Isolated Singularities and Residue Theorem

We started out with the difference quotient of a complex-valued function of a complex variable to introduce the Cauchy-Riemann equations which are used in Stokes’s theorem in real dimension 2 to get the theorem of Cauchy-Goursat. Then we artificially introduced a singularity at \( a \) to form

\[
\frac{f(z)}{(z-a)^{k+1}}
\]

from a holomorphic function \( f(z) \) to derive Cauchy’s integral formula for the \( k \)-th order derivative of a holomorphic function, which can then be applied to evaluate certain class of definite integrals by differentiating functions at certain points to certain orders. Here we would like to discuss the theory of isolated singularities so that we can develop a more systematic theory of residues for the evaluation of a certain class of definite integrals. An isolated singularity of a holomorphic function \( f(z) \) at \( z = a \) means that \( f(z) \) is holomorphic on some deleted open disk neighborhood \( \{0 < |z - a| < R\} \) (with \( R > 0 \)) of \( a \) in \( \mathbb{C} \). The theory of isolated singularities at \( z = a \) seeks to study the growth behavior of \( f(z) \) as \( z \) approaches \( a \). The tool used in the study is the Laurent series expansion of \( f \) on the degenerate open annulus \( \{0 < |z - a| < R\} \) (with \( R > 0 \)) which states that

\[
f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n
\]

with \( c_n \in \mathbb{C} \) (for \( n \in \mathbb{Z} \)) and with absolute and uniform convergence on \( 0 < r_1 \leq |z - a| \leq r_2 < R \), where

\[
c_n = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)dz}{(z-a)^{n+1}}
\]

for any \( 0 < r < R \) and for all \( n \in \mathbb{Z} \). The part

\[
\sum_{n=-\infty}^{-1} c_n (z - a)^n
\]

of the Laurent series of \( f(z) \) at \( z = a \) is known as the principal part. Isolated singularities are classified into the following three kinds in the trichotomy of zero principal part, finite number (at least one) of nonzero terms in the principal part, or infinite number of nonzero terms in the principal part.
1. **Removable Singularity.** The case of $c_n = 0$ for $n < 0$ gives a *removable singularity* which means that the function defined by the power series

$$\sum_{n=0}^{\infty} c_n(z-a)^n$$

is a holomorphic function on $\{|z| < R\}$ which extends the given function $f(z)$. An alternative characterization of removable singularity is that $f(z)$ is bounded on some deleted neighborhood of $a$. The verification of the alternative characterization is as follows.

Since

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)dz}{(z-a)^{n+1}}$$

for any $0 < r < R$ and for all $n < 0$, if $f(z)$ is bounded on $0 < |z| \leq R_0$ for some $0 < R_0 < R$, then by letting $r \to 0$ in

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)dz}{(z-a)^{n+1}},$$

we can conclude that $c_n = 0$ for $n < 0$. On the other hand, if $c_n = 0$ for $n < 0$, the function $f(z)$ can be extended to a holomorphic function on $\{|z| < R\}$ and there exists some $0 < R_0 < R$ such that $f(z)$ is bounded on $0 < |z| \leq R_0$.

2. **Pole.** The singularity at $z = a$ is a *pole* if $c_n = 0$ for $n < -k$ and $c_{-k} \neq 0$ for some $k > 0$. It is alternatively characterized by $\lim_{z \to a} |f(z)| = \infty$. The number $k$ is called the *order* of the pole of $f(z)$ at $z = a$. The order $k$ is alternatively characterized by

$$\lim_{z \to a} |(z-a)^k f(z)|$$

being a positive number. The verification of the alternative characterization is as follows.

Suppose $c_n = 0$ for $n < -k$ and $c_{-k} \neq 0$ for some $k > 0$. Since $f(z)$ can be written

$$\frac{g(z)}{(z-a)^k},$$
where
\[ g(z) = \sum_{n=0}^{\infty} c_{n-k}(z-a)^n, \]
and \( g(a) = c_{-k} \neq 0 \), it follows that
\[ \lim_{z \to a} |f(z)| = \lim_{z \to a} \frac{|g(z)|}{|z-a|^k} = \infty \]
and
\[ \lim_{z \to a} ((z-a)^k f(z)) = |g(a)| = |c_{-k}| > 0. \]

