1. Introduction

Let $M$ be a smooth $n$-dimensional manifold. Then, de Rham’s theorem states that the de Rham cohomology of $M$ is naturally isomorphic to its singular cohomology with coefficients in $\mathbb{R}$; in particular, de Rham cohomology is a purely topological invariant. This fact is a manifestation of a fundamental relationship between the analytical and topological characteristics of a smooth manifold, where the former include information about solutions to differential equations of the formulation $d\eta = \omega$ for closed forms $\omega$, and the latter, information about the “holes” in each given dimension, which is made precise by singular homology.

The natural isomorphism will be given by a version of Stokes’ theorem, which describes a duality between de Rham cohomology and singular homology. Specifically, the pairing of differential forms and singular chains, which can taken to be smooth, yields a map from the $k$th de Rham cohomology group to the $k$th singular cohomology group with coefficients in $\mathbb{R}$ for each $k \geq 0$. And in 1931, Georges de Rham himself proved that this map is in fact an isomorphism. In this exposition, we will give a proof of this beautiful and fundamental result.

2. Smooth Singular Homology

Our desired relationship between singular cohomology and de Rham cohomology will be given by the integration of differential forms over singular chains. Specifically, given a singular $k$-simplex $\sigma : \Delta_k \to M$ and a $k$-form $\omega$ on a manifold $M$, we wish to integrate over $\Delta_k$ the pullback form $\sigma^* \omega$.

However, pulling back differential forms requires the map in question to be sufficiently smooth, yet as maps, singular $k$-simplices are only continuous in general. To address this issue, we will show that singular homology can be computed using only smooth simplices instead of all continuous ones.

For a subset $C \subset \mathbb{R}^m$, we say that a continuous map $f : C \to M$ is smooth at a point $p \in C$ if there exists an open neighborhood of $p$ in which $f$ has a smooth extension. Furthermore, we say that $f$ is smooth if it is smooth at every point $p \in C$. Then, specializing to the case that $C$ is equal to the standard $k$-simplex of $\mathbb{R}^m$, defined by $\Delta_k := [\delta_0, \ldots, \delta_k] \subset \mathbb{R}^m$ for the standard basis vectors $\delta_i$, we define a smooth $k$-simplex of $M$ to be a $k$-simplex $\sigma : \Delta_k \to M$ that is smooth as a map. For the usual free abelian group $C_k(M)$ on all $k$-chains of $M$, we define $C^\infty_k(M)$ to be the subgroup of $C_k(M)$ generated by the smooth $k$-simplices of $M$. Define the smooth $k$-chains of $M$ to be the elements of $C^\infty_k(M)$, i.e., formal sums of finitely many smooth $k$-simplices. It is clear from the above definitions that the boundary of a smooth $(k+1)$-simplex is a smooth $k$-chain, so the $k$th smooth singular homology group

$$H^\infty_k(M) = \frac{\text{Ker}(\partial : C^\infty_k(M) \to C^\infty_{k-1}(M))}{\text{Im}(\partial : C^\infty_{k+1}(M) \to C^\infty_k(M))}$$

is well-defined. Since the inclusion $i : C^\infty_k(M) \hookrightarrow C_k(M)$ satisfies $i \circ \partial = \partial \circ i$, we see that it induces a map $\iota_* : H^\infty_k(M) \to H_k(M)$ defined by $\iota_*[c] = [i(c)]$.

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1This exposition is based on the proof provided in [2, Chapter 18], which in turn is based on that given in [1, Section V.9]. For another proof of de Rham’s theorem based on sheaf theory, see [3, Chapter 5].
We would like to show that this map $i_*$ is an isomorphism. We will first need to construct for an arbitrary simplex a homotopy to a smooth simplex such that when the homotopy for any simplex is restricted to a boundary face, this yields the homotopy of the simplex given by the boundary face. To do so, we will need the following lemma showing that for a map whose domain is the standard $k$-simplex, smoothness on each boundary face implies smoothness on the entire boundary simultaneously.

**Lemma 2.1.** For the standard simplex $\Delta_k \subset \mathbb{R}^m$, let $f : \partial \Delta_k \rightarrow M$ be a continuous map whose restriction to each boundary face is smooth. Then, $f$ is smooth as a map from the entire boundary $\partial \Delta_k$ to $M$.

*Proof.* Recall the notation $\Delta_k = [\delta_0, \ldots, \delta_k]$, where the standard basis vectors $\delta_0, \ldots, \delta_k$ are the vertices of the simplex. For $0 \leq i \leq k$, let $\partial_i \Delta_k = [\delta_0, \ldots, \delta_i, \ldots, \delta_k]$ denote the boundary face opposite of $\delta_i$. Our hypothesis then is precisely that for every $i$ and $x \in \partial_i \Delta_k$, there exists an open neighborhood $U_x \ni x$ and a smooth extension $f : U_x \rightarrow M$ of $f$, i.e., $f$ restricted to $U_x \cap \partial_i \Delta_k$ gives $f$. We then need to show that for every $x \in \partial \Delta_k$, a smooth extension $\tilde{f}$ can be chosen for all boundary faces, i.e., such that $\tilde{f}$ restricted to $U_x \cap \partial \Delta_k$ coincides with $f$.

