Biholomorphic Map as Orientation-Preserving Conformal Map, Maps by Mercator Projection and Complex Structure of Riemann Sphere from Stereographic Projection

We introduced a complex-differentiable (or holomorphic) function as a complex-valued function whose difference quotient with respect to the complex variable admits a limit. The Cauchy-Riemann equations follow from two different ways of taking the limit of the difference quotient, one way along the horizontal line and the other along the vertical line. Interpreted in terms of the \(\mathbb{R}\)-linear map between the tangent space of the domain space \(\mathbb{R}^2\) and the target space \(\mathbb{R}^2\) defined by the differential of the function, the Cauchy-Riemann equations are equivalent to the statement that the \(\mathbb{R}\)-linear map from \(\mathbb{R}^2 = \mathbb{C}\) to itself is \(\mathbb{C}\)-linear.

We now give yet another interpretation of the Cauchy-Riemann equations in the case of nonzero complex derivative. In this new interpretation the Cauchy-Riemann equations in the case of nonzero complex derivative are characterized by the condition that the \(\mathbb{R}\)-linear map from \(\mathbb{R}^2 = \mathbb{C}\) to itself preserves angles and the orientation. To verify this interpretation it suffices to verify that an \(\mathbb{R}\)-linear map of \(\mathbb{R}^2 = \mathbb{C}\) to itself is a \(\mathbb{C}\)-linear isomorphism if and only if it angle-preserving and orientation-preserving. Note that orientation-preserving means the determinant of the \(\mathbb{R}\)-linear map is positive. Here is the verification.

Let \(T : \mathbb{R}^2 \to \mathbb{R}^2\) be an \(\mathbb{R}\)-linear map defined by the \(2 \times 2\) matrix \((a \ b
c \ d)\).

The angle \(\theta\) between two column vectors \(\vec{\xi} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\) and \(\vec{\eta} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}\) of \(\mathbb{R}^2\) satisfies
\[
\cos \theta = \frac{\vec{\xi} \cdot \vec{\eta}}{||\vec{\xi}|| \ ||\vec{\eta}||} = \frac{\alpha \gamma + \beta \delta}{\sqrt{\alpha^2 + \beta^2} \sqrt{\gamma^2 + \delta^2}}.
\]

Suppose \(T\) preserves angles and has positive determinant. Then in particular, \(T\) maps the two orthogonal basis vectors \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\) of \(\mathbb{R}^2\) to two orthogonal vectors, which means that the dot product \(ab + cd\) of their \(T\)-images vanishes. The dot product
\[
(a + b)(-a + b) + (c + d)(-c + d)
\]
of the $T$-images of the two orthogonal vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ of $\mathbb{R}^2$ also vanishes. Thus we have the two equations

$$ab + cd = 0 \quad \text{and} \quad a^2 + c^2 = b^2 + d^2.$$ 

One of $a$ and $c$ is nonzero and without loss of generality we can assume that $a \neq 0$, because the argument for $c \neq 0$ is completely analogous. Let $\lambda \in \mathbb{R}$ such that $d = \lambda a$. From $ab + cd = 0$ we get $ab + c\lambda a = 0$ so that $b = -\lambda c$. From the equation $a^2 + c^2 = b^2 + d^2$ it follows that $a^2 + c^2 = \lambda^2 (c^2 + a^2)$ and $\lambda = \pm 1$. The case $\lambda = 1$ yields

$$T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

while the case $\lambda = -1$ yields

$$T = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$ 

When $T$ is orientation-preserving, its determinant needs to be positive. Thus $\lambda = 1$ and

$$T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

which means that $T$, as an $\mathbb{R}$-linear map of $\mathbb{R}^2 = \mathbb{C}$ to itself, is $\mathbb{C}$-linear.