On the other hand, suppose \( \lim_{z \to a} |f(z)| = \infty \). Then there exists some \( 0 < R_0 < R \) such that \( |f(z)| \geq 1 \) for \( 0 < |z| < R_0 \). Let
\[ h(z) = \frac{1}{f(z)} \]
on \( \{0 < |z| < R_0\} \). The holomorphic function \( h(z) \) is bounded by 1 in absolute value on \( \{0 < |z| < R_0\} \) and is therefore a removable singularity and can be expressed as a convergent power series
\[ h(z) = \sum_{n=0}^{\infty} d_n(z-a)^n \]
on \( \{0 < |z| < R_0\} \). Since \( \lim_{z \to a} |f(z)| = \infty \), it follows that \( \lim_{z \to a} |h(z)| = 0 \) and there exists some positive integer \( k \) such that \( d_k \neq 0 \) and \( d_n = 0 \) for \( 0 \leq n < k \). Let
\[ q(z) = \sum_{n=0}^{\infty} d_{n+k}(z-a)^n \]
such that \( h(z) = (z-a)^k q(z) \). Since the function \( q(z) \) is holomorphic on \( \{|z| < R_0\} \) with \( q(a) = d_k \neq 0 \), there exists some \( 0 < R_1 < R_0 \) such that \( q(z) \) is nowhere zero on \( \{|z| < R_1\} \). We can express the holomorphic function \( \frac{1}{q(z)} \) as a convergent power series
\[ \frac{1}{q(z)} = \sum_{n=0}^{\infty} e_n(z-a)^n \]
on \( \{|z| < R_1\} \) with \( e_0 \neq 0 \). From

\[ f(z) = \frac{1}{h(z)} = \frac{1}{(z-a)^k q(z)} = \frac{1}{(z-a)^k} \sum_{n=0}^{\infty} e_n(z-a)^n = \sum_{n=-k}^{\infty} e_{n+k}(z-a)^n \]

and the uniqueness of the coefficients of a Laurent series it follows that \( c_n = e_{n+k} \) which is 0 for \( n < -k \) and nonzero for \( n = -k \). It means that \( z = a \) is a pole of order \( k \) for \( f(z) \).

3. Essential Singularity. The isolated singularity \( z = a \) of \( f(z) \) is called an essential singularity if \( c_n \neq 0 \) for an infinite number of negative integers \( n \). An alternative characterization is that for any \( 0 < r < R \) the image of \( \{0 < |z-a| < r\} \) under the map \( z \mapsto f(z) \) is dense in \( \mathbb{C} \) (in the sense that any nonempty open subset of \( \mathbb{C} \) contains some point of the image). This alternative characterization of essential singularity is also known as the theorem of Casorati-Weierstrass. The verification of the alternative characterization of essential singularity is as follows.

If the isolated singularity of \( f(z) \) at \( z = a \) is a removable singularity so that \( f(a) \) can be defined to make \( f(z) \) holomorphic on \( \{|z| < R\} \), then there exists some \( 0 < r < R \) such that \(|f(z)| < |f(a)| + 1 \) for \( 0 < |z| < r \) and the image of \( \{0 < |z-a| < r\} \) under the map \( z \mapsto f(z) \) is not dense in \( \mathbb{C} \). If the isolated singularity of \( f(z) \) at \( z = a \) is a pole so that \( \lim_{z \to a} |f(z)| = \infty \), then there exists some \( 0 < r < R \) such that \(|f(z)| > 1 \) for \( 0 < |z| < r \) and the image of \( \{0 < |z| < r\} \) under the map \( z \mapsto f(z) \) is not dense in \( \mathbb{C} \). Thus, if for any \( 0 < r < R \) the image of \( \{0 < |z-a| < r\} \) under the map \( z \mapsto f(z) \) is dense in \( \mathbb{C} \), then the isolated singularity of \( f(z) \) at \( z = a \) must be an essential singularity.

Now assume that the isolated singularity of \( f(z) \) at \( z = a \) is an essential singularity. To prove that for any \( 0 < r < R \) the image of \( \{0 < |z-a| < r\} \) under the map \( z \mapsto f(z) \) is dense in \( \mathbb{C} \), we assume the contrary and we are going to derive contradiction. There exists some nonempty open subset \( U \) of \( \mathbb{C} \) which is disjoint from the image image of \( \{0 < |z| < r\} \) under the map \( z \mapsto f(z) \) for some \( r > 0 \). Without loss of generality we can assume that \( U = \{|z-b| < \rho\} \) for some \( \rho > 0 \). Let \( g(z) = \frac{1}{f(z)-b} \) on \( \{0 < |z-a| < r\} \). Since \(|g(z)| \leq \frac{1}{\rho} \) on \( \{0 < |z-a| > r\} \), the point \( z = a \) is a removable singularity for \( g(z) \) and as a result \( g(z) \) defines a non identically zero holomorphic function on \( \{0 < |z-a| < r\} \). Let \( k \) be the vanishing order
of \(g(z)\). Then \(f(z) = \frac{1}{g(z)} + b\) is either holomorphic on \(|z - a| < r\) when \(g(a) \neq 0\) with \(k = 0\) or having a pole of order \(k\) at \(z = a\) when \(k > 0\). This contradicts the assumption that \(z = a\) is an essential singularity for \(f(z)\).