Since $x$ is contained in at least one boundary face of $\Delta_k$ but cannot be contained in all of them, we can without loss of generality suppose that the boundary faces containing $x$ are $\partial_0 \Delta_k, \ldots, \partial_m \Delta_k$ for some $1 \leq j \leq k$, and in particular, that $x$ is not contained in $\partial_0 \Delta_k$. By our hypothesis, there exists an open neighborhood $U_i \ni x$ and a smooth map $\tilde{f}_i : U_i \rightarrow M$ whose restriction to $U_i \cap \partial_i \Delta_k$ coincides with $f$. Let $U = \bigcap_{i=1}^j U_i$. We now prove by induction on $j$ that there exists a smooth map $\tilde{f} : U \rightarrow M$ whose restriction to $U \cap \left( \bigcup_{i=1}^j \partial_i \Delta_k \right)$ coincides with $f$. Since our claim is local, we can, by replacing $U$ with a subneighborhood if necessary, suppose that $f$ maps $U$ to a coordinate chart of $f(x)$ that can be diffeomorphically identified with $\mathbb{R}^n$. The base case $j = 1$ is trivial. Now, suppose that $j > 1$ and that there exists a smooth map $\tilde{f}_0 : U \rightarrow M$ whose restriction to $U \cap \left( \bigcup_{i=1}^j \partial_i \Delta_k \right)$ coincides with $f$. Note that the boundary face $\partial_i \Delta_k$ is precisely the intersection of the hyperplane $x_i = 0$ with $\partial \Delta_k$. Consistent with this, define $\tilde{f} : U \rightarrow M$ by

$$\tilde{f}(x^1, \ldots, x^m) = \tilde{f}_0(x^1, \ldots, x^m) - \tilde{f}_0(x^1, \ldots, x^j-1, 0, x^{j+1}, \ldots, x^m) + \tilde{f}_j(x^1, \ldots, x^j-1, 0, x^{j+1}, \ldots, x^m).$$

For every $1 \leq i \leq j - 1$, the restriction of $\tilde{f}$ to $U \cap \partial_i \Delta_k$ equals

$$\tilde{f}(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n)$$

$$= \tilde{f}_0(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n) - \tilde{f}_0(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^{j-1}, 0, x^{j+1}, \ldots, x^n)$$

$$+ \tilde{f}_j(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^{j-1}, 0, x^{j+1}, \ldots, x^n)$$

$$= f(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n),$$

since $\tilde{f}_0$ coincides with $f$ for $x \in \Delta_k$ satisfying $x^i = 0$ and similarly $\tilde{f}_j$ coincides with $f$ for $x \in \Delta_k$ satisfying $x^j = 0$. An analogous argument shows that the restriction of $\tilde{f}$ to $U \cap \partial_j \Delta_k$ is

$$\tilde{f}(x^1, \ldots, x^{j-1}, 0, x^{j+1}, \ldots, x^n)$$

$$= \tilde{f}_0(x^1, \ldots, x^{j-1}, 0, x^{j+1}, \ldots, x^n) - \tilde{f}_0(x^1, \ldots, x^{j-1}, 0, x^{j+1}, \ldots, x^n)$$

$$+ \tilde{f}_j(x^1, \ldots, x^{j-1}, 0, x^{j+1}, \ldots, x^n)$$

$$= f(x^1, \ldots, x^{j-1}, 0, x^{j+1}, \ldots, x^n),$$

which proves the inductive step as needed. \(\square\)

We will also need the following, often called Whitney’s approximation theorem.
**Theorem 2.2** (Whitney). Let $X$ be a smooth manifold with or without boundary, $Y$ a smooth manifold without boundary, and $f: X \to Y$ a continuous map smooth on a closed subset $C \subset X$. We have that $F$ is homotopic relative to $C$ to a smooth map $X \to Y$.

**Proof.** [2, Theorem 6.26].

Now, we are ready to prove our desired lemma that constructs, for any simplex, a homotopy from it to a smooth simplex such that the homotopies respect restriction to boundary faces.

**Lemma 2.3.** For every $k$-simplex $\sigma: \Delta_k \to M$ from the standard $k$-simplex of $\mathbb{R}^k$ to $M$, there exists a continuous map $H_\sigma: \Delta_k \times I \to M$ satisfying the following:

(i) $H_\sigma$ is a homotopy from $\sigma(x) = H_\sigma(x,0)$ to a smooth $k$-simplex $\tilde{\sigma}(x) = H_\sigma(x,1)$.

