$.
- Let

$$s = \prod_{j=1}^{q} (w - a_j),$$

where $w$ is the inhomogeneous coordinate of the affine part $\mathbb{C}$ of $\mathbb{P}_1$.
- Define

$$\|s\| = \frac{|(w - a_1) \cdots (w - a_q)|}{C(1 + |w|^2)^{\frac{q}{2}}}$$

where $C$ is a positive constant chosen large enough such that $\|s\| < 1$ on all of $\mathbb{C}$. 
• The reason for putting \((1 + |w|^2)^{\frac{q}{2}}\) in the denominator of the definition of \(\|s\|\) is to make sure that \(\|s\|\) is continuous at the infinity point of \(\mathbb{P}_1\).

• We introduce the singular metric on \(\mathbb{P}_1\) whose associated \((1, 1)\)-form is

\[
\frac{i}{2\pi} \partial \bar{\partial} \log \frac{1}{\|s\|^2 (\log \|s\|^2)^2}.
\]

• This is motivated by the Poincaré metric of \(\mathbb{D}^*\) and the left-hand side of the equation

\[
\partial_z \partial_{\bar{z}} \log \frac{1}{|z|^2 (\log |z|)^2} = \frac{1}{2|z|^2 (\log |z|)^2}.
\]

• The reason for the factor \(\frac{i}{2\pi}\) is for the purpose of applying the differentiation version of Cauchy’s kernel to get the counting function for the points \(a_1, \cdots, a_q\).
The use $\log \|s\|^2$ in the denominator instead of $\log \|s\|$ simply adds a factor of 4 in the denominator and is introduced to avoid the use of the square root in differentiation.

As explained above in the discussion of the main idea of going from the Schwarz-Ahlfors-Pick lemma to the curvature argument of the second main theorem of Nevanlinna,

we are going to apply the divergence theorem to the pullback of

$$\frac{i}{2\pi} \partial \bar{\partial} \log \frac{1}{\|s\|^2(\log \|s\|^2)^2}$$

by a holomorphic map $f : \mathbb{C} \to \mathbb{P}_1$ on a disk of varying radius and

then integrate one more time with respect to the differential of the logarithm of the radial coordinate.

In order to expand out

$$\frac{i}{2\pi} \partial \bar{\partial} \log \frac{1}{\|s\|^2(\log \|s\|^2)^2},$$
we compute
\[\overline{\partial} \log \frac{1}{(\log \|s\|^2)^2} = -\frac{2}{\log \|s\|^2} \overline{\partial} \log \|s\|^2\]
and
\[\frac{i}{2\pi} \overline{\partial} \partial \log \frac{1}{(\log \|s\|^2)^2} = -\frac{2}{\log \|s\|^2} \frac{i}{2\pi} \overline{\partial} \partial \log \|s\|^2\]
\[+ \frac{i \partial \log \|s\|^2 \wedge \overline{\partial} \log \|s\|^2}{\pi (\log \|s\|^2)^2}\]
and
\[\frac{i}{2\pi} \overline{\partial} \partial \log \frac{1}{\|s\|^2 (\log \|s\|^2)^2} = -\left(1 + \frac{2}{\log \|s\|^2}\right) \frac{i}{2\pi} \overline{\partial} \partial \log \|s\|^2\]
\[+ \frac{i \partial \log \|s\|^2 \wedge \overline{\partial} \log \|s\|^2}{\pi (\log \|s\|^2)^2}\]
• Now when $w$ belongs to the affine part $\mathbb{C}$ of $\mathbb{P}_1$, 

$$\partial \log \|s\|^2 = \partial \log \frac{|(w - a_1) \cdots (w - a_q)|^2}{C(1 + |w|^2)^q}$$

$$= \left( \frac{1}{w - a_1} + \cdots + \frac{1}{w - a_q} + \frac{q\bar{w}}{1 + |w|^2} \right) \, dw.$$

• At the infinity point of $\mathbb{P}_1$ with local coordinate $\zeta = \frac{1}{w}$, from

$$\|s\|^2 = \frac{|(w - a_1) \cdots (w - a_q)|^2}{C(1 + |w|^2)^q} = \frac{|(1 - a_1\zeta) \cdots (1 - a_q\zeta)|^2}{C(1 + |\zeta|^2)^q}$$

it follows that

$$\partial \log \|s\|^2 = \left( \frac{a_1}{a_1\zeta - 1} + \cdots + \frac{a_q}{a_q\zeta - 1} + \frac{q\bar{\zeta}}{1 + |\zeta|^2} \right) \, d\zeta.$$

• Let $\omega_{FS}$ be the form associated to the Fubini-Study metric of $\mathbb{P}_1$. 

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From

\[ |\partial \log \|s\|^2|^2 = (1 + o(1)) \frac{|dw|^2}{|w - a_j|^2} \quad \text{as} \quad w \rightarrow a_j \quad \text{for} \quad 1 \leq j \leq a_j \]

we conclude that

\[ \sqrt{-1} \partial \log \|s\|^2 \wedge \partialbar \log \|s\|^2 + \hat{C} \omega_{FS} \geq \hat{c} \frac{\omega_{FS}}{\|s\|^2} \]

on \( \mathbb{P}_1 \) for some positive constants \( \hat{c} \) and \( \hat{C} \).

