1. Introduction

A central motivating question in analytic number theory is the following:

For a given $x > 0$, how many prime numbers are there less than $x$? More generally, how many prime numbers less than $x$ satisfy a given condition – e.g., are contained in a given arithmetic progression $\equiv a \pmod{q}$?

The last question is trivial when $a$ and $q$ are not relatively prime, but when $(a, q) = 1$ does hold, one can seek an asymptotic formula for the counting function

$$\pi(x; q, a) := \sum_{p \leq x \atop p \equiv a \pmod{q}} 1.$$  

In fact, we can reduce the problem of estimating $\pi(x; q, a)$ with that of estimating Chebyshev’s $\vartheta$-function for the arithmetic progression $\equiv a \pmod{q}$, defined by

$$\vartheta(x; q, a) := \sum_{p \leq x \atop p \equiv a \pmod{q}} \log p.$$  

Indeed, we can use a Riemann-Stieltjes integral and integrate by parts to get

$$\pi(x; q, a) = \int_{2}^{x} \frac{1}{\log t} d\vartheta(t; q, a) = \frac{\vartheta(x; q, a)}{\log x} + \int_{2}^{x} \frac{\vartheta(t; q, a)}{t \log^2 t} dt.$$  

We can further reduce to estimating a sum that is close to $\vartheta(x; q, a)$ for all intents and purposes: Chebyshev’s $\psi$-function for the arithmetic progression $\equiv a \pmod{q}$, defined by

$$\psi(x; q, a) = \sum_{n \leq x \atop n \equiv a \pmod{q}} \Lambda(n),$$  

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where \( \Lambda : \mathbb{Z}_{>0} \to \mathbb{R} \) is the von Mangoldt function defined by
\[
\Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^r \text{ for some prime } p, \\
0 & \text{otherwise}.
\end{cases}
\]

It is more natural to deal with \( \psi(x; q, a) \) than it is to deal with \( \vartheta(x; q, a) \) or \( \pi(x; q, a) \). To see this, we note that the orthogonality of characters (Proposition 2.3 in the next section) allows us to write
\[
(1.2) \quad \psi(x; q, a) = \sum_{n \leq x} \Lambda(n) \mathbb{1}_{q, a}(n) = \sum_{n \leq x} \Lambda(n) \left( \frac{1}{\phi(q)} \sum_{\chi \in X_q} \chi(n) \right) = \frac{1}{\phi(q)} \sum_{\chi \in X_q} \chi(a) \psi(x, \chi),
\]
where \( X_q \) denotes the set of Dirichlet characters with period \( q \), the indicator function \( \mathbb{1}_{q, a}(n) \) for the arithmetic progression \( \equiv a \pmod{q} \) is defined by
\[
(1.3) \quad \mathbb{1}_{q, a}(n) := \begin{cases} 
1 & \text{if } n \equiv a \pmod{q}, \\
0 & \text{otherwise},
\end{cases}
\]
and \( \psi(x, \chi) \) is defined by
\[
\psi(x, \chi) := \sum_{n \leq x} \Lambda(n) \chi(n).
\]

We can then exploit the appearance of \( \Lambda(n) \) in the logarithmic derivative of \( L(s, \chi) \) (see Definition A.1 in the appendix) to approximate \( \psi(x, \chi) \) by the integral
\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left( -\frac{L'(s, \chi)}{L(s, \chi)} \right) x^s ds = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \sum_{n=1}^{\infty} \frac{\chi(n) \Lambda(n)}{n^s} \cdot x^s ds
\]
\[
= \sum_{n=1}^{\infty} \chi(n) \Lambda(n) \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{s} \left( \frac{x}{n} \right)^s ds,
\]
since setting \( c > 0 \) makes the inner integral on the right-hand side,
\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{s} \left( \frac{x}{n} \right)^s ds,
\]
approximate the indicator function\( \mathbb{1} >_1(\frac{x}{n}) \), where
\[
(1.4) \quad \mathbb{1} >_1(y) := \begin{cases} 
0 & \text{if } 0 < y < 1, \\
\frac{1}{2} & \text{if } y = 1, \\
1 & \text{if } y > 1.
\end{cases}
\]

Thus, having the von Mangoldt function in the expression for \( \psi(x; q, a) \) allows us to use the analytic properties of \( L(s, \chi) \) to asymptotically study \( \psi(x; q, a) \) and by extension, \( \pi(x; q, a) \).

The best currently known estimate for \( \psi(x; q, a) \) that is uniform across a range of values for \( q \) is the following, proven by Walfisz [23] as an application of a theorem of Siegel [14] regarding the hypothetical exceptional zero of \( L(s, \chi) \) for real nonprincipal primitive \( \chi \).

**Theorem 1.1 (Siegel-Walfisz).** For any fixed \( A > 0 \), we have
\[
\psi(x; q, a) = \frac{x}{\phi(q)} + O \left( x e^{-c \sqrt{\log x}} \right)
\]
for all \( x \geq 2 \) and \( 1 \leq q \leq (\log x)^A \), where \( c > 0 \) and the absolute constant depends only on \( A \).

There is also a variant of the above result for the more fundamental sum \( \psi(s, \chi) \).

\( ^1 \)The proof of this fact is similar to that of (4.3). For a reference, see [3, p.105–6]
**Theorem 1.2.** For any fixed $A > 0$, we have

$$\psi(s, \chi) = 1_{\chi_0}(\chi)x + O\left(xe^{-c\sqrt{\log x}}\right)$$

for all $x \geq 2$, $1 \leq q \leq (\log x)^A$, and $\chi \in X_q$, where $c > 0$,

$$1_{\chi_0}(\chi) := \begin{cases} 1 & \text{if } \chi \text{ is principal}, \\ 0 & \text{otherwise}, \end{cases}$$

and the absolute constant depends only on $A$.

For a detailed discussion of these results and their proofs, we direct the reader to [3 §22].

The unconditional error term $O(xe^{-c\sqrt{\log x}})$ is quite bad; it is worse than $x^\varepsilon$ for every $\varepsilon < 1$. This is especially unsettling, since we expect the error term to actually behave as $O(x^{\frac{1}{2}}(\log x)^2)$, the error term conditional under the Generalized Riemann Hypothesis (stated in Conjecture A.7 and heretofore denoted as GRH). Indeed, Titchmarsh [17] has shown that if GRH holds, then

$$\psi(x, \chi) = 1_{\chi_0}(\chi)x + O(x^{\frac{1}{2}}(\log qx)^2),$$

for all $\chi \in X_q$, from which one gets

$$(1.5) \quad \psi(x; q, a) = \frac{x}{\phi(q)} + O(x^{\frac{1}{2}}(\log x)^2)$$

uniformly for all $q$.

However, one can get an improved error term for $\psi(x; q, a)$ by averaging over a range of $q$. More precisely, denote the error term for $\psi(x; q, a)$ by

$$E(x; q, a) := \psi(x; q, a) - \frac{x}{\phi(q)}.$$ 

Furthermore, let

$$E(x; q) := \max\{|E(x; q, a)| : (a, q) = 1\}$$

and

$$E^*(x; q) := \max_{y \leq x} E(y; q).$$

A result proven independently by Bombieri [2] and A. I. Vinogradov [20] states that averaging $E^*(x; q)$ over a range of $q$ gives an asymptotic order of growth that is more in line with the error term $[1.5]$ that we expect from GRH.

**Theorem 1.3** (Bombieri–Vinogradov). Fix $A > 0$. For all $x \geq 2$ and all $Q \in [x^{\frac{1}{2}}(\log x)^{-A}, x^{\frac{1}{2}}]$, we have

$$\sum_{q \leq Q} E^*(x; q) \ll x^{\frac{1}{2}}Q(\log x)^5,$$

where the absolute constant depends only on $A$.

We note that this bound is trivial for $Q$ sufficiently large. Indeed, for $y \geq 1$ and $q \in \mathbb{Z}_{>0}$, there are at most $\frac{y}{q} + 1$ positive integers $n$ such that $n \leq y$ and $n \equiv a \pmod{q}$. Thus, applying the triangle inequality to (1.1) yields $\psi(y; q, a) \ll \frac{x \log x}{q}$ for $y \leq x$ and $q \leq x^{\frac{1}{2}}$. Furthermore, since $\phi(q) \gg \frac{q}{\log(q+1)}$ (for a reference, see [7]), we have $\frac{x \log(q+1)}{q} \ll \frac{x \log x}{q}$ for $y \leq x$ and $q \leq x^{\frac{1}{2}}$. This overall shows that in the statement of Theorem 1.3, each term $E^*(x; q)$ occurring in the sum is bounded in magnitude by $\frac{x \log x}{q}$. This gives us a trivial bound under the hypotheses of the theorem:

$$(1.6) \quad \sum_{q \leq Q} E^*(x; q) \ll \sum_{q \leq Q} \frac{x \log x}{q} \ll x(\log x)(\log Q) \ll x(\log x)^2.$$
Thus, the statement of Theorem 1.3 is trivial for $Q \gg x^{\frac{3}{2}}(\log x)^{-3}$. Also, given the lower bound $x^{\frac{3}{2}}(\log x)^{-A} \leq Q$, we observe that the bound given by Theorem 1.3 is $\gg x(\log x)^{5-A}$. It follows that Theorem 1.3 is nothing more than a reduction in the exponent of $\log x$ in the trivial bound (1.6).

However, this reduction turns out to be essentially of the same strength as the estimate from assuming GRH and thus critical in applications, such as the work of Zhang [25] and Goldston et al. [6] on bounded prime gaps.

We first build up the basic theory of Dirichlet characters in Section 2, which will be necessary for using $\psi(x,\chi)$ to investigate $\psi(x; q, a)$ as in (1.2), and ultimately in our proof of Theorem 1.3. A natural follow-up topic is a discussion of the associated Dirichlet $L$-functions, which are essential for motivating our line of inquiry – in particular, they comprise the central methodology used in the proof of the Siegel-Walfisz theorem, and of any modern prime-number-theorem-type result – but are not needed for our immediate purposes and thus included as an appendix. We then proceed to introduce in Section 3 the large sieve, which will be the key tool that we use for our main proof of Theorem 1.3 in Section 4.

2. Dirichlet Characters

Definition 2.1. Let $q \in \mathbb{Z}_{>0}$. A Dirichlet character of period $q$ is a function $\chi : \mathbb{Z} \to \mathbb{C}$ induced by a homomorphism $\mathbb{Z}/q\mathbb{Z}^\times \to \mathbb{C}^\times$ (i.e, an irreducible character of $(\mathbb{Z}/q\mathbb{Z})^\times$) in the following way:

$$
\xymatrix{
\mathbb{Z}/q\mathbb{Z}^\times \ar[r]^-{\chi} & \mathbb{C}^\times \\
\mathbb{Z} \ar[u]^-{\times q} \ar[r] & \mathbb{Z}/q\mathbb{Z} \ar[u]^-{\chi}
}
$$

Equivalently [3, p.2], it is a function $\chi : \mathbb{Z} \to \mathbb{C}$ which is

1. periodic modulo $q$, i.e, $\chi(n+q) = \chi(n)$ for all $n \in \mathbb{Z}$,
2. totally multiplicative, i.e, $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$,
3. satisfies $\chi(1) = 1$, and
4. satisfies $\chi(n) = 0$ for all $n \in \mathbb{Z}$ such that $(n, q) > 1$.

As stated before, we denote the set of Dirichlet characters of period $q$ (corresponding to the set of irreducible characters of $(\mathbb{Z}/q\mathbb{Z})^\times$) by $X_q$. Since $(\mathbb{Z}/q\mathbb{Z})^\times$ is a finite abelian group, the irreducible characters of $(\mathbb{Z}/q\mathbb{Z})^\times$ are all one-dimensional and form a group (under multiplication) that is isomorphic to $(\mathbb{Z}/q\mathbb{Z})^\times$. It immediately follows that $X_q$ is a group in the same way. Note that the complex conjugate of a Dirichlet character of period $q$ is a Dirichlet character of period $q$, and also that $|X_q| = |(\mathbb{Z}/q\mathbb{Z})^\times| = \phi(q)$.

Example 2.2. The principal character of period $q$, denoted by

$$
\chi_0(n) := \begin{cases} 
1 & \text{if } (n, q) = 1, \\
0 & \text{otherwise},
\end{cases}
$$

is a Dirichlet character of period $q$. For $q = 1$ or 2, the principal character is clearly the only Dirichlet character. However, this is not true for $q > 2$. For $q = 3$, there are precisely $\phi(3) = 2$ distinct Dirichlet characters.

<table>
<thead>
<tr>
<th>$n \pmod{3}$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0(n)$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1(n)$</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

For $q = 4$, there are also precisely two distinct Dirichlet characters, since $\phi(4) = 2$. 
Since $\chi = 1$, there are precisely $\phi(5) = 4$ distinct Dirichlet characters. Note that unlike in the previous cases, we have complex (i.e, non-real) characters, $\chi_1$ and $\chi_3$, which are complex conjugates.

<table>
<thead>
<tr>
<th>$n \pmod{5}$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0(n)$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1(n)$</td>
<td>0</td>
<td>1</td>
<td>$i$</td>
<td>$-i$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\chi_2(n)$</td>
<td>0</td>
<td>1</td>
<td>$-1$</td>
<td>$-1$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3(n)$</td>
<td>0</td>
<td>1</td>
<td>$-i$</td>
<td>$i$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

In the above examples, one can observe that for any $a \in \{0, 1, \ldots, q-1\}$ (mod $q$), the function $1_{q,a}(n)$ (defined in the introduction) can be written as a linear combination of the Dirichlet characters of period $q$. For example,

$$1_{5,2}(n) = \frac{1}{4} \chi_0(n) - \frac{i}{4} \chi_1(n) - \frac{1}{4} \chi_2(n) + \frac{i}{4} \chi_3(n).$$

In fact, by the orthogonality of characters, this property holds for any period $q$.

**Proposition 2.3 (Orthogonality).** For any $q \in \mathbb{Z}_{>0}$ and $a \in \{0, \ldots, q-1\}$ (mod $q$), we have

$$1_{q,a}(n) = \frac{1}{\phi(q)} \sum_{\chi \in X_q} \overline{\chi(a)} \chi(n)$$

**Proof.** Since $(\mathbb{Z}/q\mathbb{Z})^\times$ is an abelian group, every element comprises its own conjugacy class. Thus, the claim follows immediately from column orthogonality.

It is possible for a Dirichlet character to be induced by another Dirichlet character with a smaller period. We formalize this notion as follows.

**Definition 2.4.** Let $\chi \in X_q$. For $q_1 \in \mathbb{Z}_{>0}$, we say that $\chi(n)$ restricted by $(n, q) = 1$ has period $q_1$ if it has the property that $\chi(n + q_1) = \chi(n)$ for all $n$ such that $(n, q) = (n, q_1) = 1$. Furthermore, let $c(\chi)$ denote the conductor of $\chi$, defined to be the least $q_1 \in \mathbb{Z}_{>0}$ such that $\chi(n)$ restricted by $(n, q) = 1$ has period $q_1$. We say that $\chi$ is primitive if $c(\chi) = q$, and imprimitive precisely if $c(\chi) < q$.