(ii) For each boundary face inclusion $F_{i,k}: \Delta_{k-1} = \partial_i \Delta_k \hookrightarrow \Delta_k$,

$$H_{\sigma \circ F_{i,k}}(x,t) = H_\sigma(F_{i,k}(x),t)$$

for $(x,t) \in \Delta_{k-1} \times I$.

(iii) For $\sigma$ a smooth $k$-simplex, $H_\sigma$ is the constant homotopy, i.e., $H_\sigma(x,t) = \sigma(x)$.

**Proof.** For $\sigma$ smooth, $H_\sigma$ can be taken to be the constant homotopy, which is easily seen to satisfy properties (1) and (2). We prove the problem statement for not necessarily smooth $\sigma$ by induction on the dimension of $\sigma$. An arbitrary 0-simplex $\sigma: \Delta_0 \to M$ is smooth, so the base case is trivial. Suppose now that for all $\ell < j$, we have constructed for each $\ell$-simplex $\sigma'$ a corresponding homotopy $H_{\sigma'}$ satisfying the lemma’s properties. Consider an arbitrary non-smooth $j$-simplex $\sigma: \Delta_j \to M$. Define the subset $S := (\Delta_k \times \{0\}) \cup (\partial \Delta_k \times I) \subset \Delta_k \times I$, and define $H_0: S \to M$ by

$$H_0(x,t) := \begin{cases} \sigma(x) & \text{if } x \in \Delta_k, t = 0, \\ H_{\sigma \circ F_{i,k}}(F_{i,k}^{-1}(x),t) & \text{if } x \in \partial_i \Delta_k. \end{cases}$$

To show that $H_0$ is continuous, we need to show that the functions of the above cases coincide on overlaps so that we can apply the gluing lemma. For $(x,0) \in S$ such that $x \in \partial_i \Delta_k$, the inductive hypothesis implies that

$$H_{\sigma \circ F_{i,k}}(F_{i,k}^{-1}(x),0) = \sigma \circ F_{i,k}(F_{i,k}^{-1}(x)) = \sigma(x),$$

and indeed $\sigma(x)$ and $H_{\sigma \circ F_{i,k}}(F_{i,k}^{-1}(x),0)$ agree. Next, suppose $(x,t) \in S$ such that $x \in \partial_i \Delta_k \cap \partial_j \Delta_k$, where without loss of generality we assume $0 \leq i < j \leq k$. Note that we have the homeomorphism $\Delta_{k-2} \to \partial_i \Delta_k \cap \partial_j \Delta_k$ given by $F_{i,k} \circ F_{j,k-1} = F_{j,k} \circ F_{i,k-1}$, and thus we can write $x = F_{i,k} \circ F_{j,k-1}(y) = F_{j,k} \circ F_{i,k-1}(y)$ for some $y \in \Delta_{k-2}$. Then, by the inductive hypothesis, we have

$$H_{\sigma \circ F_{i,k}}(F_{i,k}^{-1}(x),t) = H_{\sigma \circ F_{i,k}}(F_{j,k-1}^{-1}(y),t) = H_{\sigma \circ F_{i,k} \circ F_{j,k-1}}(y,t)$$

and

$$H_{\sigma \circ F_{j,k}}(F_{j,k}^{-1}(x),t) = H_{\sigma \circ F_{j,k}}(F_{j,k-1}^{-1}(y),t) = H_{\sigma \circ F_{j,k} \circ F_{i,k-1}}(y,t),$$

showing that $H_{\sigma \circ F_{j,k}}(F_{j,k}^{-1}(x),t) = H_{\sigma \circ F_{i,k} \circ F_{j,k-1}}(y,t)$ as needed.

Next, we wish to extend $H_0$ to the larger domain $\Delta_k \times I$. To do so, we compose it with a continuous retraction $r: \Delta_k \times I \to S$; note that $r$ can be defined by projecting from $(x_0,2) \in \mathbb{R}^k \times \mathbb{R}$ for an interior point $x_0 \in \Delta_K$. This gives us the continuous map $H := H_0 \circ r: \Delta_k \times I \to M$. However, while $H$ is a homotopy from $\sigma$ to some $k$-complex $\sigma' := H(\cdot,1)$ satisfying property (3), we are not guaranteed that $\sigma'$ is smooth. Thus, we need to modify $H$ to be a homotopy to a smooth $k$-simplex. First, we check that $H$ is smooth on $\partial_i \Delta_k \times \{1\}$. Indeed, note that $\partial_i \Delta_k \times \{1\}$ is contained in $S$, so $H$ coincides with $H_0$ on each such set. Every boundary simplex $\sigma \circ F_{i,k}$ satisfies property (1) by the inductive hypothesis, so we have that $H_0$ is smooth on all boundary simplices $\partial \Delta_k$. By Lemma 2.1 it follows that $H_0$ is smooth on all of $\partial \Delta_k \times \{1\}$. Next, take a continuous extension $\sigma''$ of $\sigma'$ to an open set $U \subset \mathbb{R}^k$ containing $\Delta_k$. By Theorem 2.2 there exists a homotopy $H: U \times I \to M$ from $\sigma''$ to a smooth map $H(\cdot,1): U \to M$, satisfying that $H(x,t) = \sigma''(x)$ for
all \(x \in \partial \Delta_k\). Restricting to \(\Delta_k \times I\), we obtain a homotopy \(\hat{H}: \Delta_k \times I \to M\) from \(\sigma'\) to a smooth \(k\)-simplex \(\hat{\sigma} := \hat{H}(\cdot, 1)\), similarly satisfying that \(\hat{H}(x, t) = \sigma'(x)\) for all \(x \in \partial \Delta_k\).

