Computation of Infinite Sums by Residues and Partial Fraction Expansion of Meromorphic Functions

One kind of definite integrals which we can compute by methods of residues is
\[ \int_{x=-\infty}^{\infty} \frac{P(x)}{Q(x)} \, dx, \]
where \( P(x) \) and \( Q(x) \) are polynomials with the degree of \( Q(x) \) at least 2 more than that of \( P(x) \) under the additional assumption that \( Q(x) \) is nowhere zero on \( \mathbb{R} \).

We now discuss the application of the methods of residues to discrete infinite sums instead of definite integrals, for example, to compute explicitly the infinite sum
\[ \sum_{n=-\infty}^{\infty} \frac{P(n)}{Q(n)}, \]
where \( P(x) \) and \( Q(x) \) are polynomials with the degree of \( Q(x) \) at least 2 more than that of \( P(x) \) under the additional assumption that \( Q(n) \) is nonzero for any \( n \in \mathbb{Z} \).

The idea is to use a meromorphic function \( f(z) \) on \( \mathbb{C} \) whose poles include \( z = n \) for \( n \in \mathbb{Z} \) and to use a sequence of contours \( C_n \) with the property that the domain enclosed by \( C_n \) is increasing as \( n \) increases and approach \( \mathbb{C} \) as \( n \to \infty \) and
\[ \lim_{n \to \infty} \int_{C_n} f(z) \, dz = 0 \]
and the residue of \( f(z) \) at \( z = n \) is \( \frac{P(n)}{Q(n)} \), so that
\[ \sum_{n=-\infty}^{\infty} \frac{P(n)}{Q(n)} \]
is equal to the negative of the sum of the residues of \( f(z) \) at poles other than the points of \( \mathbb{Z} \).

We have to choose a meromorphic function \( f(z) \) on \( \mathbb{C} \) whose residue at \( z = n \) is \( \frac{P(n)}{Q(n)} \). One choice is the function
\[ \frac{P(z)}{Q(z)} \pi \cot \pi z \]
because
\[ \pi \cot \pi z = \cos \pi z \frac{\pi}{\sin \pi z}, \]
has a simple pole at \( z = n \) with residue 1, from \( \cos \pi n = (-1)^n \) and
\[ \lim_{z \to n} \frac{\pi(z - n)}{\sin \pi z} = \lim_{z \to n} \frac{\pi(z - n)}{(-1)^n \sin \pi(z - n)} = (-1)^n. \]
If \( Q(n) \) is nonzero for \( n \in \mathbb{Z} \), we conclude that \( z = n \) is a simple pole for
\[ \frac{P(z)}{Q(z)} \pi \cot \pi z \]
with residue precisely equal to \( \frac{P(n)}{Q(n)} \).

For the contour \( C_n \) we use the square with vertices at \((n + \frac{1}{2})(\pm 1 \pm i)\). Observe that when \( |y| > \frac{1}{2\pi} \) we have
\[ |\cot \pi z| \leq \left| \frac{e^{2i\pi y} + 1}{e^{2i\pi y} - 1} \right| = \frac{e^{2\pi y} + 1}{e^{2\pi y} - 1} = 1 + \frac{2}{e^{2\pi y} - 1} \leq 1 + \frac{2}{e - 1} \]
and hence uniformly bounded. Also observe that \( \cot \pi z \) is bounded on the line segment joining \( \frac{1}{2}(1 - i) \) to \( \frac{1}{2}(1 + i) \) and we can use the periodicity
\[ \cot \pi(z + 1) = \cot \pi z \]
of \( \cot \pi z \) with period 1 to conclude that \( \cot \pi z \) is uniformly bounded on \( C_n \).
From the assumption that the degree of \( Q(z) \) is at least 2 more than the degree of \( P(z) \) it now follows that
\[ \lim_{n \to \infty} \int_{C_n} \frac{P(z)}{Q(z)} \pi \cot \pi z \, dz = 0. \]
Finally from the residue theorem (which now simply says that the sum of all the residues is zero) we have the formula
\[ \sum_{n=-\infty}^{\infty} \frac{P(n)}{Q(n)} = -\sum_{j=1}^{k} \text{Res}_{z=a_j} \left( \frac{P(z)}{Q(z)} \pi \cot \pi z \right), \]
where \( a_1, \ldots, a_k \) are the distinct zeroes of the polynomial \( Q(z) \) (i.e., each zero being counted only once by ignoring its multiplicity).
As a simple example, we compute

\[ \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} \]

for \( a > 0 \). The two points whose residue for

\[ \frac{1}{z^2 + a^2} \pi \cot \pi z \]

we have to compute are the two zeroes \( ai \) and \(-ai\) of \( z^2 + a^2 \). Since both poles are simple, we have

\[
\text{Res}_{z=ai} \left( \frac{1}{z^2 + a^2} \pi \cot \pi z \right) = \lim_{z \to ai} \left( \frac{z - ai}{z^2 + a^2} \pi \cot \pi z \right)
\]

\[
= \left( \frac{1}{z + ai} \pi \cot \pi z \right)_{z=ai}
\]

\[
= \pi \cot \pi ai
\]

\[
= 2ai
\]

\[
= \pi \frac{i(e^{i\pi ai} + e^{-i\pi ai})}{2ai} \frac{e^{i\pi ai} - e^{-i\pi ai}}{2a e^{-\pi a} - e^{\pi a}}
\]

\[
= -\frac{\pi}{2a} \coth \pi a
\]

and, with \( a \) replaced by \(-a\) and the odd property of the function \( z \mapsto \coth \pi z \),

\[
\text{Res}_{z=-ai} \left( \frac{1}{z^2 + a^2} \pi \cot \pi z \right) = -\frac{\pi}{2a} \coth \pi a.
\]