This is simply a comparison of the pole orders of \( \frac{1}{\|s\|^2} \) and \( |\partial \log \|s\|^2|^2 \) at any point of \( \mathbb{P}_1 \).

The use of the smooth positive definite \((1,1)\)-form \( \omega_{FS} \) on \( \mathbb{P}_1 \) is to enable us to write down, from such a pole-order comparison, an inequality which holds on all of \( \mathbb{P}_1 \).

Note that if we change the constant \( C \) in the definition of \( s \) at the beginning, the constant \( \hat{C} \) in the inequality is not affected but the constant \( \hat{c} \) will be changed by a positive factor.
From the differential version of Cauchy's integral formula and the curvature of the Fubini-Study metric of $\mathbb{P}_1$ being 1 we conclude that
\[
\frac{i}{2\pi} \partial \bar{\partial} \log \frac{1}{\|s\|^2} = q \frac{\omega_{FS}}{4\pi} - \delta_{a_1} - \cdots - \delta_{a_q}
\]
in the sense of distribution.

The reason for the $4\pi$ in the denominator on the right-hand side is from the following computation.

\[
\|s\|^2 = \frac{|(w - a_1) \cdots (w - a_q)|^2}{C(1 + |w|^2)^q}
\]

while
\[
\omega_{FS} = \frac{4du \wedge dv}{(1 + |w|^2)^2} = 2i \frac{dw \wedge d\bar{w}}{(1 + |w|^2)^2}
\]

and
\[
\frac{i}{1 + |w|^2} = -\frac{idw \wedge d\bar{w}}{(1 + |w|^2)^2}
\]
and
\[ \frac{i}{2\pi} \partial \bar{\partial} \log \left( \frac{1}{1 + |w|^2} \right)^q = -q \frac{i}{2\pi} \frac{dw \wedge d\bar{w}}{(1 + |w|^2)^2} = -q \frac{\omega_{FS}}{4\pi}. \]

Putting all the inequalities together, we get
\[ \frac{i}{2\pi} \partial \bar{\partial} \log \frac{1}{\|s\|^2(\log \|s\|^2)^2} \geq \left(1 + \frac{2}{\log \|s\|^2}\right) \left(q \frac{\omega_{FS}}{4\pi} - \delta_{a_1} - \cdots - \delta_{a_q}\right) \]
\[ - \frac{\hat{C}}{(\log \|s\|^2)^2} \omega_{FS} + \frac{\omega_{FS}}{\lambda^2\|s\|^2(\log \|s\|^2)^2} \]
on \mathbb{P}_1.

For any given \( \varepsilon > 0 \), in the definition of \( s \) we can choose \( C \) at the beginning so large that
\[ \left| \frac{2}{\log \|s\|^2} \right| \leq \varepsilon \quad \text{and} \quad \left| \frac{4\pi \hat{C}}{(\log \|s\|^2)^2} \right| \leq \varepsilon \quad \text{on} \quad \mathbb{P}_1. \]
Inequality Satisfied by Constructed Metric

- Note that $\|s\| < 1$ on $P_1$ so that $\frac{1}{\log \|s\|^2}$ is negative and approaches 0 as the constant $C$ in the definition of $\|s\|$ goes to infinity.
- The constant $\hat{c}$ depends on the choice of $C$ and we are going to add a subscript $\varepsilon$ to $\hat{c}$ to indicate that $\hat{c}$ depends on $\varepsilon$.
- We add to both sides

\[
\frac{i}{2\pi} \partial \bar{\partial} \log \omega_{FS} = -2 \left( \frac{\omega_{FS}}{2\pi} \right)
\]

to rewrite the inequality as

\[
\frac{i}{2\pi} \partial \bar{\partial} \log \frac{\omega_{FS}}{\|s\|^2(\log \|s\|^2)^2} \geq (q - 2 - (q + 1)\varepsilon) \frac{\omega_{FS}}{4\pi}
\]
\[
- (1 + \varepsilon) (\delta_{a1} + \cdots + \delta_{aq}) + \hat{c}\varepsilon \frac{\omega_{FS}}{\|s\|^2(\log \|s\|^2)^2}.
\]
• The reason for adding the identity to the inequality is to make sure that the function to which $\frac{i}{2\pi} \partial \bar{\partial} \log$ is applied matches the last term of the right-hand side of the inequality up to a positive constant.

• Note that the coefficient 2 on the right-hand side of the identity which is added to the final inequality comes from the fact that the Gaussian curvature of the Fubini-Study metric of $\mathbb{P}_1$ is 2.

• The $(1, 1)$-form $\omega_{\mathbb{P}_1, a_1, \ldots, a_q}$ for a singular metric constructed on $\mathbb{P}_1$ for a set of $q$ distinct points of $\mathbb{C}$ which is described before we discuss the implementation of the curvature-argument approach to the second main theorem of Nevanlinna theory is the $(1, 1)$-form

$$\omega_{FS} = \frac{\omega_{FS}}{\|s\|^2(\log \|s\|^2)^2}$$

on the Riemann sphere $\mathbb{P}_1$. 