Choose a continuous function \(u: \mathbb{R}^k \to M\) that has values \(0 < u(x) < 1\) for \(x \in \Delta_k^0\) and equals 1 on \(\partial \Delta_k\). We use the homotopies \(H\) and \(\hat{H}\) to define the homotopy

\[
H_\sigma(x, t) := \begin{cases} 
H \left(x, \frac{t}{u(x)}\right) & \text{if } x \in \partial \Delta_k \text{ or } (x \in \Delta_k^0 \text{ and } 0 \leq t \leq u(x)), \\
\hat{H} \left(x, \frac{t-u(x)}{1-u(x)}\right) & \text{if } x \in \Delta_k^0 \text{ and } u(x) \leq t \leq 1.
\end{cases}
\]

Since \(H(x, 1) = \sigma'(x) = \hat{H}(x, 0)\), we have that \(H_\sigma\) is continuous on \(\partial \Delta_k \times I\) by the gluing lemma. Moreover, \(H_\sigma(x, t)\) is given by the continuous function \(H(x, \frac{t}{u(x)})\) in a neighborhood of \(\partial \Delta_k \times \{0, 1\}\).

Finally, we show that \(H_\sigma\) is continuous on \(\partial \Delta_k \times \{1\}\). For an arbitrary \(x_0 \in \partial \Delta_k\), take an open neighborhood \(U \subset M\) of \(H(x_0, 1)\). By continuity, there exists \(\varepsilon_1 > 0\) such that for all \(x \in \Delta_k\) and \(t \in [0, u(x)]\), we have that \(|(x, t) - (x_0, 1)| < \varepsilon_1\) implies \(H(x, t/u(x)) \in U\). Furthermore, since \(I\) is compact and \(\hat{H}(x_0, t) = \hat{H}(x_0, 0) = H(x_0, 1) = H_\sigma(x_0, 1) \in U\) for all \(t \in I\), it follows that there exists \(\varepsilon_2 > 0\) such that for all \(x \in \Delta_k\), we have that \(|x - x_0| < \varepsilon_2\) implies \(\hat{H}(x, t) \in U\) for all \(t \in I\), and in particular, \(\hat{H}(x, \frac{t-u(x)}{1-u(x)}) \in U\). Thus, \(|(x, t) - (x_0, 1)| < \min(\varepsilon_1, \varepsilon_2)\) implies \(H_\sigma(x, t) \in U\), which shows that \(H_\sigma\) is continuous at \((x_0, 1)\) as needed.

We have that \(H_\sigma\) is a homotopy from \(H_0(\cdot, 0) = H(\cdot, 0) = \sigma\) to the \(k\)-simplex defined by \(H_\sigma(x, 1) = \hat{H}(x, 1) = \hat{\sigma}(x)\) on \(x \in \Delta_k^0\) and \(H_\sigma(x, 1) = H(x, 1) = \sigma'(x) = \hat{H}(x, t) = \hat{\sigma}(x)\) on \(x \in \partial \Delta_k\) (where \(t\) is arbitrary in the latter equality). This shows that \(H_\sigma\) is a homotopy from \(\sigma\) to the smooth \(k\)-simplex \(\hat{\sigma}\), verifying property (1). Moreover, we have \(H_\sigma = H\) on \(\partial \Delta_k \times I\) by construction, verifying property (2).

Equipped with the above lemma, we can achieve our originally stated goal of proving that smooth singular homology is isomorphic to singular homology.

**Theorem 2.4.** The map \(i_*: H_k^\infty(M) \to H_k(M)\) is an isomorphism.

**Proof.** Let \(i_0: \Delta_k \hookrightarrow \Delta_k \times I\) be defined by \(x \mapsto (x, 0)\), and similarly, let \(i_1: \Delta_k \hookrightarrow \Delta_k \times I\) be defined by \(x \mapsto (x, 1)\). Note that these are clearly smooth embeddings, and that for a \(k\)-simplex \(\sigma\), the homotopy \(H_\sigma\) defined in Lemma 2.3 satisfies that \(H_\sigma \circ i_1\) is a smooth \(k\)-simplex by property (1). This allows us to define a homomorphism \(s: C_k(M) \to C_k^\infty(M)\) by sending \(\sigma \mapsto H_\sigma \circ i_1\) and extending linearly. One observes that \(s\) is a chain map, since