**Example 2.5.** The principal character of period 1 is trivially primitive. On the other hand, any principal character of period $q > 1$ is imprimitive, since its conductor is clearly 1.

For another example, let $\chi \in X_{15}$ be the Dirichlet character given by

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi(n)$</td>
<td>0</td>
<td>1</td>
<td>$i$</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>$i$</td>
<td>$-i$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-i$</td>
<td>$-1$</td>
<td></td>
</tr>
</tbody>
</table>

Since $\chi(1) = \chi(11)$, $\chi(2) = \chi(7)$, $\chi(4) = \chi(14)$, and $\chi(8) = \chi(13)$, we have that $\chi(n)$ restricted by $(n, 15) = 1$ has period 5. It is immediate to check that this is the least such period, i.e, $c(\chi) = 5$. Thus, $\chi$ is imprimitive.

For $q_1 \in \mathbb{Z}_{>0}$ such that $\chi(n)$ restricted by $(n, q) = 1$ has period $q_1$, we have $c(\chi) \mid q_1$. Indeed, $q_2 := (c(\chi), q_1)$ also satisfies the property that $\chi(n)$ restricted by $(n, q) = 1$ has period $q_2$, but since $c(\chi)$ is the least period for which this is true, we necessarily have $q_2 = c(\chi)$, i.e, $c(\chi) \mid q_1$.

As we originally set out to do, we will now show that every imprimitive character is induced by a primitive character with a smaller period.
Proposition 2.6. Let $\chi \in X_q$. There exists a unique Dirichlet character $\chi_1 \in X_{c(\chi)}$ that induces $\chi$, i.e., such that

$$\chi(n) = \begin{cases} \chi_1(n) & \text{if } (n, q) = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Furthermore, $\chi_1$ is primitive.

Proof. Such a Dirichlet character $\chi_1$ satisfies $\chi_1(n) = \chi_1(n + tc(\chi)) = \chi(n + tc(\chi))$ for all $n, t \in \mathbb{Z}$ such that $(n, c(\chi)) = 1$, since $\chi_1$ has period $c(\chi)$. This proves uniqueness.

We prove existence by construction. For each $n \in \mathbb{Z}$ satisfying $(n, c(\chi)) = 1$, we choose $t \in \mathbb{Z}$ such that $(n + tc(\chi), q) = 1$, which allows us to set $\chi_1(n) := \chi(n + tc(\chi))$. This process completely defines $\chi_1 \in X_{c(\chi)}$, provided that such an integer $t$ exists for each $n$ with $(n, c(\chi)) = 1$. It suffices to show that given such an integer $n$, there exists $t$ such that $p \nmid n + tc(\chi)$ for every $p \in P$, where $P$ is the set of primes that divide $q$ but not $c(\chi)$. But $P$ is finite, and for each $p \in P$, the congruence $n + tc(\chi) \equiv 1 \pmod{p}$ has a (unique) solution $t \pmod{p}$. Thus, the Chinese remainder theorem states that there exists an integer $t$ such that $n + tc(\chi) \equiv 1 \pmod{p}$ for every $p \in P$, which ensures that $p \nmid n + tc(\chi)$ for every $p \in P$, as needed.

It is standard to check that $\chi_1$ is a Dirichlet character of period $c(\chi)$. It remains to show that $\chi_1$ is primitive. Suppose that $q_2 \in \mathbb{Z}_{>0}$ satisfies the property that $\chi_1$ restricted by $(n, c(\chi)) = 1$ has period $q_2$. Since $(n, q) = 1 \implies (n, c(\chi)) = 1$, it follows that $\chi(n)$ restricted by $(n, q) = 1$ has period $q_2$, which must be greater than or equal to $c(\chi)$. Thus, $q_2 \geq c(\chi)$, which shows that $\chi_1$ is primitive. \qed

Example 2.7. It is easy to check that the Dirichlet character $\chi \in X_{15}$ from Example 2.5 (with conductor $c(\chi) = 5$) is induced by the primitive character $\chi_1 \in X_5$ defined in Example 2.2.

The following is a useful criterion for ascertaining that a given positive integer $q_1$ is a period of $\chi \in X_q$ restricted by $(n, q) = 1$.

Lemma 2.8. Let $\chi \in X_q$ and $q_1 \in \mathbb{Z}_{>0}$. $\chi(n)$ restricted by $(n, q) = 1$ has period $q_1$ if and only if $\chi(n) = 1$ for all $n \in \mathbb{Z}$ such that $n \equiv 1 \pmod{q_1}$ and $(n, q) = 1$.

Proof. The forward direction is immediate, so we now prove the backward direction. Let $q_2 = (q, q_1)$. There exist integers $x, y$ such that $xq + yq_1 = q_2$. For any integer $n$ such that $n \equiv 1 \pmod{q_2}$ and $(n, q) = 1$, we can write $n = 1 + aq_2$ for some integer $a$, which yields $n = 1 + a(xq + yq_1) = 1 + aq_1 \pmod{q}$. Thus, $\chi(1 + qyq_1) = \chi(n) \neq 0$, which gives us that $(1 + qyq_1, q) = 1$. Furthermore, by our hypothesis, $\chi(1 + qyq_1) = 1$. We have shown that $\chi(n) = 1$ for all $n \in \mathbb{Z}$ such that $n \equiv 1 \pmod{q_2}$ and $(n, q) = 1$, i.e., that the hypothesis holds even when we replace $q_1$ with $q_2$.

Let $h, k$ be integers such that $(h, q) = (k, q) = 1$ and $h \equiv k \pmod{q_2}$. Viewing $h, k$ as elements of $(\mathbb{Z}/q\mathbb{Z})^\times$, there exists a unique element $m \in (\mathbb{Z}/q\mathbb{Z})^\times$ such that $h \equiv mk \pmod{q}$, which implies that $h \equiv mk \pmod{q_2}$ since $q_2 \mid q$. Since $(k, q) = 1$, it follows that $m \equiv 1 \pmod{q_2}$, and thus $\chi(m) = 1$. This shows that $\chi(h) = \chi(mk) = \chi(m)\chi(k) = \chi(k)$. Thus, $\chi(n)$ restricted by $(n, q) = 1$ has period $q_2$, and since $q_2 \mid q_1$, $\chi(n)$ restricted by $(n, q) = 1$ has period $q_1$. \qed

So far, we have discussed the properties of Dirichlet characters, which correspond precisely to the characters of the multiplicative group of $(\mathbb{Z}/q\mathbb{Z})^\times$. In a similar vein, we can also consider the characters of the additive group $\mathbb{Z}/q\mathbb{Z}$. For $x \in \mathbb{R}$, let $e(x) := \exp(2\pi ix)$. Note then that the characters of the additive group $\mathbb{Z}/q\mathbb{Z}$ are precisely given by

$$n \mapsto e\left(\frac{mn}{q}\right), \quad m \in \{0, 1, \ldots, q-1\}.$$
Definition 2.9. For \( \chi \in X_q \), define the Gauss sum of \( \chi \) by

\[
\tau(\chi) := \sum_{m \in \mathbb{Z}/q\mathbb{Z}} \chi(m) e\left(\frac{m}{q}\right).
\]

Suppose that \((n, q) = 1\), i.e., that \(n\) is invertible in \(\mathbb{Z}/q\mathbb{Z}\). Then,

\[
\chi(n) \tau(\chi) = \chi(n) \sum_{m \in \mathbb{Z}/q\mathbb{Z}} \chi(m) e\left(\frac{m}{q}\right) = \sum_{m \in \mathbb{Z}/q\mathbb{Z}} \chi(n^{-1}m) e\left(\frac{m}{q}\right) = \sum_{h \in \mathbb{Z}/q\mathbb{Z}} \chi(h) e\left(\frac{hn}{q}\right).
\]

This gives us our desired expression for \(\chi(n)\) in terms of exponentials, so long as \(\tau(\chi) \neq 0\). In fact, if \(\chi\) is primitive, we can show that the equality

\[
\chi(n) \tau(\chi) = \sum_{h \in \mathbb{Z}/q\mathbb{Z}} \chi(h) e\left(\frac{hn}{q}\right)
\]

holds for any \(n\).

Lemma 2.10. Let \(\chi \in X_q\) be primitive. For any \(n \in \mathbb{Z}/q\mathbb{Z}\) such that \((n, q) > 1\), we have

\[
\sum_{h \in \mathbb{Z}/q\mathbb{Z}} \chi(h) e\left(\frac{hn}{q}\right) = 0.
\]

In particular, the equality \((2.2)\) holds for all \(n\).

Proof. Let \(a = \frac{q}{(a, q)}\). We can write the elements \(h \in \mathbb{Z}/q\mathbb{Z}\) as \(h = ab + r\) for \(b \in \{0, 1, \ldots, (n, q) - 1\}\) and \(r \in \{0, 1, \ldots, a - 1\}\), which yields

\[
\sum_{h \in \mathbb{Z}/q\mathbb{Z}} \chi(h) e\left(\frac{hn}{q}\right) = \sum_{r=0}^{a-1} \sum_{b=0}^{(n, q)-1} \chi(ab + r) e\left(\frac{(ab + r)n}{q}\right) = \sum_{r=0}^{a-1} e\left(\frac{rn}{q}\right) \sum_{b=0}^{(n, q)-1} \chi(ab + r).
\]

Note that \(\chi(ab + r)\) is periodic with period \((n, q)\), which allows us to rewrite the inner sum as

\[
\sum_{b=0}^{(n, q)-1} \chi(ab + r) = \frac{(n, q)}{q} \sum_{b=0}^{q-1} \chi(ab + r) = \frac{(n, q)}{q} \sum_{b \in \mathbb{Z}/q\mathbb{Z}} \chi(ab + r).
\]

Thus, it suffices to show that \(S := \sum_{b \in \mathbb{Z}/q\mathbb{Z}} \chi(ab + r)\) vanishes.

For \(c \in \mathbb{Z}/q\mathbb{Z}\), the congruence \(ax + r \equiv c \pmod{q}\) is solvable if and only if \((a, q) \mid c - r\), and if this is true, the congruence is equivalent to

\[
\frac{a}{(a, q)} x \equiv \frac{c - r}{(a, q)} \pmod{\frac{q}{(a, q)}},
\]

which has a unique solution modulo \(\frac{q}{(a, q)}\), since \((\frac{a}{(a, q)}, \frac{q}{(a, q)}) = 1\). Thus, as \(x\) ranges through \(\mathbb{Z}/q\mathbb{Z}\), \(ax + r\) ranges through precisely the congruence classes modulo \(q\) that are congruent to \(r\pmod{(a, q)}\), and each such congruence class occurs precisely \((a, q)\) times. Thus,

\[
S = \sum_{b \in \mathbb{Z}/q\mathbb{Z}} \chi(ab + r) = \frac{(a, q)}{q} \sum_{d \in \mathbb{Z}/q\mathbb{Z}} \chi(d)
\]

Let \(m \in (\mathbb{Z}/q\mathbb{Z})^\times\) such that \(m \equiv 1 \pmod{(a, q)}\). Then,

\[
\chi(m)S = (a, q) \sum_{d \in \mathbb{Z}/q\mathbb{Z}} \chi(m) \chi(d) = (a, q) \sum_{d \in \mathbb{Z}/q\mathbb{Z}} \chi(md) = (a, q) \sum_{c \in \mathbb{Z}/q\mathbb{Z}} \chi(c) = S
\]
Suppose for the sake of a contradiction that $S \neq 0$. Then, the above shows that $\chi(m) = \chi(m) = 1$ for all $m \in (\mathbb{Z}/q\mathbb{Z})^*$ such that $m \equiv 1 \pmod{(a,q)}$. Since $\chi$ is primitive, it follows from Lemma 2.8 that $q = (a,q)$, i.e, $q \mid a$. But this is impossible, since $(n,q) > 1$. Thus, $S = 0$, as needed. □

For primitive $\chi$, we can also show that $\tau(\chi) \neq 0$, which means we can divide both sides of the equality (2.2) by $\tau(\chi)$ to get our desired identity expressing $\chi$ in terms of exponentials.

**Lemma 2.11.** Let $\chi \in X_q$ be primitive. Then

$$|\tau(\chi)| = q^{\frac{1}{2}}.$$ 

**Proof.** Multiplying the equality (2.2) with its complex conjugate counterpart, we get

$$|\chi(n)|^2 |\tau(\chi)|^2 = \sum_{h \in \mathbb{Z}/q\mathbb{Z}} \sum_{k \in \mathbb{Z}/q\mathbb{Z}} \chi(h) \chi(k) e\left(\frac{n(h-k)}{q}\right).$$

Sum both sides of this equality over $n \in \mathbb{Z}/q\mathbb{Z}$. By row orthogonality of characters, the resulting left-hand side is $\phi(q) |\tau(\chi)|^2$. For each $(h,k) \in (\mathbb{Z}/q\mathbb{Z})^2$, the sum over $n \in \mathbb{Z}/q\mathbb{Z}$ of the summand corresponding to $(h,k)$, i.e,

$$\sum_{n \in \mathbb{Z}/q\mathbb{Z}} \chi(h) \chi(k) e\left(\frac{n(h-k)}{q}\right) = \chi(h) \chi(k) \sum_{n \in \mathbb{Z}/q\mathbb{Z}} e\left(\frac{n(h-k)}{q}\right),$$

is equal to 0 if $h - k \not\equiv 0 \pmod{q}$ and equal to $\chi(h) \chi(k) q = q$ if $h - k \equiv 0 \pmod{q}$; this is again a consequence of row orthogonality. Thus, summing over all $(h,k)$ yields the equality

$$\phi(q) |\tau(\chi)|^2 = q \sum_{h \in \mathbb{Z}/q\mathbb{Z}} \chi(h) \chi(h) = q \sum_{h \in (\mathbb{Z}/q\mathbb{Z})^*} 1 = q \phi(q)$$

This proves our claim. □

Suppose that $\chi \in X_q$ is nonprincipal. Then, since $\sum_{n=0}^{q-1} \chi(n) = 0$ by row orthogonality, it immediately follows that $\sum_{n=M+1}^{M+N} \chi(n) < q$ for any $M \in \mathbb{Z}$ and $N \in \mathbb{Z}_{>0}$. However, it is possible to obtain a tighter bound on the character sum. In 1918, Pólya [10] and I. M. Vinogradov [21] (not to be confused with A. I. Vinogradov, who is the “Vinogradov” in the subject of this exposition) independently proved the better upper bound of $O(q^{\frac{1}{2}} \log q)$, which will be useful for our main proof.