On the other hand, suppose

$$T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

with $\det T = a^2 + b^2 > 0$, so that $T$, as an $\mathbb{R}$-linear map of $\mathbb{R}^2 = \mathbb{C}$ to itself, is $\mathbb{C}$-linear. Then

$$T^t T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (a^2 + b^2) I,$$

where the superscript $^t$ denotes taking the transpose of a matrix and $I$ means the identity $2 \times 2$ matrix. From

$$\frac{(T\tilde{\xi}) \cdot (T\tilde{\eta})}{\|T\tilde{\xi}\| \|T\tilde{\eta}\|} = \frac{\tilde{\xi}^t T^t T\tilde{\eta}}{\sqrt{\tilde{\xi}^t T^t T\tilde{\xi} \sqrt{\tilde{\eta}^t T^t T\tilde{\eta}}}} = \frac{\tilde{\xi}^t \tilde{\eta}}{\sqrt{\tilde{\xi}^t \tilde{\xi} \sqrt{\tilde{\eta}^t \tilde{\eta}}}} = \frac{\tilde{\xi} \cdot \tilde{\eta}}{\|\tilde{\xi}\| \|\tilde{\eta}\|}$$
it follows that the cosine of the angle between $\vec{\xi}$ and $\vec{\eta}$ is equal to the cosine of the angle between $T\vec{\xi}$ and $T\vec{\eta}$. The positivity of $\det T = a^2 + b^2$ means that $T$ is orientation-preserving. This finishes our verification of the equivalence of the $\mathbb{C}$-linearity of the (injective) $\mathbb{R}$-linear map $T$ from $\mathbb{R}^2 = \mathbb{C}$ to itself and the angle-preserving and orientation property of $T$.

Yet another characterization of a biholomorphic map $w = f(z)$ is that the orientation-preserving map $(x, y) \mapsto (u(x, y), v(x, y))$ with $z = x + iy$ and $w = u + iv$ pulls back the Euclidean metric $du^2 + dv^2$ of the target $(u, v)$-space to some positive function $\gamma(x, y)$ times the Euclidean metric $dx^2 + dy^2$ of the domain $(x, y)$-space. The positive function $\gamma(x, y)$ is known as a conformal factor. The justification for this characterization is as follows.

If $w = f(z)$ is biholomorphic, then the Jacobian matrix

$$T = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix} = \begin{pmatrix}
a & -b \\
b & a
\end{pmatrix}$$

with $a = \frac{\partial u}{\partial x}$ and $b = \frac{\partial v}{\partial x}$ so that

$$du^2 + dv^2 = (du \ dv) \begin{pmatrix} du \\ dv \end{pmatrix}$$

$$= (dx \ dy) T^t T \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$= (dx \ dy) (a^2 + b^2 I) \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$= (a^2 + b^2) (dx \ dy) \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$= \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right) (dx^2 + dy^2)$$

$$= |f'(z)|^2 (dx^2 + dy^2)$$

with $|f'(z)|^2$ as the conformal factor.

On the other hand, if

$$du^2 + dv^2 = \gamma(x, y) (dx^2 + dy^2)$$

for some $\gamma(x, y) > 0$ and $T$ denotes the Jacobian matrix

$$
\begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix},
$$

then from

$$
du^2 + dv^2 = (du \ dv) \begin{pmatrix} du \\ dv \end{pmatrix} = (dx \ dy) \ T^T \begin{pmatrix} dx \\ dy \end{pmatrix}
$$

and

$$
\gamma(x, y) (dx^2 + dy^2) = (dx \ dy) (\gamma(x, y)I) \begin{pmatrix} dx \\ dy \end{pmatrix}
$$

it follows that

$$
(dx \ dy) \ T^T \begin{pmatrix} dx \\ dy \end{pmatrix} = (dx \ dy) (\gamma(x, y)I) \begin{pmatrix} dx \\ dy \end{pmatrix}
$$

and $T^T = \gamma(x, y)I$. As we have just seen, with the use of the dot product, this implies that $T$ is angle-preserving and, together with the condition that $\det T > 0$ from its orientation-preserving property, yields the Cauchy-Riemann equations

$$
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
$$

This finishes the verification of the characterization of a biholomorphic map as map which pulls back the Euclidean metric of the target space to the Euclidean metric of the domain space up to a positive conformal factor.