\[
s \circ \partial(\sigma) = s \left( \sum_{i=0}^{k} (-1)^i \sigma \circ F_{i,k} \right) = \sum_{i=0}^{k} (-1)^i H_\sigma \circ F_{i,k} \circ i_1 = \sum_{i=0}^{k} (-1)^i H_\sigma \circ (F_{i,k} \times \text{Id}_I) \circ i_1
\]

\[
= \sum_{i=0}^{k} (-1)^i H_\sigma \circ i_1 \circ F_{i,k} = \partial(H_\sigma \circ i_1) = \partial \circ s(\sigma),
\]

where we have used property (2) of \(H_\sigma\). Thus, \(s\) induces a homomorphism \(s_*: H_k(M) \to H_k^\infty(M)\), which we will show to be the inverse of \(i_*: H_k^\infty(M) \to H_k(M)\). Note first that property (3) of \(H_\sigma\) shows that \(s \circ i = \text{Id}_{C_k^\infty(M)}\), so that in particular, \(s_* \circ i_* = \text{Id}_{H_k(M)}\).

For the converse result that \(i_* \circ s_* = \text{Id}_{H_k^\infty(M)}\), it suffices to show that a chain homotopy \(h: C_k(M) \to C_{k+1}(M)\) such that \(\partial \circ h + h \circ \partial = \iota \circ s - \text{Id}_{C_k(M)}\) exists. For a map from \(\Delta_k\) to some subset of a Euclidean space \(V\), denote the affine transformation sending \(\delta_i\) to \(v_i \in V\) by \(A(v_0, \ldots, v_k)\). Then, we define

\[
G_{i,k} := A((\delta_0, 0), \ldots, (\delta_i, 0), (\delta_i, 1), \ldots, (\delta_k, 1)),
\]

which is clearly a well-defined map \(G_{i,k} : \Delta_{k+1} \to \Delta_k \times I\). One can compute that the maps \(G_{i,k} \circ F_{i,k+1}, G_{i-1,k} \circ F_{i,k+1}: \Delta_k \to \Delta_k \times I\) coincide as

\[
G_{i,k} \circ F_{i,k+1} = G_{i-1,k} \circ F_{i,k+1} = A((\delta_0, 0), \ldots, (\delta_i-1, 0), (\delta_i, 1), \ldots, (\delta_k, 1)).
\]
Also, one can compute that the map \((F_{i,k} \times \text{Id}_I) \circ G_{j,k-1} : \Delta_k \to \Delta_k \times I\) is given by
\[
(F_{i,k} \times \text{Id}_I) \circ G_{j,k-1} = \begin{cases} 
G_{j+1,k} \circ F_{i,k+1} & \text{if } i \leq j, \\
G_{j,k} \circ F_{i+1,k+1} & \text{if } i > j.
\end{cases}
\]

Define \(h : C_k(M) \to C_{k+1}(M)\) by
\[
h(\sigma) := \sum_{i=0}^k (-1)^i H_\sigma \circ G_{i,k}.
\]

We see that
\[
h \circ \partial(\sigma) = \partial \left( \sum_{i=0}^k (-1)^i H_\sigma \circ G_{i,k} \right) = \sum_{i=0}^k \sum_{j=0}^{k-1} (-1)^{i+j} H_\sigma \circ F_{i,k} \circ G_{j,k-1}
\]
\[
= \sum_{i=0}^k \sum_{j=0}^{k-1} (-1)^{i+j} H_\sigma \circ (F_{i,k} \times \text{Id}_I) \circ G_{j,k-1}
\]
\[
= \left( \sum_{0 \leq i < j \leq k-1} (-1)^{i+j} H_\sigma \circ G_{j+1,k} \circ F_{i,k+1} \right) + \left( \sum_{0 \leq j < i \leq k-1} (-1)^{i+j} H_\sigma \circ G_{j,k} \circ F_{i+1,k+1} \right),
\]
where we have used property (2) of Lemma \(\text{(2.3, 2.1, and 2.2)}.\) Also, we have
\[
\partial \circ h(\sigma) = \partial \left( \sum_{i=0}^k (-1)^i H_\sigma \circ G_{i,k} \right) = \sum_{i=0}^k \sum_{j=0}^{k+1} (-1)^{i+j} H_\sigma \circ G_{i,k} \circ F_{j,k+1}
\]
\[
= \left( \sum_{0 \leq i < j-1} \sum_{j=0}^{k+1} (-1)^{i+j} H_\sigma \circ G_{i,k} \circ F_{j,k+1} \right) - \left( \sum_{j=1}^{k+1} H_\sigma \circ G_{j-1,k} \circ F_{j,k+1} \right)
\]
\[
+ \left( \sum_{j=1}^{k+1} H_\sigma \circ G_{j,k} \circ F_{j,k+1} \right) + \left( \sum_{0 \leq j < k} (-1)^{i+j} H_\sigma \circ G_{i,k} \circ F_{j,k+1} \right),
\]
where we have split into subsums \(i < j-1, (j \neq 0 \text{ and } i = j-1), (j \neq k \text{ and } i = j), \text{ and } i > j.\) Note however that the first and last subsums of the above together equal the negative of the expression for \(h \circ \partial(\sigma).\) Also, by applying \(\text{(2.1)}\), one gets that the second and fourth subsums cancel out to
\[
\partial \circ h(\sigma) = H_\sigma \circ G_{0,k} \circ F_{0,k+1} - H_\sigma \circ G_{k,k} \circ F_{k+1,k+1}
\]
\[
= H_\sigma \circ A((\delta_0, 0), \ldots, (\delta_k, 0)) - H_\sigma \circ A((\delta_0, 1), \ldots, (\delta_k, 1)) = H_\sigma \circ i_1 - H_\sigma \circ i_0.
\]
Thus, overall we have
\[
h \circ \partial(\sigma) + \partial \circ h(\sigma) = H_\sigma \circ i_1 - H_\sigma \circ i_0 = \iota(\sigma) - \sigma = s(\iota \circ s(\sigma)) - \sigma,
\]
as needed. \(\square\)