Recall that the Ahlfors-Shimizu version of the Nevanlinna characteristic function $T(R, f)$ is just

$$
\mathcal{I}_R \left( f^* \left( \frac{\omega_{FS}}{4\pi} \right) \right),
$$

because

$$
\mathcal{I}_R \left( f^* \left( \frac{\omega_{FS}}{\pi} \right) \right) = \int_{r=0}^{R} \frac{dr}{r} \int_{|z|<r} f^* \left( \frac{1}{4\pi} \frac{2idw \wedge d\bar{w}}{(1 + |w|^2)^2} \right)
\quad = \frac{1}{4\pi} \int_{r=0}^{R} \frac{dr}{r} \int_{|z|<r} \frac{4|f'|^2}{(1 + |f|^2)^2}.
$$
Using Divergence Theorem to Pullback of Constructed Metric, Concavity of Logarithm and Calculus Lemma

- Pulling back by $f$ to $\mathbb{C}$ by $f$ and applying the divergence theorem and then integrating with respect to the differential of the logarithm of the radial coordinate,
- we obtain

$$\mathcal{A}_r \left( \log f^* \left( \frac{\omega_{FS}}{2\pi \|s\|^2 (\log \|s\|^2)^2} \right) \right) + O(1)$$

$$= \mathcal{I}_r \left( \log f^* \left( \frac{i}{2\pi} \partial \bar{\partial} \log \frac{\omega_{FS}}{\|s\|^2 (\log \|s\|^2)^2} \right) \right) + O(1)$$

$$\geq \mathcal{I}_r \left( f^* \left[ (q - 2 - (q + 1)\varepsilon) \frac{\omega_{FS}}{4\pi} - (1 + \varepsilon) (\delta_{a_1} + \cdots + \delta_{a_q}) ight.ight.$$

$$\left. + \hat{c}_\varepsilon \frac{\omega_{FS}}{\|s\|^2 (\log \|s\|^2)^2} \right) \right).$$
• By the concavity of the logarithmic function we have

\[ A_r \left( \log f^* \left( \frac{\omega_{FS}}{2\pi \| s \|^2 (\log \| s \|^2)^2} \right) \right) \leq \log A_r \left( f^* \left( \frac{\omega_{FS}}{2\pi \| s \|^2 (\log \| s \|^2)^2} \right) \right). \]

• By the Calculus Lemma

\[
\log A_r \left( f^* \left( \frac{\omega_{FS}}{2\pi \| s \|^2 (\log \| s \|^2)^2} \right) \right) \\
\leq \hat{\epsilon} \log r + (1 + \delta) \log I_r \left( f^* \left( \frac{\omega_{FS}}{2\pi \| s \|^2 (\log \| s \|^2)^2} \right) \right) \| \hat{\epsilon}, \delta. \\
\]

• It follows that

\[
\hat{\epsilon} \log r + (1 + \delta) \log I_r \left( f^* \left( \frac{\omega_{FS}}{2\pi \| s \|^2 (\log \| s \|^2)^2} \right) \right) \\
\geq I_r \left( f^* \left[ (q - 2 - (q + 1)\epsilon) \frac{\omega_{FS}}{4\pi} - (1 + \epsilon) (\delta a_1 + \cdots + \delta a_q) \\
+ \hat{\epsilon} \frac{\omega_{FS}}{\| s \|^2 (\log \| s \|^2)^2} \right] \right) \| \hat{\epsilon}, \delta \quad \text{for any } \hat{\epsilon} > 0 \text{ and } \delta > 0, \\
\]
and

\[ I_r \left( f^* \left( (q - 2 - (q + 1)\varepsilon) \frac{\omega_{FS}}{4\pi} \right) \right) \]
\[ \leq I_r \left( f^* \left( (1 + \varepsilon) (\delta_{a_1} + \cdots + \delta_{a_q}) \right) \right) + O(\log r) \ |. \]

In terms of the Nevanlinna characteristic function \( T(r, f) \) and the counting function \( N(r, f, a_j) \), this means

\[ ((1 - \varepsilon)q - 2) T(r, f) \leq (1 + \varepsilon) \sum_{j=1}^{q} N(r, f, a_j) + O(\log r) \ |. \]

for any \( \varepsilon > 0 \),

which is a slightly weaker form of the second main theorem of Nevanlinna (because of the use of any \( \varepsilon > 0 \)). Moreover, we have not used the truncated counting function.
Now, by taking limit inferior as $r \to \infty$, we obtain the defect relation
\[
\sum_j \delta(f, a_j) \leq 2
\]
by dividing the rewritten inequality
\[
\sum_{j=1}^{q} (T(r, f) - N(r, f, a_j)) \leq \frac{2\varepsilon + 2}{1 + \varepsilon} T(r, f) + O(\log r)
\]
by $T(r, f)$ and taking limit inferior as $r \to \infty$

• to get and
\[
\sum_j \delta(f, a_j) \leq \frac{2\varepsilon + 2}{1 + \varepsilon} \text{ for any } \varepsilon > 0.
\]