We now apply this last characterization to the stereographic projection of the 2-sphere to its tangent plane and discuss how two such stereographic projections, one to the tangent plane at the south pole and another to the tangent plane at the north pole, define two biholomorphically related coordinates charts on the 2-sphere to make it into a complex manifold of complex dimension 1 which is known as the Riemann sphere.
We put the 2-sphere $S$ into the position in $\mathbb{R}^3$ with $(0, 0, \frac{1}{2})$ as its center and $(0, 0, 0)$ as its south pole $S$ and $(0, 0, 1)$ as its north pole $N$ so that the equation for the 2-sphere $S$ is

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

or equivalently

$$x^2 + y^2 + z^2 - z = 0.$$  

The stereographic projection with $N$ as the light-source and the $(x, y)$-coordinate plane as the target plane is described by

$$X = \frac{x}{1 - z}, \quad Y = \frac{y}{1 - z},$$

where the point $(X, Y)$ in the $(x, y)$-coordinate plane is the image of the point $(x, y, z)$ in $S$ under the stereographic projection, because we have two similar triangles, one with vertices $N = (0, 0, 1), (x, y, z)$ and $(0, 0, 0)$ and the other with vertices $N = (0, 0, 1), (X, Y, 0)$ and $S = (0, 0, 0)$, so that

$$dX = \frac{dx}{1 - z} + \frac{xdz}{(1 - z)^2}, \quad dY = \frac{dy}{1 - z} + \frac{ydz}{(1 - z)^2}$$

and

$$dX^2 = \frac{1}{(1 - z)^4} \left((1 - z)^2 \, dx^2 + 2x (1 - z) \, dx \, dz + x^2 dz^2\right),$$

$$dY^2 = \frac{1}{(1 - z)^4} \left((1 - z)^2 \, dy^2 + 2y (1 - z) \, dy \, dz + y^2 dz^2\right).$$

Using $x^2 + y^2 + z^2 = z$ and $2x \, dx + 2y \, dy + (2z - 1) \, dz = 0$, we get

$$dX^2 + dY^2 = \frac{1}{(1 - z)^2} \left(dx^2 + dy^2 + dz^2\right).$$
Though in this case the stereographic projection is from \( S - \{ N \} \) to \( \mathbb{R}^2 \) and not between two open subsets of \( \mathbb{R}^2 \), yet the above argument for maps between two open subsets of \( \mathbb{R}^2 \) also works to yield the conclusion that the stereographic projection from \( S - \{ N \} \) to the \((x, y)\)-coordinate plane is conformal in the sense that it preserves the angles between two curves, when \( S \) is given the metric \( dx^2 + dy^2 + dz^2 \), which means that in the coordinate chart \((x, y)\) the metric is
\[
dx^2 + dy^2 + \left( \frac{2xdx + 2ydy}{2z - 1} \right)^2,
\]
in the coordinate chart \((x, z)\) the metric is
\[
dx^2 + \left( \frac{2xdx + (2z - 1)dz}{2y} \right)^2 + dz^2,
\]
and in the coordinate chart \((y, z)\) the metric is
\[
\left( \frac{2ydy + (2z - 1)dz}{2x} \right)^2 + dy^2 + dz^2.
\]

We now look at the stereographic projection with \( S \) as light-source and the plane tangential to \( S \) at the north pole \( N \) as the target plane, which is described by
\[
X' = \frac{x}{z}, \quad Y' = \frac{y}{z},
\]
where \((X', Y', 1)\) is the image of the point \((x, y, z)\), because of the two similar triangles, one with vertices \( S = (0, 0, 0), (x, y, z) \) and \((0, 0, z)\) and the other with vertices \( S = (0, 0, 0), (X', Y', 1) \) and \( N = (0, 0, 1) \). This stereographic projection is also conformal. The composite map
\[
(X, Y) \mapsto (x, y, z) \mapsto (X', Y')
\]
is orientation-reversing. To make the orientation right, we consider the map
\[
(X, Y) \mapsto (x, y, z) \mapsto (X', -Y'),
\]
which is now orientation-preserving and conformal and therefore defines a biholomorphic map from \( \mathbb{C} - \{0\} \) to \( \mathbb{C} - \{0\} \). We would like to determine the relation between the two complex numbers \( X + iY \) and \( X' - iY' \). We
claim that one is the reciprocal of the other. We now verify the claim by straightforward computation as follows.

\[(X + iY)(X' - iY') = XX' + YY' + i(-XY' + YX')\]

\[= \frac{x^2}{(1 - z)z} + \frac{y^2}{(1 - z)z} + i\left(-\frac{xy}{(1 - z)z} + \frac{yx}{(1 - z)z}\right)\]

\[= \frac{x^2 + y^2}{(1 - z)z} = \frac{z - z^2}{(1 - z)z} = 1.\]

With the claim verified, we now have the coordinate chart \(S - \{N\}\) with complex coordinate \(X + iY\) and the coordinate chart \(S - \{S\}\) with complex coordinate \(X' - iY'\), which are biholomorphically related by the equation \((X + iY)(X' - iY') = 1\). This makes \(S\) into a complex manifold of complex dimension 1, called the Riemann sphere, which is naturally identified with the extended Gauss plane \(\mathbb{C} \cup \{\infty\}\).