3. The Main Proof

Consider a closed \(k\)-form \(\omega \in \Omega^k(M)\) and a smooth \(k\)-simplex \(\sigma : \Delta_k \to M.\) Define the integral of \(\omega\) over \(\sigma\) by
\[
\int_\sigma \omega := \int_{\Delta_k} \sigma^* \omega,
\]
considering $\Delta_k \subset \mathbb{R}^k$ as the domain of integration. Naturally, for a smooth chain $c = \sum_{i=1}^{\ell} c_i \sigma_i \in C^\infty_k(M)$ we define the integral of $\omega$ over $\sigma$ by

$$\int_c \omega := \sum_{i=1}^{\ell} c_i \int_{\sigma_i} \omega.$$ 

One can prove an analogue of Stokes’ theorem for this notion of integral.

**Theorem 3.1 (Stokes).** For $c \in C^\infty_k(M)$ and $\omega \in \Omega^{k-1}(M)$, we have

$$\int_{\partial c} \omega = \int_c d\omega.$$ 

**Proof.** This is a straightforward derivation from the usual Stokes’ theorem. See [2, Theorem 18.12] for a detailed proof. □

As mentioned in the introduction, the above version of Stokes’ theorem allows us to define a map $\mathcal{I} : H^k_{dR}(M) \to H^k(M; \mathbb{R})$. First, note that by the universal coefficient theorem, the natural map $H^k(M; \mathbb{R}) \to \text{Hom}(H_k(M), \mathbb{R})$ is an isomorphism, which allows us to define $\mathcal{I}$ as follows. Let $[\omega] \in H^k_{dR}(M)$ and $[c] \in H_k(M) \cong H^\infty_k(M)$, where throughout the remainder of this exposition, we take $c$ to be smooth by appealing to Theorem 2.4. Then, define $\mathcal{I}[\omega]$ to be the cohomology class corresponding to the element of $\text{Hom}(H_k(M), \mathbb{R})$ defined by $[c] \mapsto \int_c \omega$.

Note that this map $H_k(M) \to \mathbb{R}$ is well-defined, since for another smooth $k$-cycle $c'$ that represents $[c]$, it follows from Theorem 2.4 that $c - c' = \partial c''$ for a smooth $(k+1)$-chain $c''$, so

$$\int_c \omega - \int_{c'} \omega = \int_{\partial c''} \omega = \int_{c''} d\omega = 0$$

by Theorem 3.1. Moreover, $\mathcal{I}$ is well-defined as a map from $H^k_{dR}(M)$, since for exact $\omega = d\eta$, we have again by Theorem 3.1 that

$$\int_c \omega = \int_c d\eta = \int_{\partial c} \eta = 0,$$

and linearity is clear. We name $\mathcal{I}$ the *de Rham map*, which is natural in the following way.

**Proposition 3.2.** The de Rham map $\mathcal{I}$ satisfies the following properties:

(a) For a smooth map of smooth manifolds $f : M \to N$, we have the commutative diagram

$$H^k_{dR}(N) \xrightarrow{f^*} H^k_{dR}(M) \xrightarrow{\mathcal{I}} H^k(N; \mathbb{R}) \xrightarrow{\mathcal{I}} H^k(M; \mathbb{R}).$$

(b) For open sets $U, V$ of $M$ such that $U \cup V = M$, we have the commutative diagram

$$H^{k-1}_{dR}(U \cap V) \xrightarrow{\delta} H^k_{dR}(M) \xrightarrow{\mathcal{I}} H^{k-1}(U \cap V; \mathbb{R}) \xrightarrow{\xi} H^k(M; \mathbb{R}),$$

where $\delta$ (respectively, $\xi$) are the boundary maps in the Mayer–Vietoris sequences of de Rham cohomology (respectively, of singular cohomology).
Proof. (a) For a smooth $k$-simplex $\sigma$ and $\omega \in \Omega^k(N)$, we have
\[
\mathcal{I}(f^*\omega)([\sigma]) = \int_{\sigma} F^*\omega = \int_{\Delta_k} \sigma^* f^* \omega = \int_{\Delta_k} (f \circ \sigma)^* \omega = \int_{f \circ \sigma} \omega = \mathcal{I}([\omega]) \circ f_*([\sigma]) = f^*(\mathcal{I}([\omega])([\sigma])
\]
Extend linearly to verify the desired property.