**Theorem 2.12** (Pólya–Vinogradov). Let $\chi \in X_q$ be nonprincipal. For any $M \in \mathbb{Z}$ and $N \in \mathbb{Z}_{>0}$,

$$\left| \sum_{n=M+1}^{M+N} \chi(n) \right| < 2q^{\frac{1}{2}} \log q.$$ 

**Proof.** We follow the elementary argument of Schur [13].

Suppose that $\chi$ is primitive. Since it is also nonprincipal, we necessarily have $q \geq 3$ and $\chi(q) = 0$. By Lemma 2.10 we have

$$\chi(n) = \frac{1}{\tau(\chi)} \sum_{a=0}^{q-1} \chi(a) e\left(\frac{an}{q}\right) = \frac{1}{\tau(\chi)} \sum_{a=1}^{q-1} \chi(a) e\left(\frac{an}{q}\right),$$
which implies that
\[
\sum_{n=M+1}^{M+N} \chi(n) = \frac{1}{\tau(\chi)} \sum_{a=1}^{q-1} \chi(a) \sum_{n=M+1}^{M+N} e\left(\frac{an}{q}\right) = \frac{1}{\tau(\chi)} \sum_{a=1}^{q-1} \chi(a) \cdot \frac{e\left(\frac{a(M+N+1)}{q}\right) - e\left(\frac{a(M+1)}{q}\right)}{e\left(\frac{a}{q}\right) - 1}.
\]

Recall from Lemma 2.11 that \(|\tau(\chi)| = q^{\frac{1}{2}}\). Thus, taking absolute values and applying the triangle inequality, we have
\[
\left|\sum_{n=M+1}^{M+N} \chi(n)\right| \leq q^{-\frac{1}{2}} \sum_{a=1}^{q-1} \frac{2}{e\left(\frac{a}{q} - 1\right)} = q^{-\frac{1}{2}} \sum_{a=1}^{q-1} \frac{1}{\frac{1}{2} \left( e\left(\frac{a}{2q}\right) - e\left(-\frac{a}{2q}\right) \right) e\left(\frac{a}{2q}\right)}
\]
\[
= q^{-\frac{1}{2}} \sum_{a=1}^{q-1} \frac{1}{\sin\left(\frac{\pi a}{q}\right)} = q^{-\frac{1}{2}} \sum_{a=1}^{q-1} \frac{1}{\sin\left(\frac{\pi a}{q}\right)}.\]

Since \(\frac{1}{\sin(\pi x)}\) is convex on \([0, 1]\), its definite integral is lower-bounded by any midpoint Riemann sum approximation. So, since \(\frac{1}{q} \sum_{a=1}^{q-1} \frac{1}{\sin\left(\frac{\pi a}{q}\right)}\) is a midpoint Riemann sum approximation of \(\int_{1/2q}^{1-1/2q} \frac{1}{\sin(\pi x)}\), we can make the upper bound
\[
\left|\sum_{n=M+1}^{M+N} \chi(n)\right| \leq q^{-\frac{1}{2}} \cdot q \cdot \frac{1}{q} \int_{1/2q}^{1-1/2q} \frac{1}{\sin(\pi x)} dx = q^{\frac{1}{2}} \int_{1/2q}^{1-1/2q} \frac{1}{\sin(\pi x)} = 2q^{\frac{1}{2}} \int_{1/2q}^{1/2q} \frac{1}{\sin(\pi x)} dx.
\]

Note that \(\sin(\pi x) < 2x\) for \(0 < x < \frac{1}{2}\), so the above is
\[
< 2q^{\frac{3}{2}} \int_{\frac{1}{2q}}^{\frac{1}{2q}} \frac{1}{2x} dx = q^{\frac{3}{2}} \log q.
\]

This proves the claim for \(\chi\) primitive. In fact, we have the tighter bound
\[
(2.3) \quad \left|\sum_{n=M+1}^{M+N} \chi(n)\right| < q^{\frac{1}{2}} \log q.
\]

Now, suppose that \(\chi\) is not primitive. Let \(\chi_1 \in X_{q_1}\) be the primitive character that induces \(\chi\), and write \(q = q_1 r\). Then,
\[
(2.4) \quad \sum_{n=M+1}^{M+N} \chi(n) = \sum_{n=M+1}^{M+N} \chi_1(n).
\]
We recall the following identity\footnote{To prove this, note that for $p_1^{j_1} \cdots p_r^{j_r} \neq 1$ (i.e, $j \geq 1$), we have}

\[
\sum_{d \mid m} \mu(d) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m > 1, \end{cases}
\]

where the Möbius function $\mu : \mathbb{Z}_{>0} \to \{-1, 0, 1\}$ is defined by

\[
\mu(n) := \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ has repeated prime factors}, \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct prime factors}. \end{cases}
\]

Using the above, we see that (2.4) is equal to

\[
\sum_{n=M+1}^{M+N} \chi_1(n) \sum_{d \mid (n,r)} \mu(d) = \sum_{d \mid r} \mu(d) \sum_{n=M+1}^{M+N} \chi_1(n) = \sum_{d \mid r} \mu(d) \chi_1(d) \sum_{\frac{M+1}{d} \leq m \leq \frac{M+N}{d}} \chi_1(m).
\]

Taking absolute values and applying (2.3) to the inner sum of the right-hand side above, we obtain

\[
\left| \sum_{n=M+1}^{M+N} \chi_1(n) \right| \leq \sum_{d \mid r} |\mu(d)\chi_1(d)| : \sum_{\frac{M+1}{d} \leq m \leq \frac{M+N}{d}} \chi_1(m) < q_1^{\frac{1}{2}} \log q_1 \sum_{d \mid r} 1 \leq q_1^{\frac{1}{2}} \log q_1 \cdot 2 \sum_{d \leq \sqrt{r}} 1,
\]

where we use the bijection between the divisors of $r$ that are less than $\sqrt{r}$ and the divisors that are greater than $\sqrt{r}$, given by $d \mapsto \frac{r}{d}$. We can use the trivial bound on the inner sum of the right-hand side above to get

\[
\left| \sum_{n=M+1}^{M+N} \chi_1(n) \right| < q_1^{\frac{1}{2}} \log q_1 \cdot 2 \sum_{d \leq \sqrt{r}} 1 \leq q_1^{\frac{1}{2}} \log q_1 \cdot 2 r^{\frac{1}{2}} = 2q_1^{\frac{1}{2}} \log q_1 \leq 2q_1^{\frac{1}{2}} \log q. \quad \square
\]

### 3. The Large Sieve

Recall Bessel’s inequality, the statement that given orthonormal vectors $\phi_1, \ldots, \phi_R$ of an inner product space $V$ over $\mathbb{C}$, the inequality

\[
\sum_{r=1}^{R} |\langle \xi, \phi_r \rangle|^2 \leq \|\xi\|^2
\]

holds for any $\xi \in V$. However, in number theory, one often deals with vectors that are not necessarily orthonormal. Thus, it makes sense to seek a general bound analogous to Bessel’s inequality, i.e, an inequality of the form

\[
\sum_{r=1}^{R} |\langle \xi, \phi_r \rangle|^2 \leq A \|\xi\|^2
\]
for some $A$ that depends on $\phi_1, \ldots, \phi_R$. Ideally, we would like the constant $A$ to be close to 1 when the vectors $\phi_r$ are close to being orthonormal in some sense. To this end, we prove the following.

**Proposition 3.1.** Let $\phi_1, \ldots, \phi_R$ be arbitrary vectors of an inner product space $V$ over $\mathbb{C}$. Then, for

$$A := \max_{r \in \{1, \ldots, R\}} \sum_{s=1}^{R} |(\phi_r, \phi_s)|,$$

the inequality

$$\sum_{r=1}^{R} |(\xi, \phi_r)|^2 \leq A \|\xi\|^2$$

holds for all $\xi \in V$.

**Proof.** Let $u_1, \ldots, u_R \in \mathbb{C}$. Since $0 \leq (|u_r| - |u_s|)^2 = |u_r|^2 - |2u_r \overline{u_s}| + |u_s|^2$, we have

$$\frac{1}{2} \left( |u_r|^2 + |u_s|^2 \right) |(\phi_r, \phi_s)| \leq \frac{1}{2} \sum_{r=1}^{R} \sum_{s=1}^{R} \frac{1}{2} \left( |u_r|^2 + |u_s|^2 \right) |(\phi_r, \phi_s)|.$$

The summands such that $r = s$ are $|u_r|^2 |(\phi_r, \phi_r)|$, and the summands such that $r \neq s$ can be symmetrically paired off so that the sum of each pair’s summands is

$$\frac{1}{2} \left( |u_r|^2 + |u_s|^2 \right) |(\phi_r, \phi_s)| + \frac{1}{2} \left( |u_s|^2 + |u_r|^2 \right) |(\phi_s, \phi_r)| = |u_r|^2 |(\phi_r, \phi_s)| + |u_s|^2 |(\phi_s, \phi_r)|$$

Thus, we can rearrange the double sum as

$$\sum_{r=1}^{R} \sum_{s=1}^{R} \frac{1}{2} \left( |u_r|^2 + |u_s|^2 \right) |(\phi_r, \phi_s)| = \sum_{r=1}^{R} u_r^2 \sum_{s=1}^{R} |(\phi_r, \phi_s)|,$$

which overall gives us

$$\left| \sum_{r=1}^{R} \sum_{s=1}^{R} u_r \overline{u_s} (\phi_r, \phi_s) \right| \leq \sum_{r=1}^{R} u_r^2 \sum_{s=1}^{R} |(\phi_r, \phi_s)| \leq \left( \max_{r \in \{1, \ldots, R\}} \sum_{s=1}^{R} |(\phi_r, \phi_s)| \right) \sum_{r=1}^{R} u_r^2 = A \sum_{r=1}^{R} |u_r|^2.$$

But by Boas’ criterion \[1\], the above implies that the inequality

$$\sum_{r=1}^{R} |(\xi, \phi_r)|^2 \leq A \|\xi\|^2$$

holds for all $\xi \in V$. Indeed, following Boas’ argument, we have

$$0 \leq \left\| \xi - \sum_{r=1}^{R} u_r \phi_r \right\|^2 = \left\| \xi - \sum_{r=1}^{R} u_r \phi_r, \xi - \sum_{r=1}^{R} u_r \phi_r \right\|^2$$

$$= \|\xi\|^2 - \sum_{r=1}^{R} u_r (\xi, \phi_r) - \sum_{r=1}^{R} \overline{u_r} (\xi, \phi_r) + \sum_{r=1}^{R} \sum_{s=1}^{R} u_r \overline{u_s} (\phi_r, \phi_s)$$

$$\leq \|\xi\|^2 - 2 \text{Re} \left( \sum_{r=1}^{R} \overline{u_r} (\xi, \phi_r) \right) + A \sum_{r=1}^{R} |u_r|^2,$$

upon which we can take $u_r = \frac{1}{A} (\xi, \phi_r)$ to see that

$$0 \leq \|\xi\|^2 - \frac{2}{A} \sum_{r=1}^{R} |(\xi, \phi_r)|^2 + \frac{1}{A} \sum_{r=1}^{R} |(\xi, \phi_r)|^2 = \|\xi\|^2 - \frac{1}{A} \sum_{r=1}^{R} |(\xi, \phi_r)|^2,$$
which is equivalent to our desired statement.

We now introduce a notion that is closely related to the above discussion: the large sieve. This idea was first proposed by Linnik in [8], and soon after, he used the large sieve to investigate the distribution of quadratic nonresidues. However, it was Rényi who methodically developed the large sieve, using a probabilistic point of view. The large sieve eventually became a standard tool for number theorists, owing largely to Roth’s fundamental paper [12], which obtained essentially optimal estimates for the large sieve, and to Bombieri’s application of a further refined large sieve to the distribution of primes in arithmetic progressions [2], which is the subject of this exposition. Davenport and Halberstam [4] have since formalized the definition of the large sieve in the following analytic language.

**Definition 3.2.** Let \( N \in \mathbb{Z}_{>0} \) and \( \delta > 0 \). For a real-valued function \( \Delta := \Delta(N, \delta) > 0 \), we say that the large sieve inequality holds with \( \Delta \) precisely if the following holds for any integer \( M \) and sequence of complex numbers \( \{a_n\}_{n \in \mathbb{Z}_{>0}} \):

Denoting

\[
S(\alpha) := \sum_{n=M+1}^{M+N} a_n e(n\alpha),
\]

for any \( R \in \mathbb{Z}_{>0} \) and any \( \alpha_1, \ldots, \alpha_R \in \mathbb{R} \) such that \( r \neq s \implies \|\alpha_r - \alpha_s\| \geq \delta \), we have

\[
\sum_{r=1}^{R} |S(\alpha_r)|^2 \leq \Delta \sum_{n=M+1}^{M+N} |a_n|^2.
\]

In fact, to show that the large sieve inequality holds with \( \Delta \), it suffices to check that the above condition holds for one choice of \( M \). This can be seen by defining

\[
T_K(\alpha) := \sum_{n=K+1}^{K+N} a_{M-K+n} e(n\alpha) = c((K-M)\alpha)S(\alpha)
\]

for each \( K \in \mathbb{Z} \) and noting that \( |S(\alpha)| = |T_K(\alpha)| \). Thus, showing that the aforementioned condition holds for \( M \) is equivalent to showing it for all \( K \in \mathbb{Z} \).

A quick argument shows that a large sieve \( \Delta \) must necessarily be \( \gg N + \frac{1}{\delta} \).

**Proposition 3.3.** Suppose that \( \Delta \) satisfies the large sieve inequality for \( N \in \mathbb{Z}_{>0} \) and \( \delta > 0 \). Then,

\[
\Delta \geq \max \left\{ N, \frac{1}{\delta} - 1 \right\}.
\]

**Proof.** Setting \( M = 0 \), \( a_n = 1 \) for all \( n \in \mathbb{Z} \), \( R = 1 \), and \( \alpha_1 = 0 \) in Definition 3.2, \( \Delta \) necessarily satisfies \( |S(0)|^2 \leq \Delta \cdot N \). Since \( S(0) = N \), we have \( N \leq \Delta \), as needed.

To prove the other required bound \( \Delta \geq \frac{1}{\delta} - 1 \), we can suppose that \( \delta < 1 \), since otherwise we are done. For any \( R \in \mathbb{Z}_{>0} \) and any \( \alpha_1, \ldots, \alpha_R \in \mathbb{R} \), we have

\[
\int_0^1 \sum_{r=1}^{R} |S(\alpha_r + \beta)|^2 \, d\beta = R \int_0^1 |S(\beta)|^2 \, d\beta = R \sum_{n=M+1}^{M+N} |a_n|^2.
\]

Furthermore, by the mean value theorem for integrals, there exists \( \gamma \in [0, 1] \) such that

\[
\sum_{r=1}^{R} |S(\alpha_r + \gamma)|^2 = \int_0^1 \sum_{r=1}^{R} |S(\alpha_r + \beta)|^2 \, d\beta = R \sum_{n=M+1}^{M+N} |a_n|^2.
\]

\[\text{In a departure from prior notation, } \|\cdot\| \text{ will heretofore denote the distance to the closest integer.}\]
Note that \( \| \alpha_r + \gamma - (\alpha_s + \gamma) \| = \| \alpha_r - \alpha_s \| \). Thus, the above implies that \( R \leq \Delta \) as long as \( \alpha_1, \ldots, \alpha_R \) satisfy \( r \neq s \implies \| \alpha_r - \alpha_s \| \geq \delta \). In particular, note that \( R = \left\lfloor \frac{1}{\delta} \right\rfloor \) allows for such a choice of \( \alpha_1, \ldots, \alpha_R \); for example, one can define \( \alpha_r = r\delta \). Thus, \( \Delta \geq \left\lfloor \frac{1}{\delta} \right\rfloor \geq \frac{1}{\delta} - 1 \). \( \square \)

The above proposition shows that the following large sieve has the optimal order of magnitude.