Hence

\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = - \left( \text{Res}_{z=ai} \left( \frac{1}{z^2 + a^2} \pi \cot \pi z \right) + \text{Res}_{z=-ai} \left( \frac{1}{z^2 + a^2} \pi \cot \pi z \right) \right)
\]

\[
= - \left( -\frac{\pi}{2a} \coth \pi a - \frac{\pi}{2a} \coth \pi a \right) = \frac{\pi \coth \pi a}{a}.
\]

Another kind of infinite sum which can be similarly computed in an explicit way is

\[
\sum_{n=-\infty}^{\infty} (-1)^n \frac{P(n)}{Q(n)}.
\]
where \( P(x) \) and \( Q(x) \) are polynomials with the degree of \( Q(x) \) at least 2 more than that of \( P(x) \) under the additional assumption that \( Q(n) \) is nonzero for any \( n \in \mathbb{Z} \). For this kind of infinite sum the meromorphic function \( f(z) \) to be used is modified to be

\[
\frac{P(z)}{Q(z)} \pi \csc \pi z,
\]

because \( \cos \pi z \) is \((-1)^n\) at \( z = n \). The same contour \( C_n \) of the square with vertices at \( (n + \frac{1}{2})(\pm 1 \pm i) \) is used.

\[
\text{Res}_{z=n} \left( \frac{P(z)}{Q(z)} \pi \csc \pi z \right) = \lim_{z \to n} \frac{P(z)}{Q(z)} \frac{\pi(z - n)}{\sin \pi z} = \frac{P(n)}{Q(n)} \lim_{z \to n} \frac{\pi(z - n)}{(-1)^n \sin \pi(z - n)} = (-1)^n \frac{P(n)}{Q(n)}.
\]

When \( y > \frac{1}{\pi} \) we have

\[
|\csc \pi z| = \left| \frac{2i}{e^{i\pi z} - e^{-i\pi z}} \right| = \left| \frac{2}{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}} \right| \leq \frac{2}{e^{\pi y} - e^{-\pi y}} \leq \frac{2}{e - e^{-1}}.
\]

When \( y < -\frac{1}{\pi} \) we have

\[
|\csc \pi z| = \left| \frac{2i}{e^{i\pi z} - e^{-i\pi z}} \right| = \left| \frac{2}{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}} \right| \leq \frac{2}{e^{-\pi y} - e^{\pi y}} \leq \frac{2}{e - e^{-1}}.
\]

Hence \( \csc \pi z \) is uniformly bounded on \( |y| > \frac{1}{\pi} \). Moreover, \( \csc \pi z \) is bounded on the line segment joining \( \frac{1}{2}(1 - i) \) to \( \frac{1}{2}(1 + i) \) and we can use the periodicity

\[
|\csc \pi (z + 1)| = |\csc \pi z|
\]
of $|\csc \pi z|$ with period 1 to conclude that $\csc \pi z$ is uniformly bounded on $C_n$. From the assumption that the degree of $Q(z)$ is at least 2 more than the degree of $P(z)$ it now follows that
\[
\lim_{n \to \infty} \int_{C_n} P(z) \frac{Q(z)}{\csc \pi z} \, dz = 0.
\]
Finally from the residue theorem (which now simply says that the sum of all the residues is zero) we have the formula
\[
\sum_{n=-\infty}^{\infty} (-1)^n \frac{P(n)}{Q(n)} = -\sum_{j=1}^{k} \text{Res}_{z=a_j} \left( \frac{P(z)}{Q(z)} \pi \csc \pi z \right),
\]
where $a_1, \cdots, a_k$ are the distinct zeroes of the polynomial $Q(z)$ (i.e., each zero being counted only once by ignoring its multiplicity).

Infinite Product Expansion of Sine Function. The identity
\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi \coth \pi a}{a},
\]
which was derived for $a > 0$, holds when $a$ is replaced by any complex number $z$, because the left-hand side of
\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + z^2} = \frac{\pi \coth \pi z}{z}
\]
defines a meromorphic function on $\mathbb{C}$ and the identity simply follows from applying the identity theorem the meromorphic function which is the difference of the two sides. Note that the coefficients of a Laurent series with an isolated singularity are computed along a circle centered at the isolated singularity so that the identity theorem applied to the complement of its poles determine a meromorphic function completely. To make it more convenient to factor the denominator of each term on the left-hand side, we replace $z$ by $iz$ to get
\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 - z^2} = \frac{\pi \coth \pi iz}{iz} = \frac{\pi(e^{\pi iz} + e^{-\pi iz})}{iz(e^{\pi iz} - e^{-\pi iz})} = -\frac{\pi \cot \pi z}{z}.
\]
or

\[ \pi \cot \pi z = \sum_{n=-\infty}^{\infty} \frac{z}{z^2 - n^2}, \]

which can be rewritten as

\[
\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \\
= \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z + n} + \frac{1}{z - n} \right) \\
= \frac{1}{z} + \sum_{n \in \mathbb{Z} - \{0\}} \left( \frac{1}{z - n} + \frac{1}{n} \right).
\]

The identity

\[ \pi \cot \pi z = \frac{1}{z} + \sum_{n \in \mathbb{Z} - \{0\}} \left( \frac{1}{z - n} + \frac{1}{n} \right) \]

is the partial fraction expansion of the cotangent function or the expansion of the cotangent function into a sum of its principal parts.

TO BE CONTINUED ...