(b) Let $[\omega] \in H^{k-1}_{dR}(U \cap V)$. Appealing to the well-definedness of the Mayer–Vietoris sequence maps, choose $\eta_U \in H^{k}_{dR}(U)$ and $\eta_V \in H^{k}_{dR}(V)$ such that $\omega = \eta_U|_{U \cap V} - \eta_V|_{U \cap V}$, and in particular, $\delta([\omega])$ is equal to $d\eta_U$ on $U$ and $d\eta_V$ on $V$. Likewise, let $[b] \in H^k(M)$, where the representative chain $b$ can be chosen to be smooth and satisfy $b = b_U + b_V$ for smooth chains $b_U \in C_k(U)$ and $b_V \in C_k(V)$, so that $\partial_*([b]) = [\partial b_U] = -[\partial b_V]$, where $\partial_*$ denotes the map induced by the boundary map $\partial : C_k(M) \to C_{k-1}(U \cap V)$ on chains. We have
\[
\mathcal{I}(\delta([\omega])([b]) = \mathcal{I}(\delta([\omega]))([b_U]) + \mathcal{I}(\delta([\omega]))([b_V]) = \left(\int_{b_U} d\eta_U\right) + \left(\int_{b_V} d\eta_V\right)
\]
and
\[
\mathcal{I}([\omega])(\partial_*([b])) = \left(\int_{\partial b_U} \omega\right) - \left(\int_{\partial b_V} \omega\right) = \left(\int_{\partial b_U} \eta_U\right) - \left(\int_{\partial b_V} \eta_V\right) = \left(\int_{\partial b_U} \eta_U\right) + \left(\int_{\partial b_V} \eta_V\right)
\]
But the left integrals of both expressions are equal by Theorem 3.3 as are the right integrals. This holds for arbitrary $b$, so $\mathcal{I}(\delta([\omega])] = \mathcal{I}([\omega]) \circ \partial_* = \xi \circ \mathcal{I}([\omega])$, as needed. \hfill \Box

We are now ready to prove our main theorem. Before we do so, we introduce the following terminology. A smooth manifold $M$ is de Rham if $\mathcal{I} : H^k_{dR}(M) \to H^k(M; \mathbb{R})$ is an isomorphism for all $k \geq 0$. An open cover $\{U_i\}_{i \in I}$ of $M$ is a de Rham open cover if all finite intersections $U_{i_1} \cap \cdots \cap U_{i_r}$ are de Rham. A de Rham basis is a de Rham open cover that is also a basis. With this terminology, de Rham’s theorem can be stated concisely.

**Theorem 3.3 (de Rham).** Every smooth manifold $M$ is de Rham.

**Proof.** First, we show that every smooth manifold $M$ with a finite de Rham cover, say $\{U_1, \ldots, U_r\}$ is de Rham. We prove this by induction on $r$. The base case $r = 1$ is trivial. Suppose that $M$ has a de Rham cover $\{U, V\}$. We have the following commutative diagram of Mayer–Vietoris sequences:

\[
\begin{array}{ccc}
H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) & \longrightarrow & H^{k-1}_{dR}(U \cap V) \\
\mathcal{I} \oplus \mathcal{I} & \downarrow & \mathcal{I} \\
H^{k-1}_{dR}(U; \mathbb{R}) \oplus H^{k-1}_{dR}(V; \mathbb{R}) & \longrightarrow & H^{k-1}(U \cap V; \mathbb{R}) \\
\mathcal{I} \oplus \mathcal{I} & \downarrow & \mathcal{I} \\
H^k_{dR}(U) \oplus H^k_{dR}(V) & \longrightarrow & H^k_{dR}(U \cap V) \\
\mathcal{I} \oplus \mathcal{I} & \downarrow & \mathcal{I} \\
H^k(U; \mathbb{R}) \oplus H^k(V; \mathbb{R}) & \longrightarrow & H^k(U \cap V; \mathbb{R}).
\end{array}
\]

The commutativity of the second-to-left square follows from Proposition 3.2(b), and commutativity for the other squares of the diagram follows from Proposition 3.2(a). The inductive hypothesis implies that the first, second, fourth, and fifth vertical maps are isomorphisms, so by the five lemma, the third vertical map is also an isomorphism, as needed.