**Theorem 3.4.** The large sieve inequality holds with \( \Delta = N + \frac{3}{\delta} \).

**Proof.** If \( R = 1 \), then by the Cauchy-Schwarz inequality,

\[
|S(1)|^2 = \left| \sum_{n=M+1}^{M+N} a_n e(n\alpha_1) \right|^2 \leq N \sum_{n=M+1}^{M+N} |a_n|^2,
\]

from which the claim is trivial. So, we can suppose that \( R \geq 2 \). Note that \( \| \alpha_r - \alpha_s \| \geq \delta \) for \( r \neq s \), and also that the large sieve inequality only depends on \( \alpha_r \) \((\text{mod } 1)\). So, without loss of generality, we can suppose \( 0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_R < 1 \leq \alpha_1 + 1 \), where the difference between any two adjacent terms in this inequality is greater than or equal to \( \delta \). It follows that \( \delta \leq \frac{1}{R} \leq \frac{1}{2} \).

As remarked earlier, it suffices to show the claim for one value of \( M \), so we have reduced to showing

\[
\sum_{r=1}^{R} \left| \sum_{k=-K}^{K} a_k e(k\alpha_r) \right|^2 \leq \left( 2K + \frac{3}{\delta} \right) \sum_{k=-K}^{K} |a_k|^2
\]

for arbitrary \( K \in \mathbb{Z}_{\geq 0} \). Indeed, this proves the claim for when \( N \) is odd (by setting \( N = 2K + 1 \)) as well as for when \( N \) is even (by setting \( N = 2K \) and \( a_{-K} = 0 \)).

We proceed by applying Theorem 3.1 to

\[
\ell^2 = \left\{ v = (v_k)_{k \in \mathbb{Z}} : v_k \in \mathbb{C} \text{ such that } \sum_{k \in \mathbb{Z}} |v_k|^2 < \infty \right\}
\]

with inner product

\[
\langle x, y \rangle = \sum_{k \in \mathbb{Z}} x_k \overline{y}_k.
\]

We will define \( \xi \) and the vectors \( \phi_r \) in terms of nonnegative real numbers \( \{b_k\}_{k \in \mathbb{Z}} \) such that \( \sum_{k \in \mathbb{Z}} b_k < \infty \) and \( b_k > 0 \) for \( k \in \{-K, \ldots, K\} \). Specifically, we define \( \xi = (\xi_k)_{k \in \mathbb{Z}} \in \ell^2 \) by

\[
\xi_k = \begin{cases} \frac{a_k}{\sqrt{b_k}} & \text{if } k \in \{-K, \ldots, K\}, \\ 0 & \text{otherwise}, \end{cases}
\]

and \( \phi_r = (\phi_{r,k})_{k \in \mathbb{Z}} \in \ell^2 \) by \( \phi_{r,k} = \sqrt{b_k} e(-k\alpha_r) \), for \( r \in \{1, \ldots, R\} \). Then, by Theorem 3.1, we have

\[
\sum_{r=1}^{R} \left| \sum_{k=-K}^{K} a_k e(k\alpha_r) \right|^2 \leq A \sum_{k=-K}^{K} \frac{|a_k|^2}{b_k},
\]

where

\[
A = \max_{r \in \{1, \ldots, R\}} \left| \langle \phi_r, \phi_s \rangle \right| = \max_{r \in \{1, \ldots, R\}} \left| \sum_{k \in \mathbb{Z}} b_k e(k(\alpha_s - \alpha_r)) \right|.
\]

By writing \( B(\alpha) := \sum_{k \in \mathbb{Z}} b_k e(k\alpha) \), we can write the above as \( A = \max_{r \in \{1, \ldots, R\}} \sum_{s=1}^{R} |B(\alpha_s - \alpha_r)| \).
To prove (3.1), it suffices to show that there exists a choice of \( \{b_k\}_{k \in \mathbb{Z}} \) such that \( A \leq 2K + \frac{3}{\delta} \), i.e., that for all \( r \in \{1, \ldots, R\} \), we have

\[
\sum_{s=1}^{R} |B(\alpha_s - \alpha_r)| \leq 2K + \frac{3}{\delta}.
\]

With this goal in mind, we set

\[
b_k = \begin{cases} 
1 & \text{if } |k| \leq K, \\
1 - \frac{|k| - K}{L} & \text{if } K \leq |k| \leq K + L, \\
0 & \text{otherwise.}
\end{cases}
\]

Note that \( B(0) = \sum_{k \in \mathbb{Z}} b_k = 2K + L \).

It will be useful to obtain a general expression for \( B(\alpha) \). To this end, we introduce the identity

\[
\sum_{|j| \leq J} (J - |j|)e(j\alpha) = \sum_{j_1=1}^{J} \sum_{j_2=1}^{J} e((j_1 - j_2)\alpha) = \sum_{j_1=1}^{J} \sum_{j_2=1}^{J} e(j_1\alpha)e(j_2\alpha) = \left| \sum_{j=1}^{J} e(j\alpha) \right|^2 = \left| e(J\alpha) - \frac{1}{1 - e(-\alpha)} \right|^2
\]

which holds for \( J \in \mathbb{Z}_{\geq 0} \) and \( \alpha \notin \mathbb{Z} \). We apply the identity first for \( J = K + L \) and then for \( J = K \).

By subtracting the latter from the former, we obtain

\[
B(\alpha) = \frac{\sin^2(\pi(K + L)\alpha) - \sin^2(\pi K \alpha)}{L \sin^2(\pi \alpha)} \quad (\alpha \notin \mathbb{Z}),
\]

from which we observe that

\[
|B(\alpha)| \leq \frac{1}{L \sin^2(\pi \alpha)} \leq \frac{1}{4L \|\alpha\|^2} \quad (\alpha \notin \mathbb{Z}),
\]

since \( \sin(\pi x) \geq 2x \) for \( x \in [0, \frac{1}{2}] \).

Recall that \( ||\alpha_s - \alpha_r|| \geq \delta \) for \( r \neq s \). Thus, it follows from (3.3) that

\[
\sum_{s=1}^{R} |B(\alpha_s - \alpha_r)| = B(0) + 2 \sum_{j=1}^{\infty} \frac{1}{4Lj^2\delta^2} = 2K + L + \frac{\pi^2}{12L\delta^2} \leq 2K + L + \frac{1}{L\delta^2}.
\]

Set \( L = \left\lfloor \frac{1}{\delta} \right\rfloor \) to obtain

\[
\sum_{s=1}^{R} |B(\alpha_s - \alpha_r)| \leq 2K + \frac{1}{\delta} + 1 + \frac{1}{\delta} \leq 2K + \frac{3}{\delta},
\]

where we have used the fact that \( \delta \leq \frac{1}{2} \). Thus, (3.1) holds, which proves our claim. \( \square \)

While Theorem 3.4 has the optimal order of magnitude, we remark that constants can be improved. To illustrate, Selberg has shown that the large sieve inequality holds with \( \Delta = N + \frac{1}{2} - 1 \); for a detailed discussion and proof, see the survey article [9].

We conclude our discussion of the large sieve with an important application to estimating averages of character sums (due to Gallagher in [5]), which will be the cornerstone of our main proof.
Theorem 3.5 (Gallagher). For $M \in \mathbb{Z}$, $Q, N \in \mathbb{Z}_{\geq 0}$, and complex numbers $\{a_n\}_{n \in \mathbb{Z}_{\geq 0}}$, we have:

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \in X_q^*} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \leq (N + 3Q^2) \sum_{n=M+1}^{M+N} |a_n|^2.$$ 

Proof. Recall that we used Lemma 2.10 and Lemma 2.11 to show that for $\chi \in X_q^*$, 

$$\chi(n) = \frac{1}{\tau(\chi)} \sum_{h \in \mathbb{Z}/q\mathbb{Z}} \chi(h) e \left( \frac{hn}{q} \right)$$ 

for any integer $n$. Multiplying both sides by $a_n$ and summing over all $n$, we have 

$$\sum_{n=M+1}^{M+N} a_n \chi(n) = \frac{1}{\tau(\chi)} \sum_{h \in \mathbb{Z}/q\mathbb{Z}} \chi(h) \sum_{n=M+1}^{M+N} a_n \cdot e \left( \frac{hn}{q} \right) = \frac{1}{\tau(\chi)} \sum_{h \in \mathbb{Z}/q\mathbb{Z}} \chi(h) S \left( \frac{h}{q} \right),$$

where $S(\alpha) := \sum_{n=M+1}^{M+N} a_n e(n\alpha)$ as before.

Recall from Lemma 2.11 that $|\tau(\chi)| = q^{1/2}$. Thus, taking the squared absolute value of the above equality and summing over all $\chi \in X_q^*$, we see that

$$\sum_{\chi \in X_q^*} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 = \frac{1}{q} \sum_{\chi \in X_q^*} \left| \sum_{h \in \mathbb{Z}/q\mathbb{Z}} \chi(h) S \left( \frac{h}{q} \right) \right|^2 \leq \frac{1}{q} \sum_{\chi \in X_q^*} \sum_{h \in \mathbb{Z}/q\mathbb{Z}} \left| \chi(h) S \left( \frac{h}{q} \right) \right|^2$$

$$= \frac{1}{q} \sum_{\chi \in X_q^*} \sum_{h=0}^{q-1} \sum_{k=0}^{q-1} \chi(h) \chi(k) S \left( \frac{h}{q} \right) S \left( \frac{k}{q} \right)$$

$$= \frac{1}{q} \sum_{h=0}^{q-1} \sum_{k=0}^{q-1} S \left( \frac{h}{q} \right) S \left( \frac{k}{q} \right) \sum_{\chi \in X_q^*} \chi(h) \chi(k).$$

By orthogonality (Proposition 2.3), $\sum_{\chi \in X_q^*} \chi(h) \chi(k)$ equals $\phi(q)$ if $h \equiv k \pmod{q}$ and $(h,q) = 1$, and vanishes otherwise. Thus, 

$$\sum_{\chi \in X_q^*} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \leq \frac{1}{q} \sum_{h=0}^{q-1} \sum_{k=0}^{q-1} S \left( \frac{h}{q} \right) S \left( \frac{k}{q} \right) \sum_{\chi \in X_q^*} \chi(h) \chi(k) = \frac{\phi(q)}{q} \frac{q-1}{(h,q)=1} \left| S \left( \frac{h}{q} \right) \right|^2.$$

It follows that 

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \in X_q^*} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \leq \sum_{q=1}^{Q} \sum_{h=0}^{q-1} \left| S \left( \frac{h}{q} \right) \right|^2.$$

Note that for $h_1, h_2 \in \{0, \ldots, q-1\}$ and $q_1, q_2 \in \{1, \ldots, Q\}$ such that $(h_1, q_1) = (h_2, q_2) = 1$,

$$\left| h_1 - h_2 \right| = \left| h_1 q_2 - h_2 q_1 \right| \geq \frac{1}{q_1 q_2} \geq \frac{1}{Q^2}.$$ 

\footnote{We define $X_q^*$ to be the set of primitive Dirichlet characters of period $q$.}
Thus, we can apply the large sieve of Theorem 3.4 with $\delta = \frac{1}{Q^2}$ to conclude that
\[
\sum_{q=1}^{Q} \sum_{\substack{h=0 \\ (h,q)=1}}^{q-1} \left| S\left( \frac{h}{q} \right) \right|^2 \leq (N + 3Q^2) \sum_{n=M+1}^{M+N} |a_n|^2,
\]
which proves our claim. \qed

4. The Main Proof

We now set out to prove our main result, Theorem 1.3. Recall from (1.2) that for $a, q$ such that $(a,q) = 1$, we have
\[
\psi(y; a, q) = \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \overline{\chi(a)} \psi(y, \chi).
\]
Let $\psi'(y, \chi) := \psi(y, \chi) - \frac{y}{\phi(q)} \chi_0(\chi)$, Then,
\[
\psi(y; a, q) - \frac{y}{\phi(q)} = \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \overline{\chi(a)} \psi'(y, \chi),
\]
to which we can apply the triangle inequality to obtain
\[
|E(y; a)| \leq \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \left| \psi'(y, \chi) \right|.
\]
The right-hand side is independent of $a$, so in fact, we have
\[
E(y; q) \leq \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \left| \psi'(y, \chi) \right|.
\]
For $\chi \in \mathcal{X}_q$, let $\chi_1$ denote the primitive character inducing $\chi$, and let $q_1$ denote the period of $\chi_1$. Then, we see that
\[
\psi'(y, \chi_1) - \psi'(y, \chi) = \psi(y, \chi_1) - \psi(y, \chi) = \sum_{p|q} \sum_{k \leq \log p \leq y} \chi_1(p^k) \log p = O \left( \log y \sum_{p|q} \log p \right)
\]
\[
= O \left( (\log y)(\log q) \right) = O \left( (\log qy)^2 \right).
\]
Therefore,
\[
E(y; q) \ll (\log qy)^2 + \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \left| \psi'(y, \chi_1) \right|,
\]
from which it follows that
\[
\sum_{q \leq Q} E^*(x; q) \ll \sum_{q \leq Q} \left( (\log qx)^2 + \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \max_{y \leq x} \left| \psi'(y, \chi_1) \right| \right)
\]
\[
\leq Q(\log Qx)^2 + \sum_{q \leq Q} \sum_{\chi \in \mathcal{X}_q} \max_{y \leq x} \left| \psi'(y, \chi_1) \right|.
\]
We can count the Dirichlet characters of period $q \leq Q$ by considering all characters induced by primitive characters of period $q_1 \leq Q$. Counting in this way, we can rewrite the above as
\[
(4.1) \sum_{q \leq Q} E^*(x; q) \ll Q(\log Qx)^2 + \sum_{q_1 \leq Q} \sum_{\chi \in \mathcal{X}_{q_1}} \max_{y \leq x} \left| \psi'(y, \chi_1) \right| \left( \sum_{1 \leq m \leq Q/q_1} \frac{1}{\phi(mq_1)} \right).
\]
Thus, under the hypothesis \( \phi(m)\phi(q_1) \leq \phi(mq_1) \), we have for any \( r \in \mathbb{R}_{>1} \) that
\[
\sum_{1 \leq m \leq r} \frac{1}{\phi(mq_1)} \leq \frac{1}{\phi(q_1)} \sum_{1 \leq m \leq r} \frac{1}{\phi(m)}.
\]
To bound the sum in the right-hand side, we use the fact that \( \phi \) is multiplicative (note, however, that it is not totally multiplicative) to observe that
\[
\sum_{1 \leq m \leq r} \frac{1}{\phi(m)} \leq \prod_{p \leq r} \left( \sum_{k=0}^{\infty} \frac{1}{\phi(p^k)} \right) = \prod_{p \leq r} \left( 1 + \sum_{k=0}^{\infty} \frac{1}{p^k(p-1)} \right) = \prod_{p \leq r} \left( 1 + \frac{1}{(p-1)(1-p^{-1})} \right)
\]
and taking the logarithm, we have
\[
\log \sum_{1 \leq m \leq r} \frac{1}{\phi(m)} = \sum_{p \leq r} \log \left( 1 + \frac{p}{(p-1)^2} \right) \leq \sum_{p \leq r} \frac{p}{(p-1)^2} = \sum_{p \leq r} \left( \frac{1}{p} + O(p^{-2}) \right) = \log \log(r + 1) + O(1),
\]
where we have applied Mertens’ estimate \( \sum \frac{1}{p} = \log \log(r + 1) + O(1) \) (for a proof, see [3, p.56–57]), as well as the fact that \( \sum \frac{1}{p^2} = O(1) \). Taking the exponential of the above, we arrive at
\[
\sum_{1 \leq m \leq r} \frac{1}{\phi(m)} \ll \log (r + 1).
\]
Thus, under the hypothesis \( Q \leq x^{\frac{1}{2}} \) of Theorem 1.3, we have for all \( q_1 \leq Q \) that
\[
\sum_{1 \leq m \leq \frac{Q}{q_1}} \frac{1}{\phi(mq_1)} \leq \frac{1}{\phi(q_1)} \log \left( \frac{Q}{q_1} + 1 \right) \ll \frac{\log x}{\phi(q_1)}.
\]
Applying this bound within [4.1] and replacing \( q_1 \) and \( \chi_1 \) respectively with \( q \) and \( \chi \) for notational convenience, we obtain
\[
\sum_{q \leq Q} E^* (x; q) \ll Q (\log Q x)^2 + (\log x) \sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \max_{y \leq x} |\psi(y, \chi)|.
\]
We have therefore reduced to showing that for all \( x \geq 2 \) and \( Q \in [x^{\frac{1}{2}} (\log x)^{-A}, x^{\frac{1}{2}}] \), we have
\[
\sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \max_{y \leq x} |\psi(y, \chi)| \ll x^{\frac{3}{2}} Q (\log x)^4.
\]
To this end, assume for now that the following bound holds.