Now we discuss the conformal cylindrical projection map between the 2-sphere and the plane \(\mathbb{R}^2\) which Gerardus Mercator introduced in 1569 (more than a century before the discovery of calculus around 1670) to produce maps for marine navigation. This conformal map which has the least distortion along the equator is known as the Mercator projection. We describe this conformal map by wrapping a cylinder (which is equal to the unit circle in the \((x, y)\)-plane times the \(z\)-axis) around the unit 2-sphere in the \((x, y, z)\)-space with the origin as center and with radius 1 so that the cylinder touches the sphere along the equator of the sphere.

Let \(\lambda\) be the longitude and \(\varphi\) be the latitude of the unit sphere. The square of arc-length on the unit sphere is

\[\cos^2 \varphi \, d\lambda^2 + d\varphi^2,\]

because the radius of the horizontal circle at latitude \(\varphi\) is \(\cos \varphi\) and the angle along the horizontal circle is measured by \(\lambda\), while the angle along the vertical circle at longitude \(\lambda\) is measured by \(\varphi\). A more rigorous derivation of the metric

\[\cos^2 \varphi \, d\lambda^2 + d\varphi^2\]

on the unit sphere is to use the parametrization of the unit sphere

\[
\begin{cases}
  x = \cos \varphi \cos \lambda \\
  y = \cos \varphi \sin \lambda \\
  z = \sin \varphi
\end{cases}
\]
by the longitude $\lambda$ and the latitude $\varphi$ to write

$$\begin{align*}
\frac{dx^2}{2} + \frac{dy^2}{2} + \frac{dz^2}{2} \\
= ( - \sin \varphi \cos \lambda d\varphi - \cos \varphi \sin \lambda d\lambda )^2 + (- \sin \varphi \sin \lambda d\varphi + \cos \varphi \cos \lambda d\lambda )^2 + ( \cos \varphi d\varphi )^2 \\
= ( \sin^2 \varphi \cos^2 \lambda d\varphi^2 + 2 \sin \varphi \cos \lambda \cos \varphi \sin \lambda d\varphi d\lambda + \cos^2 \varphi \sin^2 \lambda d\lambda^2 ) \\
+ ( \sin^2 \varphi \sin^2 \lambda d\varphi^2 - 2 \sin \varphi \sin \lambda \cos \varphi \cos \lambda d\varphi d\lambda + \cos^2 \varphi \cos^2 \lambda d\lambda^2 ) + \cos^2 \varphi d\varphi^2 \\
= \sin^2 \varphi \left( \cos^2 \lambda + \sin^2 \lambda \right) d\varphi^2 + \cos^2 \varphi \left( \sin^2 \lambda + \cos^2 \lambda \right) d\lambda^2 + \cos^2 \varphi d\varphi^2 \\
= \sin^2 \varphi d\varphi^2 + \cos^2 \varphi d\lambda^2 + \cos^2 \varphi d\varphi^2 \\
= \cos^2 \varphi d\varphi^2 + d\varphi^2.
\end{align*}$$

Let $X$ be the coordinate of the point on the cylinder which measures the arc-length of the unit circle in the $(x,y)$-plane and let $Y$ be the height of the point on the cylinder measured from the $(x,y)$-plane. The Mercator projection is defined by $X = \lambda$ and $Y = \log (\tan \varphi + \sec \varphi)$ so that $dY = \sec \varphi d\varphi$ and

$$dX^2 + dY^2 = d\lambda^2 + \sec^2 \varphi d\varphi^2 = \sec^2 \varphi \left( \cos^2 \varphi d\lambda^2 + d\varphi^2 \right)$$

guarantees conformality with conformal factor $\sec^2 \varphi$. The course of a vessel which is represented by a straight line on the chart of the flattened cylindrical surface corresponds to a great circle on the globe represented by the 2-sphere. The steering angle used on the flat chart corresponds to the steering angle on the globe.