Second, we show that every smooth manifold $M$ with a de Rham basis $\{U_i\}_{i \in I}$ is de Rham. Take a smooth positive exhaustion $f : M \to \mathbb{R}$, i.e., a map such that $f^{-1}((\infty, c])$ is compact for all $c \in \mathbb{R}$. Then, for $\ell \in \mathbb{Z}_{\geq 0}$, we can define $A_{\ell} := f^{-1}([\ell, \ell + 1])$ and note that $M = \bigcup_{\ell \geq 0} A_{\ell}$. The open set $A'_{\ell} := f^{-1}((\ell - \frac{1}{2}, \ell + \frac{3}{2}))$ contains $A_{\ell}$, and can be written as a union of basis open sets $U_i$.
Since these cover \( A_\ell \), finitely many of them (the union of which we denote as \( B_\ell \)) cover \( A_\ell \). By our work above, \( B_\ell \) is de Rham. Since \( B_\ell \subseteq A_\ell \), the open sets

\[
U := \bigcup_{\text{odd } \ell \geq 0} B_\ell \quad \text{and} \quad V := \bigcup_{\text{even } \ell \geq 0} B_\ell
\]

are each a disjoint union of de Rham smooth manifolds \( B_\ell \). However, for a disjoint union \( \bigcup_{j \in J} N_j \) of countably many de Rham manifolds \( N_j \), the inclusion maps \( \iota_j : N_j \to \bigcup_{j \in J} N_j \) induce an isomorphism between the direct product of de Rham (respectively, singular) cohomology groups of \( N_j \) with the de Rham (respectively, singular) cohomology groups of \( \bigcup_{j \in J} N_j \), and it follows from Proposition \ref{prop:direct_product_isomorphism} that these isomorphisms commute with the de Rham map, showing that \( \bigcup_{j \in J} N_j \) is de Rham. This verifies that \( U \) and \( V \) are de Rham. Furthermore, \( U \cap V \) is de Rham, since it is the disjoint union of \( B_\ell \cap B_{\ell+1} \) for \( \ell \geq 0 \), and \( B_\ell \cap B_{\ell+1} \) has a finite de Rham open cover given by \( U_\ell \cap U_{\ell+1} \) such that \( U_\ell \) is one of the finitely many basis open sets chosen in defining \( B_\ell \) and \( U_{\ell+1} \), one of the finitely many basis open sets chosen in defining \( B_{\ell+1} \). Thus, \( \{ U, V \} \) is a finite de Rham cover of \( M \), so our previous work shows that \( M \) is de Rham.

We are now ready to prove the theorem. First, note that every convex open set \( U \subset \mathbb{R}^n \) is de Rham. Indeed, by the Poincaré lemma, \( H^k_{dR}(U) \) is 1-dimensional (generated by the constant function \( 1 : M \to \mathbb{R} \) of value 1) for \( k = 0 \) and trivial for \( k > 0 \), and similarly, \( H^k(U; \mathbb{R}) \cong \text{Hom}(H_k(U), \mathbb{R}) \) is 1-dimensional (generated by the dual of any singular 0-simplex \( \sigma : \Delta_0 = \{0\} \to M \), which is automatically smooth and a choice for which we fix) for \( k = 0 \) and trivial for \( k > 0 \). Then,

\[
\mathcal{J}([1])([\sigma]) = \int_{\Delta_0} \sigma^*1 = (1 \circ \sigma)(0) = 1,
\]

showing that the map \( \mathcal{J} \) is nontrivial and thus an isomorphism. Second, every open set \( U \subset \mathbb{R}^n \) is de Rham. Indeed, \( U \) has a basis of Euclidean open balls, which are convex (and thus, de Rham) and consequently, whose finite intersections are convex (and thus, de Rham). Thus, this basis is de Rham, verifying that \( U \) is de Rham. Finally, every smooth manifold \( M \) has a basis of atlas charts, and each finite intersection of such charts is diffeomorphic to an open set of \( \mathbb{R}^n \) and is thus de Rham, so the basis is de Rham. This concludes our proof that \( M \) is de Rham. \( \square \)

4. Example: de Rham Cohomology of \( S^n \)

We mention a brief application of de Rham’s theorem regarding differential forms on \( S^n \). Since \( S^n \) is a connected orientable \( n \)-manifold, there exists a choice of (nonvanishing) volume form of \( S^n \); this is a closed, but non-exact \( n \)-form, and thus, its cohomology class generates the top de Rham cohomology group \( H^n_{dR}(S^n) \cong \mathbb{R} \). Likewise, the cohomology class of the constant function \( 1 : S^n \to \mathbb{R} \) of value 1 generates the 0th de Rham cohomology group \( H^0_{dR}(S^n) \cong \mathbb{R} \). One can then ask the question: for \( 0 < k < n \), are there closed but non-exact \( k \)-forms on \( S^n \)? By de Rham’s theorem, we know that \( \mathcal{J} : H^k_{dR}(S^n) \to H^k(S^n; \mathbb{R}) \cong 0 \) is an isomorphism, showing that all closed \( k \)-forms on \( S^n \) are exact.

References


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