**Proposition 4.1.** For all \( x \geq 2 \) and \( Q \geq 1 \), we have
\[
\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \max_{y \leq x} |\psi(y, \chi)| \ll (x + x^\frac{5}{8} Q + x^{\frac{3}{2}} Q^2) (\log Q x)^4.
\]

Then, for any \( U \geq 1 \), we can apply the above proposition with \( Q = 2U \) to show that
\[
\sum_{U < q \leq 2U} \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \max_{y \leq x} |\psi(y, \chi)| \leq \sum_{U < q \leq 2U} \frac{q/U}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \max_{y \leq x} |\psi(y, \chi)| \ll \frac{x}{U} + x^\frac{5}{8} + x^{\frac{3}{2}} U (\log U x)^4.
\]
We now apply dyadic decomposition. For any $Q_1 \in [1, Q]$, we can sum the above inequality over all $U = 2^k$ for integers $k$ such that $\frac{1}{2} Q_1 < 2^k \leq 2Q$ to obtain

$$
\sum_{Q_1 < q \leq Q} \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q^*} \max_{y \leq x} |\psi'(y, \chi)| \leq \sum_{\log_2(\frac{1}{2} Q_1) < k \leq \log_2(2Q)} \sum_{2^k < q \leq 2^{k+1}} \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q^*} \max_{y \leq x} |\psi(y, \chi)|
$$

\[
\ll \sum_{\log_2(\frac{1}{2} Q_1) < k \leq \log_2(2Q)} \left( \frac{x}{2^k} + \frac{x}{2^k} + \frac{x}{2^k} \right) (\log 2Qx)^4
\]

\[
\ll \left( \frac{x}{Q_1} + \frac{x}{\log Q} + \frac{x}{Q} \right) (\log Qx)^4,
\]

where we have used the fact that $\psi(y, \chi) = \psi'(y, \chi)$ for nonprincipal primitive characters $\chi$.

Setting $Q_1 = (\log x)^A$, we have that

$$
\frac{x}{Q_1} + \frac{x}{\log Q} + \frac{x}{Q} \ll x^{\frac{1}{2}} Q
$$

for $Q \in [x^{\frac{1}{2}} (\log x)^{-A}, x^{\frac{1}{2}}]$, which overall gives

$$
\sum_{(\log x)^A < q \leq Q} \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q^*} \max_{y \leq x} |\psi'(y, \chi)| \ll x^{\frac{1}{2}} Q (\log x)^4
$$

To bound the remaining sum over $q \leq (\log x)^A$, we can apply the variant of Siegel-Walfisz for $\psi(s, \chi)$ (Theorem 1.2) to see that $\max_{y \leq x} |\psi(y, \chi)| \ll x e^{-c \log x}$, and we can sum this over all $q \leq (\log x)^A$ to obtain

$$
\sum_{1 \leq q \leq (\log x)^A} \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{X}_q^*} \max_{y \leq x} |\psi'(y, \chi)| \ll (\log x)^A \cdot x e^{-c \log x} \ll x (\log x)^{-A} \ll x^{\frac{1}{2}} Q (\log x)^4.
$$

Thus, we have shown that Theorem 1.3 holds, conditional on the truth of Proposition 4.1.

A key ingredient for proving Proposition 4.1 is applying the large sieve to estimate averages of sums of Dirichlet characters with weighted coefficients, as in the following.

**Proposition 4.2.** For real numbers $Q, M, N \geq 1$ and complex numbers $\{a_m\}_{m \in \mathbb{Z}_{\geq 0}}$ and $\{b_n\}_{n \in \mathbb{Z}_{\geq 0}}$, we have

$$
\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \in \mathcal{X}_q^*} \max_{u \in [1, MN]} \left| \sum_{m \leq M} \sum_{n \leq N \atop mn \leq u} a_m b_n \chi(mn) \right|
$$

\[
\ll (M + Q^2)^{\frac{1}{2}} (N + Q^2)^{\frac{1}{2}} \left( \sum_{m \leq M} |a_m|^2 \right)^{\frac{1}{2}} \left( \sum_{n \leq N} |b_n|^2 \right)^{\frac{1}{2}} \log(2MN)
\]

**Proof.** Recall from Theorem 3.5 that

$$
\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \in \mathcal{X}_q^*} \left| \sum_{m \leq M} a_m \chi(m) \right|^2 \ll (M + Q^2) \sum_{m \leq M} |a_m|^2,
$$
and the analogous inequality holds for \( \{b_n\}_{n \in \mathbb{Z}^\geq 0} \). Then, we can apply the Cauchy-Schwarz inequality to obtain

\[
\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \in X_q^*} \left| \sum_{m \leq M} \sum_{n \leq N} a_m b_n \chi(mn) \right| \leq \left( \sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \in X_q^*} \left| \sum_{m \leq M} a_m \chi(m) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \in X_q^*} \left| \sum_{n \leq N} b_n \chi(n) \right|^2 \right)^{\frac{1}{2}} \leq (M + Q^2)^{\frac{1}{2}} (N + Q^2)^{\frac{1}{2}} \left( \sum_{m \leq M} |a_m|^2 \right)^{\frac{1}{2}} \left( \sum_{n \leq N} |b_n|^2 \right)^{\frac{1}{2}}.
\]

(4.2)

We wish to insert the condition \( mn \leq u \) into the inner sum. To do so, we construct a well-behaved indicator function similar to (1.4). For \( \alpha > 0 \), define

\[
\delta(x) := \begin{cases} 
1 & \text{if } |x| < \alpha, \\
0 & \text{if } |x| > \alpha,
\end{cases}
\]
for \( x \in \mathbb{R} \setminus \{-\alpha, \alpha\} \), as well as

\[
C = \int_{-\infty}^{\infty} \frac{\sin y}{y} dy,
\]
which is convergent by the alternating series test. Then, note that

\[
\int_{-\infty}^{\infty} e^{iyx} \frac{\sin \alpha y}{Cy} dy = \int_{-\infty}^{\infty} \frac{(\cos xy + i \sin xy) \sin \alpha y}{Cy} dy = \int_{-\infty}^{\infty} \frac{\cos xy \sin \alpha y}{Cy} dy = \int_{-\infty}^{\infty} \frac{1}{2Cy} \left( \sin((\alpha + x)y) + \sin((\alpha - x)y) \right) dy
\]

\[
= \frac{1}{2C} \left( \int_{-\infty}^{\infty} \left\{ \begin{array}{ll}
1 & \text{if } \alpha + x > 0, \\
-1 & \text{if } \alpha + x < 0,
\end{array} \right. \right) \sin y_1 y_1 dy_1 + \int_{-\infty}^{\infty} \left\{ \begin{array}{ll}
1 & \text{if } \alpha - x > 0, \\
-1 & \text{if } \alpha - x < 0,
\end{array} \right. \right) \sin y_2 y_2 dy_2 = \delta(x),
\]

where we have used the fact that the integral of an odd function over \( \mathbb{R} \) vanishes. Furthermore, it follows from integration by parts that

\[
\int_{-\infty}^{\infty} \frac{\sin \lambda x}{x} dx = \left[ \frac{-\cos \lambda x}{\lambda x} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\sin \lambda x}{x^2} dx \ll \frac{1}{\lambda T},
\]

which gives us the useful estimate

(4.3) \[
\delta(x) = \int_{-T}^{T} e^{ixy} \frac{\sin \alpha y}{Cy} dy + O \left( \frac{1}{T |\alpha - |x||} \right).
\]

Setting \( \alpha = \log u \) (where without loss of generality, we can suppose that \( u \) is of the form \( u = k + \frac{1}{2} \) for an integer \( k \in [1, MN] \), since the inner sum

\[
\sum_{m \leq M} \sum_{n \leq N} a_m b_n \chi(mn)
\]

depends only on the integer part of \( u \) and \( x = -\log mn \), we have

\[
\left\{ \begin{array}{ll}
1 & \text{if } mn < u, \\
0 & \text{if } mn > u
\end{array} \right. = \int_{-T}^{T} m^{-i} n^{-iy} \frac{\sin(y \log u)}{Cy} dy + O \left( T^{-1} |\log \frac{mn}{u}|^{-1} \right).
\]
Multiplying this estimate to the inner sum yields

$$\sum_{m \leq M} \sum_{n \leq N} a_m b_n \chi(mn) = \int_{-T}^{T} A(y, \chi) B(y, \chi) \frac{\sin(y \log u)}{Cy} \, dy + O \left( T^{-1} \sum_{m \leq M} \sum_{n \leq N} |a_m b_n| \left| \log \frac{mn}{u} \right|^{-1} \right),$$

where

$$A(y, \chi) = \sum_{m \leq M} a_m \chi(m) m^{-iy} \quad \text{and} \quad B(y, \chi) = \sum_{n \leq N} b_n \chi(n) n^{-iy}.$$

Since

$$\left| \log \frac{mn}{u} \right| \gg \min \left\{ 1, \left| \frac{mn}{u} - 1 \right| \right\} \geq \min \left\{ 1, \left| \frac{u + \frac{1}{2}}{u} - 1 \right|, \left| \frac{u - \frac{1}{2}}{u} - 1 \right| \right\} \gg \frac{1}{u} \geq \frac{1}{MN},$$

it follows that

$$\max_{u \in [1, MN]} \left| \sum_{m \leq M} \sum_{n \leq N} a_m b_n \chi(mn) \right| \ll \int_{-T}^{T} |A(y, \chi) B(y, \chi)| \min \left\{ \frac{1}{|y|}, \log(2MN) \right\} \, dy + \frac{MN}{T} \sum_{m \leq M} \sum_{n \leq N} |a_m b_n|,$$

$$\leq \int_{-T}^{T} |A(y, \chi) B(y, \chi)| \min \left\{ \frac{1}{|y|}, \log(2MN) \right\} \, dy + \frac{MN}{T} \left( \sum_{m \leq M} |a_m|^2 \right)^{\frac{1}{2}} \left( \sum_{n \leq N} |b_n|^2 \right)^{\frac{1}{2}},$$

where we have applied the Cauchy-Schwarz inequality. Multiplying the above by $\frac{q}{\phi(q)}$ and summing over $q \leq Q$ and $\chi \in X_q^*$, we get

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \max_{\chi \in X_q^*} \left| \sum_{u \in [1, MN]} \sum_{m \leq M} \sum_{n \leq N} a_m b_n \chi(mn) \right| \ll \int_{-T}^{T} \sum_{q \leq Q} \frac{q}{\phi(q)} |A(y, \chi) B(y, \chi)| \min \left\{ \frac{1}{|y|}, \log(2MN) \right\} \, dy$$

$$+ \frac{MN}{T} \left( \sum_{m \leq M} |a_m|^2 \right)^{\frac{1}{2}} \left( \sum_{n \leq N} |b_n|^2 \right)^{\frac{1}{2}}.$$
Setting $T = MN$, we can use (4.2) and $\sum_{q=1}^{Q} \frac{q}{\phi(q)} \leq \sum_{q=1}^{Q} q \ll Q^2$ to bound the first summand by

$$\ll (M + Q^2)^{\frac{1}{2}} (N + Q^2)^{\frac{1}{2}} \left( \sum_{m \leq M} |a_m|^2 \right)^{\frac{1}{2}} \left( \sum_{n \leq N} |b_n| \right)^{\frac{1}{2}} \int_{-T}^{T} \min \left\{ \frac{1}{|y|}, \log(2MN) \right\} dy,$$

Moreover, setting $T = MN$ makes the second summand bounded by the above expression as well. \hfill \Box

**Proof of Proposition 4.1.** If $Q^2 > x$, then we are done by applying Proposition 4.2 with $M = 1$, $a_1 = 1$, $b_n = \Lambda(n)$, and $N = x$. So, suppose $Q^2 < x$. For our starting point, we follow Vaughan’s method [19] of estimating sums of the form $\sum_{n \leq N} f(n)\Lambda(n)$; inspired by I. M. Vinogradov’s original method [22] for the same purpose, Vaughan developed an analogue in which the details are significantly simpler.