**Residue of Isolated Singularity.** The purpose of introducing the notion of isolated singularities is to seek a more systematic treatment of the application of Cauchy’s integral formula for derivatives of holomorphic functions to the evaluation of a certain class of definite integrals. Cauchy’s integral formula is obtained by using the theorem of Cauchy-Goursat to equate a contour integral to another integral over a small circle and then letting the radius of the small circle approach 0. The center of the small circle is at an isolated singularity of a holomorphic function. To get a more systematic treatment of using complex analysis to evaluate definite integrals, we introduce now a new terminology for the integral over a small circle whose center is an isolated singularity. For an isolated singularity \(z = a\) of a holomorphic function \(f(z)\) defined on \(\{0 < |z - a| < R\}\) for some \(R > 0\), we define the residue of \(f(z)\) at \(z = a\), denoted by \(\text{Res}_a f\), as

\[
\text{Res}_a f = \frac{1}{2\pi i} \int_{|z|=r} f(z)dz
\]

for any \(0 < r < R\). By integrating the Laurent series

\[
f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n
\]

of \(f(z)\) term by term and using

\[
\int_{|z-a|=r} (z - a)^k dz = \int_{\theta=0}^{2\pi} ((a + re^{i\theta}) - a)^k d(a + re^{i\theta})
\]

\[
= \int_{\theta=0}^{2\pi} r^{k+1} e^{i(k+1)\theta} d\theta = \begin{cases} 
2\pi i & \text{for } k = -1 \\
0 & \text{for } k \neq -1,
\end{cases}
\]

we conclude that

\[
\text{Res}_a f = c_{-1}.
\]

In words, for a holomorphic function with an isolated singularity, the residue of the function at the isolated singularity is equal to the coefficient, with index
minus one, of the Laurent series expansion of the function on the degenerate annulus of zero inside radius centered at the isolated singularity. In the case where \( z = a \) is a pole of order \( k \) for \( f(z) \), the Laurent series is

\[
f(z) = \sum_{n=-k}^{\infty} c_n(z-a)^n
\]

with \( c_{-k} \neq 0 \) so that

\[
(z - a)^k f(z) = \sum_{n=0}^{\infty} c_{n-k}(z-a)^n
\]

and

\[
c_{-1} = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left( (z - a)^k f(z) \right) \bigg|_{z=a}.
\]

Thus for a pole of \( f(z) \) of order \( k > 0 \) at \( z = a \),

\[
\text{Res}_a f = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left( (z - a)^k f(z) \right) \bigg|_{z=a}.
\]

In other words, this formula comes from Taylor expansion after the Laurent series with a pole is converted to a power series by multiplying it by a monomial factor.

We can also arrive at the same formula by applying, to the holomorphic function \((z - a)^k f(z)\) on \(|z - a| < R\), Cauchy’s formula for its \(k\)-th order derivative at \(z = a\). We now summarize what we have introduced for the systematic treatment of the application of Cauchy’s integral formula for derivatives of holomorphic functions to the evaluation of definite integrals.

**Residue Theorem.** Suppose \( \Omega \) is a bounded open subset of \( \mathbb{C} \) with piecewise smooth boundary \( \partial \Omega \) and \( U \) is an open neighborhood of the topological closure \( \bar{\Omega} \) of \( \Omega \) in \( \mathbb{C} \). Suppose \( a_1, \cdots, a_p \) are distinct points in \( \Omega \) and \( f(z) \) is a holomorphic function on \( U - \{a_1, \cdots, a_p\} \). Then

\[
\int_{\partial \Omega} f(z)dz = 2\pi i \sum_{j=1}^{p} \text{Res}_{a_j} f.
\]

**Meromorphic Functions.** In order to be able to more easily describe an important class of functions in complex analysis, we now introduce the notion
of a meromorphic function. Suppose $U$ is an open subset of $\mathbb{C}$ and $E$ is a discrete subset of $U$ (in the sense that every point of $E$ admits a deleted open neighborhood in $U - E$). A holomorphic function $f(z)$ on $U - E$ is called a meromorphic function on $U$ if each $a$ in $E$ is a pole of $f(z)$.

Remark. The name holomorphic combines two Greek words ὅλος (meaning “whole” or “complete”) and μορφή (meaning “shape” or “form”). The name meromorphic combines two Greek words μέρος (meaning “part”) and μορφή (meaning “shape” or “form”).