We begin with the following identity for the logarithmic derivative of the Riemann zeta function $\zeta(s)$ (see Definition A.2): for

$$F(s) = \sum_{m \leq U} \Lambda(m)m^{-s} \quad \text{and} \quad G(s) = \sum_{d \leq V} \mu(d)d^{-s},$$

we have

$$-\frac{\zeta'(s)}{\zeta(s)} = F(s) - \zeta(s)F(s)G(s) - \zeta'(s)G(s) + \left( -\frac{\zeta'(s)}{\zeta(s)} - F(s) \right) \cdot (1 - \zeta(s)G(s)).$$

Comparing Dirichlet series coefficients, we discern that

$$\Lambda(n) = a_1(n) + a_2(n) + a_3(n) + a_4(n),$$

where $a_1(n)$ is the Dirichlet series coefficient of $F(s)$,

$$a_1(n) = \begin{cases} \Lambda(n) & \text{if } n \leq U, \\ 0 & \text{if } n > U, \end{cases}$$

$a_2(n)$ is the Dirichlet series coefficient of $-\zeta(s)F(s)G(s)$,

$$a_2(n) = -\sum_{md=n, \; m \leq U, \; d \leq V} \Lambda(m)\mu(d),$$

$a_3(n)$ is the Dirichlet series coefficient of $-\zeta'(s)G(s)$,

$$a_3(n) = \sum_{hd=n, \; d \leq V} \mu(d)\log h,$$

and $a_4(n)$ is the Dirichlet series coefficient of $\left( -\frac{\zeta'(s)}{\zeta(s)} - F(s) \right) \cdot (1 - \zeta(s)G(s))$,

$$a_4(n) = -\sum_{mk=n, \; k > 1, \; m > U} \Lambda(m) \left( \sum_{d|k} \mu(d) \right).$$
Thus, for any \( y \geq 1 \) and \( \chi \in X^*_q \), we have
\[
\psi(y, \chi) = \sum_{j=1}^{4} S_j(y, \chi, U, V)
\]
for
\[
S_j(y, \chi, U, V) := \sum_{n \leq y} a_j(n) \chi(n).
\]
It follows that
\[
\sum_{q \leq Q} \phi(q) \sum_{\chi \in X^*_q} \max_{y \leq x} |\psi(y, \chi)| \leq \sum_{j=1}^{4} \sum_{q \leq Q} \phi(q) \sum_{\chi \in X^*_q} \max_{y \leq x} |S_j(y, \chi, U, V)|.
\] 
Hence, it suffices to set out to bound each sum
\[
\sum_{q \leq Q} \phi(q) \sum_{\chi \in X^*_q} \max_{y \leq x} |S_j(y, \chi, U, V)|
\]
under the hypotheses that \( x \geq 2 \), \( Q \leq x^{\frac{1}{2}} \), \( U, V \geq 1 \), and \( UV \leq x \). The contribution from \( S_1(y, \chi, U, V) \) is \( \ll Q^2 U \), since clearly \( S_1(y, \chi, U, V) = O(U) \).

We will first bound the contribution from
\[
S_4(y, \chi, U, V) = -\sum_{n \leq y} \chi(n) \sum_{\substack{m \leq x \\ M<n \leq 2M}} \Lambda(m) \left( \sum_{\substack{d|k \\ d \leq V}} \mu(d) \right) \chi(mk),
\]
where we have used the fact that \( \sum_{d|n} \mu(d) = 0 \) for \( k \leq V \), since \( \sum_{d|n} \mu(d) = 0 \) for \( n > 1 \). To this end, we will apply dyadic decomposition to \( m \). Note that for any \( M \in [2, x] \),
\[
\sum_{q \leq Q} \phi(q) \sum_{\chi \in X^*_q} \max_{y \leq x} \left| \sum_{\substack{U<m \leq x \\ M<n \leq 2M}} \Lambda(m) \sum_{\substack{V<k \leq \frac{x}{m} \\ m \leq \frac{y}{M}}} \sum_{\substack{d|k \\ d \leq V}} \mu(d) \chi(mk) \right|
\]
\[
= \sum_{q \leq Q} \phi(q) \sum_{\chi \in X^*_q} \max_{y \leq x} \left| \sum_{1 \leq m \leq 2M} \sum_{1 \leq k \leq \frac{y}{M}} \sum_{m \leq \frac{y}{k}} a_m b_k \chi(mk) \right|
\]
for
\[
a_m = \begin{cases} 
\Lambda(m) & \text{if } \max\{U, M\} < m \leq \min\{\frac{x}{U}, 2M\}, \\
0 & \text{otherwise},
\end{cases}
\]
\[
b_k = \begin{cases} 
\sum_{d|k} \mu(d) & \text{if } k > V, \\
0 & \text{otherwise}.
\end{cases}
\]
This formulation allows us to apply Proposition 4.2, from which one deduces that (4.5) is
\[ \ll (Q^2 + M)^{\frac{1}{2}} \left( Q^2 + \frac{x}{M} \right)^{\frac{1}{2}} \left( \sum_{M < m \leq 2M} \Lambda(m)^2 \right)^{\frac{1}{2}} \left( \sum_{V < k \leq \frac{x}{M}} d(k)^2 \right)^{\frac{1}{2}} \log x. \]

Note that \( \sum_{M < m \leq 2M} \Lambda(m)^2 \ll \log M \sum_{M < m \leq 2M} \Lambda(m) \ll M \log M. \) Furthermore, it is true that \( \sum_{k \leq z} d(k)^2 \ll z (\log 2z)^3. \) To show this, observe that \( d(k)^2 = \sum_{d | k} f(d), \) where \( f \) is the multiplicative function defined by \( f(p^n) = 2a + 1. \) Then,
\[
\sum_{k \leq z} d(k)^2 = \sum_{k \leq z} \sum_{d | k} f(d) = \sum_{d \leq z} f(d) \left\lfloor \frac{z}{d} \right\rfloor \leq z \sum_{d \leq z} \frac{f(d)}{d} \leq z \prod_{p \leq z} \left( 1 + \frac{3}{p} + \frac{5}{p^2} + \cdots \right)
\leq z \prod_{p \leq z} \left( 1 - \frac{1}{p} \right)^{-3} \ll z (\log 2z)^3,
\]
as claimed. Thus, we use these two bounds to conclude that (4.5) is
\[
\ll (Q + M^{\frac{1}{2}}) \left( Q + x^{\frac{1}{2}} M^{-\frac{1}{2}} \right)^{\frac{1}{2}} (M \log M)^{\frac{1}{2}} \left( \frac{x}{M} \log^3 x \right)^{\frac{1}{2}} \log x
\ll (Q^2 x^{\frac{3}{2}} + Qx M^{-\frac{1}{2}} + Qx^{\frac{3}{2}} M^{\frac{1}{2}} + x)(\log x)^3,
\]
where we have applied Cauchy-Schwarz.

Finally, we apply dyadic decomposition. Setting \( M = 2^\ell U \) for all nonnegative integers \( \ell \) such that \( 2^\ell U \leq \frac{x}{y}, \) we have that
\[
\sum_{q \leq Q} \frac{\phi(q)}{q} \sum_{\chi(x)} \max_{y \leq x} |S_4(y, \chi, U, V)| \ll \sum_{0 \leq \ell \leq \log_2 \left( \frac{x}{y} \right)} (Q^2 x^{\frac{1}{2}} + Qx M^{-\frac{1}{2}} + Qx^{\frac{3}{2}} M^{\frac{1}{2}} + x)(\log x)^3
\ll (Q^2 x^{\frac{3}{2}} + Qx U^{-\frac{1}{2}} + Qx V^{-\frac{1}{2}} + x)(\log x)^4,
\]
(4.6)
since there are \( \ll \log x \) such integers \( \ell. \) This is our desired bound for the contribution from \( S_4. \)

Next, we seek to bound the contribution from
\[
S_2(y, \chi, U, V) = -\sum_{n \leq y} \sum_{\chi(n) \Lambda(m) \mu(d)} = -\sum_{m \leq U} \Lambda(m) \mu(d) \sum_{r \leq y} \chi(rmd)
\]
\[ = -\sum_{t \leq UV} \left( \sum_{m \leq U \atop md = t} \Lambda(m) \mu(d) \right) \sum_{r \leq y \atop rmd = t} \chi(rt). \]

To obtain tighter bounds, we split the sum as \( S_2(y, \chi, U, V) = S_2'(y, \chi, U, V) + S_2''(y, \chi, U, V), \) where
\[
S_2'(y, \chi, U, V) = -\sum_{t \leq UV} \left( \sum_{m \leq U \atop md = t} \Lambda(m) \mu(d) \right) \sum_{r \leq y \atop rmd = t} \chi(rt).
\]
and

$$S''_2(y, \chi, U, V) = - \sum_{U < t \leq UV} \left( \sum_{md=t \atop m \leq U \atop d \leq V} \Lambda(m) \mu(d) \right) \sum_{r \leq \frac{t}{2}} \chi(rt).$$

Just as before, we can deal with $S''_2$ applying dyadic decomposition, to the variable $t$ in this case. Analogously to before, we note that for $T \in [2, x]$, we have

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \max_{y \leq x} \left| \sum_{U < t \leq UV} \left( \sum_{md=t \atop m \leq U \atop d \leq V} \Lambda(m) \mu(d) \right) \sum_{r \leq \frac{t}{4}} \chi(rt) \right|$$

for

$$a_t = \begin{cases} \sum_{md=t \atop m \leq U \atop d \leq V} \Lambda(m) \mu(d) & \text{if } \max\{U, T\} < t \leq \min\{UV, 2T\}, \\ 0 & \text{otherwise,} \end{cases}$$

and $b_r = 1$ for all $r$. Applying Proposition 4.2, we deduce that (4.7) is

$$\ll (Q^2 + T)^{\frac{1}{2}} \left( Q^2 + \frac{x}{T} \right)^{\frac{1}{2}} \left( \sum_{T < t \leq 2T} \left( \sum_{md=t \atop m \leq U \atop d \leq V} \Lambda(m) \mu(d) \right)^2 \right)^{\frac{1}{2}} \left( \frac{x}{T} \right)^{\frac{1}{2}} \log x.$$

For positive integer $t$, we have

$$\sum_{md=t \atop m \leq U \atop d \leq V} \Lambda(m) \mu(d) \left| \sum_{\chi \in \mathcal{X}_q} \max_{y \leq x} \left| S''_2(y, \chi, U, V) \right| \right|$$

Thus, since $\sum_{T < t \leq 2T} (\log t)^2 \ll T(\log T)^2$, it follows that (4.8) is

$$\ll (Q + T^{\frac{1}{2}}) \left( Q + x^{\frac{1}{2}} T^{-\frac{1}{2}} \right) \left( T(\log T)^2 \right)^{\frac{1}{2}} \left( x^{\frac{1}{2}} T^{-\frac{1}{2}} \right) \log x$$

$$\ll (Q^2 x^{\frac{1}{2}} + Q x T^{-\frac{1}{2}} + Q x^{\frac{1}{2}} T^{\frac{1}{2}} + x)(\log x)^2$$

Summing the above bound over $T = 2^\ell U$ for all nonnegative integers $\ell$ such that $2^\ell U \leq UV$, we see that

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \in \mathcal{X}_q} \max_{y \leq x} \left| S''_2(y, \chi, U, V) \right| \ll (Q^2 x^{\frac{1}{2}} + Q x U^{-\frac{1}{2}} + Q x^{\frac{1}{2}} U^{\frac{1}{2}} V^{\frac{1}{2}} + x)(\log x)^3.$$
We now move on to bounding the contribution from $S_2'$. Applying (4.9), we have

$$|S_2'(y, \chi, U, V)| \ll \sum_{t \leq U} (\log t) \left| \sum_{r \leq y} \chi(t)\chi(r) \right| \leq (\log U) \sum_{t \leq U} \left| \sum_{r \leq y} \chi(r) \right|$$

By the Pólya–Vinogradov inequality (Theorem 2.12), we have that the inner sum is $\ll q^{1/2} \log q$ for $\chi$ nonprincipal. Thus, for $q > 1$, we have

$$|S_2'(y, \chi, U, V)| \ll (\log U) q^{1/2} \log q \ll q^{3/2} U(\log qU)^2.$$  

As for $\chi$ principal (i.e., $q = 1$), we can use the bound

$$|S_2'(y, \chi, U, V)| \ll (\log U) \sum_{t \leq U} \frac{y}{t} \ll y(\log U)^2.$$  

Thus, since $Q \leq x^{1/3}$ and $U \leq UV \leq x$, we overall have

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \max_{y \leq x} |S_2'(y, \chi, U, V)| \ll x(\log U)^2 + \sum_{1 < q \leq Q} \frac{q^{1/2}}{q} U(\log qU)^2$$

(4.11) \[\ll x(\log U)^2 + Q^{3/2}U(\log QU)^2 \ll (x + Q^{3/2}U)(\log x)^2.\]

Finally, we bound the contribution from

$$S_3(y, \chi, U, V) = \sum_{n \leq y} \chi(n) \sum_{d \leq V} \mu(d) \log h = \sum_{d \leq V} \mu(d) \sum_{h \leq \frac{y}{d}} \chi(h) \log h \ll (\log y) \sum_{d \leq V} \max_{1 \leq w \leq \frac{y}{d}} \left| \sum_{h \leq w} \chi(h) \right|.$$  

Again, we apply the Pólya–Vinogradov inequality for $q > 1$ to get

$$|S_3(y, \chi, U, V)| \ll (\log y) \sum_{d \leq V} q^{1/2} \log q \leq q^{3/2}V(\log y)(\log q) \leq q^{3/2}V(\log qy)^2,$$

whereas if $q = 1$, we have

$$|S_3(y, \chi, U, V)| \ll (\log y) \sum_{d \leq V} \frac{y}{d} = y(\log y)(\log V) \leq y(\log yV)^2.$$  

Thus, we overall have

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \max_{y \leq x} |S_3(y, \chi, U, V)| \ll x(\log Vx)^2 + \sum_{1 < q \leq Q} \frac{q^{3/2}}{q} V(\log qx)^2$$

(4.12) \[\ll (x + Q^{3/2}V)(\log x)^2.\]

By applying the bounds (4.6), (4.10), (4.11), and (4.12) to (4.4), we obtain

$$\sum_{q \leq Q} \sum_{\chi \in X_q} \max_{y \leq x} |\psi(y, \chi)| \ll (Q^2 x^{3/4} + x + QxU^{-1/2} + QxV^{-1/2} + Qx^{3/2}U^{3/2}V^{1/2} + Q^{3/2}U + Q^{3/2}V)(\log x)^4.$$  

If $x^{1/3} \leq Q \leq x^{1/2}$, then setting $U = V = x^{5/6}Q^{-1}$ yields

$$\sum_{q \leq Q} \sum_{\chi \in X_q} \max_{y \leq x} |\psi(y, \chi)| \ll (Q^2 x^{3/4} + x + Q^{3/2}x^{3/2} + x^{5/2})(\log x)^4 \ll Q^2 x^{3/4}(\log x)^4,$$  

whereas if \( Q < x^{1/3} \), then setting \( U = V = x^{1/3} \) yields
\[
\sum_{q \leq Q} \sum_{\chi \in \mathcal{X}_q^*} \max_{y \leq x} |\psi(y, \chi)| \ll (Q^2 x^{1/3} + x + Q x^{5/6} + Q^2 x^{4/3}) (\log x)^4
\]
\[
\ll (Q^2 x^{1/3} + x + Q x^{5/6}) (\log x)^4.
\]
Thus, our claimed bound holds in both cases, and we are done. \( \square \)

This concludes our proof of the Bombieri–Vinogradov theorem.

**Appendix A. Dirichlet \( L \)-Functions**

**Definition A.1.** Let \( \chi \) be a Dirichlet character of period \( q \). The corresponding Dirichlet \( L \)-function is defined by
\[
L(s, \chi) := \sum_{n=1}^{\infty} \chi(n) n^{-s},
\]
for \( s \in \mathbb{C} \) such that \( \sigma > 1 \).

**Example A.2.** Let \( \chi_0 \) be the principal character of period 1. Then, \( L(s, \chi_0) \) is given by the well-known Riemann zeta function,
\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]

Since \( |\chi(n)| \leq 1 \) for all \( n \in \mathbb{Z}_{>0} \), it follows from the triangle inequality and the Weierstrass \( M \)-test that \( L(s, \chi) \) is uniformly convergent in any compact subset of \( \{ \sigma > 1 \} \), which shows that \( L(s, \chi) \) is analytic in its domain \( \{ \sigma > 1 \} \). Furthermore, in this domain, one can show that the identity
\[
L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}
\]
holds, by appealing to the fundamental theorem of arithmetic (unique prime factorization); we call this expression the Euler product of the \( L \)-function. In particular, if \( \chi \) is a nonprimitive character of period \( q \) induced by primitive character \( \chi_1 \), comparing their Euler products gives us
\[
L(s, \chi) = L(s, \chi_1) \prod_{p \mid q} (1 - \chi_1(p)p^{-s}).
\]
Thus, to understand the behavior of an arbitrary \( L \)-function, it suffices to understand that of \( L \)-functions corresponding to primitive characters.

While the domain of Dirichlet \( L \)-functions as defined above is restricted to \( \sigma > 1 \), we can in fact analytically continue such \( L \)-functions in the following manner.

**Theorem A.3.** Let \( \chi \) be a primitive character of period \( q \). Then, the completed \( L \)-function corresponding to \( \chi \), defined by
\[
\xi(s, \chi) := \left( \frac{\pi}{q} \right)^{-\frac{1}{2}(s + \alpha(\chi))} \Gamma \left( \frac{1}{2} (s + \alpha(\chi)) \right) L(s, \chi),
\]
for the principal character \( \chi_0 \in X_1 \).
satisfies the functional equation
\[ \xi(1 - s, \chi) = \frac{j_0(\chi) q^{\frac{s}{2}}}{\tau(\chi)} \xi(s, \chi), \]
where \( a(\chi) = 0 \) if \( \chi(-1) = 1 \) and \( a(\chi) = 1 \) if \( \chi(-1) = -1 \). If \( \chi \) is principal, \( L(s, \chi) \) can be analytically continued to \( \mathbb{C} \setminus \{1\} \) with a simple pole of residue 1 at \( s = 1 \). If \( \chi \) is nonprincipal, \( \xi(s, \chi) \) is entire, and so \( L(s, \chi) \) can be analytically continued to \( \mathbb{C} \).

**Proof.** The definition of the gamma function gives us
\[ \Gamma(\frac{1}{2}s) = \int_0^\infty e^{-t \frac{1}{2}s - 1} dt, \quad (\sigma > 0), \]
and by substituting \( t = \frac{n^2\pi x}{q} \), we obtain
\[ \pi^{\frac{1}{2}} q^{\frac{s}{2}} \Gamma(\frac{1}{2}s) n^{-s} = \int_0^\infty x^{\frac{1}{2}s - 1} e^{-\frac{n^2\pi x}{q}} dx, \quad (\sigma > 0). \]
This gives us the identity
\[ \pi^{\frac{1}{2}} q^{\frac{s}{2}} \Gamma(\frac{1}{2}s) L(s, \chi) = \sum_{n=1}^\infty \pi^{\frac{1}{2}} q^{\frac{s}{2}} \Gamma(\frac{1}{2}s) n^{-s} = \sum_{n=1}^\infty \chi(n) \int_0^\infty x^{\frac{1}{2}s - 1} e^{-\frac{n^2\pi x}{q}} dx = \int_0^\infty x^{\frac{1}{2}s - 1} \left( \sum_{n=1}^\infty \chi(n) e^{-\frac{n^2\pi x}{q}} \right) dx, \quad (\sigma > 0), \]
where absolute convergence justifies switching the summation with the integral, as checked by
\[ \sum_{n=1}^\infty \int_0^\infty x^{\frac{1}{2}s - 1} e^{-\frac{n^2\pi x}{q}} dx = \sum_{n=1}^\infty \int_0^\infty x^{\frac{1}{2}s - 1} e^{-\frac{n^2\pi x}{q}} dx = \sum_{n=1}^\infty \pi^{\frac{1}{2}} q^{\frac{s}{2}} \Gamma(\frac{1}{2}s) n^{-\sigma} < \infty. \]
We now consider the following two cases.

**Case 1:** \( \chi(-1) = 1 \). Note that for \( \sigma > 0 \), we have the equality
\[ (A.2) \]
\[ \xi(s, \chi) = \pi^{\frac{1}{2}} q^{\frac{s}{2}} \Gamma(\frac{1}{2}s) L(s, \chi) = \int_0^\infty x^{\frac{1}{2}s - 1} \omega(x) dx = \int_1^\infty x^{\frac{1}{2}s - 1} \omega(x) dx + \int_1^\infty x^{-\frac{1}{2}s - 1} \omega(x^{-1}) dx, \]
where
\[ \omega(x) = \omega(x, \chi) := \sum_{n=1}^\infty \chi(n) e^{-\frac{n^2\pi x}{q}}. \]

First, suppose that \( q = 1 \), which means that \( \chi \) is the principal character and \( L(s, \chi) = \xi(s) \). We have the equality \( 2\omega(x) = \theta(x) - 1 \), where the series
\[ \theta(x) := \sum_{n=-\infty}^\infty e^{-n^2\pi x} \]
is known to satisfy the functional equation \( \theta(x^{-1}) = x^{\frac{1}{2}} \theta(x) \). Indeed, this is a special case of the following fact.

**Lemma A.4.** For any \( a \in \mathbb{C} \) and \( t > 0 \), we have
\[ \sum_{n=-\infty}^\infty e^{-\pi t(n+a)^2} = \sum_{n=-\infty}^\infty t^{-\frac{1}{2}} e^{-\frac{\pi^2 a^2}{4t}} e^{2\pi i na}. \]

**Proof.** By Poisson summation. For details, see [15, p.120].
Using the functional equation of $\theta(x)$, we obtain
\[
\omega(x^{-1}) = -\frac{1}{2} + \frac{1}{2} \theta(x^{-1}) = -\frac{1}{2} + \frac{1}{2} x^{\frac{1}{2}} \theta(x) = -\frac{1}{2} + \frac{1}{2} x^{\frac{1}{2}} + x^{\frac{1}{2}} \omega(x),
\]
from which we see that
\[
\int_1^\infty x^{-\frac{1}{2}s-1} \omega(x^{-1}) \, dx = \int_1^\infty x^{-\frac{1}{2}s-1} \omega(x^{-1}) \, dx = \int_1^\infty x^{-\frac{1}{2}s-1} \left( -\frac{1}{2} + \frac{1}{2} x^{\frac{1}{2}} + x^{\frac{1}{2}} \omega(x) \right) \, dx
\]
\[
= \int_1^\infty -\frac{1}{2} x^{-\frac{1}{2}s-1} + \frac{1}{2} x^{-\frac{1}{2}s-\frac{1}{2}} + x^{-\frac{1}{2}s-\frac{1}{2}} \omega(x) \, dx = -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty x^{-\frac{1}{2}s-\frac{1}{2}} \omega(x) \, dx.
\]
Thus,
\[
(A.3) \quad \xi(s, \chi) = \pi^{\frac{1}{2}+s} \Gamma\left(\frac{1}{2}s\right) \zeta(s) = -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \left( x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}} \right) \omega(x) \, dx.
\]
Note that the integral in the above expression is uniformly convergent in every compact subset of $\mathbb{C}$ by the Weierstrass $M$-test, since $\omega(x)$ decays at least exponentially, as one can see from
\[
\omega(x) \leq \sum_{n=1}^\infty e^{-n\pi x} = \frac{e^{-\pi x}}{1 - e^{-\pi x}} = O(e^{-\pi x}), \quad x > 1, x \to \infty.
\]
It follows that $[A.3]$ is an analytic continuation of $L(s, \chi) = \zeta(s)$ to all of $\mathbb{C}$. To identify the pole(s) of $\zeta(s)$, note that $\pi^{\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right)$ is entire, which means that the only candidates for poles of $\zeta(s)$ are 0 and 1. However, $\pi^{\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right)^{-1} = 0$ at $s = 0$, which means 0 cannot be a pole of $\zeta(s)$. Since $\pi^{\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right)^{-1} = 1$ at $s = 1$, it follows that $\zeta(s)$ has a simple pole of residue 1 at $s = 1$. Thus, $[A.3]$ gives an analytic continuation of $L(s, \chi) = \zeta(s)$ to $\mathbb{C} \setminus \{1\}$, and substituting $s$ with $1 - s$ preserves the right-hand side of $[A.3]$, showing that our proposed functional equation holds.

Next, suppose that $q > 1$. Then, since $\chi(n) = \chi(-n)$ for all $n \in \mathbb{Z}$ and $\chi(0) = 0$, we have
\[
\omega(x, \chi) = \frac{1}{2} \sum_{n=-\infty}^\infty \chi(n)e^{-\frac{\pi^2 n x}{q}}.
\]
In fact, by applying $[2.2]$ and Lemma $[A.4]$ with $t = \frac{q}{x}$, we have
\[
\tau(x) \omega(x, \chi) = \frac{1}{2} \sum_{n=-\infty}^\infty \left( \sum_{m=0}^{q-1} \chi(m) e^{\left(\frac{mn}{q}\right)} \right) e^{-\frac{\pi^2 n x}{q}} = \frac{1}{2} \sum_{m=0}^{q-1} \chi(m) \sum_{n=-\infty}^\infty e^{-\frac{\pi^2 n x}{q} + \frac{2\pi imn}{q}}
\]
\[
= \frac{1}{2} \sum_{n=-\infty}^\infty \chi(m) \left( \frac{q}{x} \right)^{\frac{1}{2}} \sum_{n=-\infty}^\infty e^{-\left( \frac{n+m}{q} \right)^2 \frac{\pi^2}{x}} = \frac{1}{2} \left( \frac{q}{x} \right)^{\frac{1}{2}} \sum_{m=0}^{q-1} \chi(m) \sum_{n=-\infty}^\infty e^{-\left( qn + m \right)^2 \frac{\pi^2}{x}}
\]
\[
= \left( \frac{q}{x} \right)^{\frac{1}{2}} \sum_{k=-\infty}^\infty \chi(k) e^{-\frac{\pi^2 k^2}{x}} = \left( \frac{q}{x} \right)^{\frac{1}{2}} \omega(x^{-1}, \chi).
\]
We apply this functional equation to split the integral as we have in $[A.2]$, which yields
\[
\xi(s, \chi) = \pi^{\frac{1}{2}s} q^{\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) L(s, \chi) = \int_0^\infty x^{\frac{1}{2}s-1} \omega(x, \chi) \, dx
\]
\[
= \int_1^\infty x^{\frac{1}{2}s-1} \omega(x, \chi) \, dx + \int_1^\infty x^{-\frac{1}{2}s-1} \omega(x^{-1}, \chi) \, dx
\]
\[
= \int_1^\infty x^{\frac{1}{2}s-1} \omega(x, \chi) \, dx + \frac{q^2}{\tau(\chi)} \int_1^\infty x^{-\frac{1}{2}s-\frac{1}{2}} \omega(x, \chi) \, dx.
\]
Similarly as before, the two integrals in the right-hand side above are both uniformly convergent in every compact subset of \( \mathbb{C} \) by the Weierstrass \( M \)-test, so \( \xi(s, \chi) \) is entire. Since \( \pi^{\frac{1}{2} s} q^{-\frac{1}{2} s} \Gamma(\frac{1}{2} s)^{-1} \) is entire, we have that \( L(s, \chi) \) is entire. Furthermore, we can replace \( s \) with \( 1 - s \) and \( \chi \) with \( \overline{\chi} \) to get

\[
\xi(1 - s, \overline{\chi}) = \int_1^\infty x^{-\frac{1}{2} s - \frac{1}{2}} \omega(x, \overline{\chi}) dx + \frac{q^{\frac{1}{2}}}{\tau(\chi)} \int_1^\infty x^{\frac{1}{2} s - 1} \omega(x, \chi) dx.
\]

Note that for any primitive \( \chi \) with \( \chi(-1) = 1 \), since

\[
\tau(\chi) = \sum_{m=0}^{q-1} \chi(m)e\left(\frac{m}{q}\right) = \sum_{m=0}^{q-1} \chi(-m)e\left(-\frac{m}{q}\right) = \overline{\tau(\chi)},
\]

it follows from Lemma 2.11 that \( \tau(\chi)\tau(\overline{\chi}) = |\tau(\chi)|^2 = q \). Thus, we overall have

\[
\xi(1 - s, \overline{\chi}) = \frac{q^{\frac{1}{2}}}{\tau(\chi)} \xi(s, \chi),
\]

as needed.

**Case 2:** \( \chi(-1) = -1 \). Replace \( s \) with \( s + 1 \) in (A.1) to get

\[
\pi^{-\frac{1}{2}(s+1)} q^{\frac{1}{2}(s+1)} \Gamma(\frac{1}{2}(s + 1)) n^{-s} = \int_0^\infty n x^{\frac{1}{2} s - \frac{1}{2}} e^{-\frac{n^2 \pi x}{q}} dx, \quad (\sigma > -1).
\]

This gives us the identity

\[
\xi(s, \chi) = \pi^{-\frac{1}{2}(s+1)} q^{\frac{1}{2}(s+1)} \Gamma(\frac{1}{2}(s + 1)) L(s, \chi) = \int_0^\infty x^{\frac{1}{2} s - \frac{1}{2}} \omega_1(x, \chi) dx, \quad (\sigma > -1),
\]

where

\[
\omega_1(x, \chi) := \sum_{n=1}^{\infty} n \chi(n) e^{-\frac{n^2 \pi x}{q}}, \quad (x > 0).
\]

As before, we can split the integral to obtain

(A.4) \[
\xi(s, \chi) = \int_1^\infty x^{\frac{1}{2} s - \frac{1}{2}} \omega_1(x, \chi) dx + \frac{q^{\frac{1}{2}}}{\tau(\chi)} \int_1^\infty x^{-\frac{1}{2} s - \frac{1}{2}} \omega_1(x^{-1}, \overline{\chi}) dx.
\]

We wish to prove a functional equation for \( \omega_1(x, \chi) \). To do so, we take \( \frac{\partial}{\partial a} \) of the identity given in Lemma [A.4] which yields

\[
-2\pi t \sum_{n=-\infty}^{\infty} (n + a)e^{-(n+a)^2 \pi t} = \frac{2\pi i}{t^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} ne^{-\frac{n^2 \pi x}{q} + 2\pi i a}.
\]

Set \( t = \frac{q}{x} \) and \( a = \frac{m}{q} \) to obtain

\[
\sum_{n=-\infty}^{\infty} ne^{-\frac{n^2 \pi x}{q} + 2\pi i an} = i \left(\frac{q}{x}\right)^{\frac{3}{2}} \sum_{n=-\infty}^{\infty} \left(n + \frac{m}{q}\right) e^{-\frac{(n + m q)^2}{\pi} \frac{1}{x}}.
\]
This allows us to derive a functional equation for \( \omega_1(x, \chi) \):

\[
\tau(\chi) \omega_1(x, \chi) = \sum_{m=0}^{q-1} \chi(m)e\left(\frac{m}{q}\right) \left( \frac{1}{2} \sum_{n=-\infty}^{\infty} ne^{-\frac{n^2 \pi x}{q}} \right) = \frac{1}{2} \sum_{m=0}^{q-1} \chi(m) \sum_{n=-\infty}^{\infty} ne^{-\frac{n^2 \pi x + 2\pi i(mn)}{q}}
\]

\[
= \frac{1}{2} i \left( \frac{q}{x} \right)^{\frac{3}{2}} \sum_{m=0}^{q-1} \chi(m) \sum_{n=-\infty}^{\infty} (n + \frac{m}{q}) e^{-(n + \frac{m}{q})^2 \frac{\pi x}{q}}
\]

\[
= \frac{1}{2} i q^{\frac{1}{2}} x^{-\frac{3}{2}} \sum_{m=0}^{q-1} \chi(m) \sum_{n=-\infty}^{\infty} (qn + m) e^{-(mn + m)^2 \frac{\pi x}{q}}
\]

\[
= iq^{\frac{1}{2}} x^{-\frac{3}{2}} \left( \frac{1}{2} \sum_{k=\infty}^{\infty} k \chi(k)e^{-\frac{\pi k^2}{q}} \right) = iq^{\frac{1}{2}} x^{-\frac{3}{2}} \omega_1(x^{-1}, \chi).
\]

Applying the above to (A.4), we see that

\[
\xi(s, \chi) = \pi^{-\frac{1}{2}(s+1)} q^{\frac{1}{2}(s+1)} \Gamma\left(\frac{1}{2}(s + 1)\right) L(s, \chi)
\]

\[
= \int_1^{\infty} x^{\frac{1}{2}s - \frac{1}{2}} \omega_1(x, \chi) dx + \frac{q^{\frac{3}{2}}}{\tau(\chi)} \int_1^{\infty} x^{\frac{1}{2}s - \frac{1}{2}} \omega_1(x^{-1}, \chi) dx
\]

\[
= \int_1^{\infty} x^{\frac{1}{2}s - \frac{1}{2}} \omega_1(x, \chi) dx + \frac{iq^{\frac{1}{2}}}{\tau(\chi)} \int_1^{\infty} x^{-\frac{1}{2}s} \omega_1(x, \chi) dx.
\]

Note that \( \omega_1(x, \chi) \) decays at least exponentially, since for sufficiently large \( x \), we have

\[
\omega_1(x, \chi) \leq \sum_{n=1}^{\infty} ne^{-\frac{n^2 \pi x}{q}} \leq e^{-\frac{\pi x}{q}} + \int_1^{\infty} te^{-\frac{t^2 \pi x}{q}} dt = O\left(e^{-\frac{\pi x}{q}}\right).
\]

Here, we have used that \( \sum_{n=1}^{\infty} ne^{-\frac{n^2 \pi x}{q}} \) is a right-hand Riemann sum approximation of \( \int_1^{\infty} te^{-\frac{t^2 \pi x}{q}} dt \),

and thus bounds the integral from below, since \( te^{-\frac{t^2 \pi x}{q}} \) is decreasing in \( t \in [1, \infty) \) for sufficiently large \( x \). Thus, both integrals in the above expression for \( \xi(s, \chi) \) are uniformly convergent in every compact subset of \( \mathbb{C} \) by the Weierstrass M-test, proving that \( \xi(s, \chi) \) is entire and consequently, that \( L(s, \chi) \) is entire. Furthermore, since any primitive \( \chi \) with \( \chi(-1) = -1 \) satisfies

\[
\tau(\chi) = \sum_{m=0}^{q-1} \chi(m)e\left(\frac{m}{q}\right) = \sum_{m=0}^{q-1} \chi(m)e\left(-\frac{m}{q}\right) = \sum_{m=0}^{q-1} \chi(-m)e\left(-\frac{m}{q}\right) = -\tau(\chi),
\]

it follows from Lemma 2.11 that \( \tau(\chi) \tau(\overline{\chi}) = -|\tau(\chi)|^2 = -q \). Thus, we overall have

\[
\xi(1 - s, \overline{\chi}) = \frac{iq^{\frac{1}{2}}}{\tau(\chi)} \xi(s, \chi),
\]

as needed.

\( \square \)

The functional equation allows us to classify the zeroes of a Dirichlet L-function.

**Corollary A.5.** Let \( \chi \) be a primitive character of period \( q \). Any zero of \( \xi(s, \chi) \) is contained in the critical strip \( \{ 0 \leq \sigma \leq 1 \} \) (but neither \( s = 0 \) or \( 1 \) is a zero), and such zeroes are placed symmetrically with respect to the critical line \( \sigma = \frac{1}{2} \).

The zeroes of \( L(s, \chi) \) are precisely given by the zeroes of \( \xi(s, \chi) \) (denoted as the nontrivial zeroes) and simple zeroes at certain nonpositive integers (denoted as the trivial zeroes). If \( \chi \) is principal, the
trivial zeroes are at \( s = -2, -4, \ldots \), whereas otherwise, they are at \( s = -a(\chi), -a(\chi) - 2, -a(\chi) - 4, \ldots \), where \( a(\chi) = 0 \) if \( \chi(-1) = 1 \) and \( a(\chi) = 1 \) if \( \chi(-1) = -1 \).

**Proof.** The Euler product shows that \( L(s, \chi) \) has no zeros in \( \{ \sigma > 1 \} \), and \( \pi^{-\frac{s+a(\chi)}{2}} q^{-\frac{s+a(\chi)}{2}} \Gamma\left(\frac{s+a(\chi)}{2}\right) \) is nonvanishing for all \( s \in \mathbb{C} \). Thus, \( \xi(s, \chi) \) has no zeros in \( \{ \sigma > 1 \} \). By the functional equation

\[
\xi(1-s, \chi) = \frac{i^{a(\chi)} q^\frac{1}{2}}{\pi(\chi)} \xi(s, \chi),
\]

it follows that \( \xi(s, \chi) \) also has no zeros in \( \{ \sigma < 0 \} \).

We invoke the important fact that \( L(1, \chi) \) does not vanish; one way of discerning this is to see that the Dedekind zeta function of \( K = \mathbb{Z}[\zeta_q] \) (where \( \zeta_q \) is a \( q \)th root of unity) satisfies

\[
\zeta_K(s) = \prod_{\chi \in X_q} L(s, \chi),
\]

and has a pole at \( s = 1 \), which means that none of the \( L(1, \chi) \) can vanish, since among \( X_q \), only the principal character \( \chi_0 \) of period \( q \) satisfies that \( L(s, \chi_0) \) has a pole, which is simple (for a detailed discussion of the Dedekind zeta function and proofs of the aforementioned claims regarding it, we refer the reader to [24, Theorem 4.3]). Thus, \( L(1, \chi) \) and \( L(1, \overline{\chi}) \) are nonvanishing, which implies that \( \xi(1, \chi), \xi(1, \overline{\chi}), \xi(0, \chi), \) and \( \xi(0, \overline{\chi}) \) are nonvanishing as well.

Note that \( \overline{\xi(1-s, \chi)} = \xi(1-s, \chi) = \frac{i^{a(\chi)} q^\frac{1}{2}}{\pi(\chi)} \xi(s, \chi) \), which shows that \( s \) is a zero of \( \xi(s, \chi) \) if and only if \( 1-s \) is also a zero, and these have the same multiplicity. This is precisely the fact that the zeroes of \( \xi(s, \chi) \) are placed symmetrically with respect to the critical line \( \sigma = \frac{1}{2} \).

Finally, note that \( \pi^{-\frac{s+a(\chi)}{2}} q^{-\frac{s+a(\chi)}{2}} \Gamma\left(\frac{s+a(\chi)}{2}\right) \) has no zeroes, and this function’s poles are given by simple poles at all \( s \) such that \( \frac{s+a(\chi)}{2} \in \mathbb{Z}_{<0} \), i.e., at \( s \in \{-a(\chi), -a(\chi) - 2, -a(\chi) - 4, \ldots \} \). If \( \chi \) is principal, then \( \xi(s, \chi) \) has a simple pole at \( s = 0 \) and is analytic and nonzero for \( \sigma < 0 \), whereas if \( \chi \) is principal, then \( \xi(s, \chi) \) is analytic and nonzero at \( s = 0 \) and for \( \sigma < 0 \). Thus, our final claim in the proposition holds.

While we know that the zeroes of \( \xi(s, \chi) \) (i.e., the nontrivial zeroes of \( L(s, \chi) \)) are located in the critical strip \( \{ 0 < \sigma < 1 \} \), it is also important to know if such zeroes exist, and if so, how many. In fact, we see that there are infinitely many such zeroes.

**Theorem A.6.** Let \( \chi \) be a primitive Dirichlet character. Define \( f(s) = \xi(s) = \frac{1}{2} s(s-1) \xi(s, \chi_0) \) if \( \chi \) is principal and \( f(s) = \xi(s, \chi) \) otherwise. The entire function \( f(s) \) has order 1 and has infinitely many zeroes. In particular, \( L(s, \chi) \) has infinitely many nontrivial zeroes.

**Proof.** First, we show that there exists \( C > 0 \) such that \( |f(s)| = O(e^{C|s|\log|s|}) \) for sufficiently large \( |s| \). By the functional equation for \( f \), it suffices to show this claim for \( \sigma > \frac{1}{2} \). The factor \( \frac{1}{2} s(s-1) \), if present in the expression for \( f(s) \), is polynomial and thus \( O(e^{C|s|\log|s|}) \) for sufficiently large \( |s| \). The factor \( \frac{1}{2} (s+a(\chi)) \) is exponential and thus also \( O(e^{C|s|\log|s|}) \) for sufficiently large \( |s| \). Recall Stirling’s formula for \( \Gamma(s) \) (see [15, p.120]), which states that for a fixed \( \delta > 0 \), we have

\[
\Gamma(s) = e^{s \log s} e^{-s} \frac{\sqrt{2\pi}}{s^\frac{1}{2}} \left( 1 + O\left( \frac{1}{|s|^\frac{1}{2}} \right) \right)
\]

for sufficiently large \( |s| \) such that \( |\arg s| \leq \pi - \delta \). Since \( \sigma > \frac{1}{2} \) implies that \( |\arg \frac{1}{2} s| < \frac{\pi}{2} \) It follows that for sufficiently large \( C > 0 \), we have

\[
\Gamma\left(\frac{s+a(\chi)}{2}\right) = e^{\frac{1}{2} (s+a(\chi)) \left( \log(s+a(\chi)) + \log\frac{1}{2} \right)} \frac{\sqrt{2\pi}}{(s+a(\chi))^\frac{1}{2}} \left( 1 + O\left( \frac{1}{|s|^\frac{1}{2}} \right) \right) = O(e^{C|s|\log|s|})
\]

7The classical way to show that \( L(1, \chi) \neq 0 \) is by appealing to the class number formula, which is essentially equivalent to the above method. See [3] [6] for more details.
for \( \sigma > \frac{1}{2} \) and sufficiently large \( |s| \).

We only have the factor \( L(s, \chi) \) left to bound. We note that \( L(s, \chi) \) is bounded in the half-plane \( \{ \sigma \geq 2 \} \) by \( \sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6} \), so it suffices to show that \( L(s, \chi) \) is bounded by \( e^{C|s| \log |s|} \) for sufficiently large \( |s| \) such that \( \frac{1}{2} < \sigma < 2 \). Indeed, letting \( (x) := x - \lfloor x \rfloor \) be the fractional part of \( x \), we check for an arbitrary \( X \geq 1 \) that

\[
|L(s, \chi)| \leq \left| \sum_{1 \leq n \leq X} n^{-s} \right| + \left| \sum_{n > X} n^{-s} \right| = \left| \sum_{1 \leq n \leq X} n^{-s} \right| + \int_{X}^{\infty} x^{-s} \, d|x|
\]

Using the fact that \( \frac{1}{2} < \sigma < 2 \) and imposing the lower bound \( t \geq 1 \), we further obtain by the triangle inequality

\[
|L(s, \chi)| \leq \sum_{1 \leq n \leq X} n^{-\frac{1}{2}} + \frac{1}{X^{\frac{1}{2}}} + \frac{1}{tX^{\frac{1}{2} - 1}} + (2 + t) \cdot \frac{1}{X^{\frac{3}{2}} + 1} \leq \int_{0}^{\lfloor X \rfloor} \frac{1}{x^{\frac{1}{2}}} \, dx + \frac{1}{X^{\frac{1}{2}}} + \frac{X^{\frac{1}{2}}}{t} + \frac{2 + t}{X^{\frac{3}{2}}}
\]

Setting \( X = t \), we get that the above is \( O(t^{\frac{1}{2}}) \). This clearly implies that \( L(s, \chi) \) is \( O(|s|^{\frac{1}{2}}) \) for \( \sigma > \frac{1}{2} \). Thus, we overall have that \( f(s) = O(e^{C|s| \log |s|}) \).

On the other hand, it follows from our work above that \( \frac{f(s)}{\Gamma(\frac{s}{2})} \) decays at most exponentially, whereas \( \Gamma\left(\frac{s + a(\chi)}{2}\right) \geq \Gamma\left(\frac{s}{2}\right) > e^{0.49|s| \log |s|} \) for sufficiently large \( s \in \mathbb{R}_{>0} \), so \( f(s) \) cannot be \( O(e^{C_{1}|s|}) \) for any \( C_{1} > 0 \). So, \( f \) is an entire function of order 1.

Suppose for the sake of a contradiction that \( f \) has finitely many zeroes \( z_{1}, \ldots, z_{N} \), counted with multiplicity. Then, since \( f(0) \neq 0 \), Hadamard’s factorization theorem (see [15, p.147]) implies

\[
f(z) = e^{g(z)} \prod_{n=1}^{N} \left( 1 - \frac{z}{z_{n}} \right)
\]

for a polynomial \( g(z) \) of degree \( \leq 1 \). But this contradicts the fact that \( f(s) \) cannot be \( O(e^{C_{1}|s|}) \) for any \( C_{1} > 0 \). Thus, \( f \) has infinitely many zeroes, i.e, \( L(s, \chi) \) has infinitely many nontrivial zeroes.

\[\square\]

We fittingly conclude our discussion of Dirichlet \( L \)-functions with a famous conjecture regarding the placement of their infinitely many nontrivial zeroes, perhaps the most famous open problem in mathematics.
Conjecture A.7 (Generalized Riemann Hypothesis). The nontrivial zeroes of $L(s, \chi)$ all lie on the critical line $\sigma = \frac{1}{2}$.

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