HODGE FILTRATION, MINIMAL EXPONENT, AND LOCAL VANISHING

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ABSTRACT. We bound the generation level of the Hodge filtration on the localization along a hypersurface in terms of its minimal exponent. As a consequence, we obtain a local vanishing theorem for sheaves of forms with log poles. These results are extended to **Q**-divisors, and are derived from a result of independent interest on the generation level of the Hodge filtration on nearby and vanishing cycles.

A. INTRODUCTION

Let X be a smooth complex variety of dimension n, and $\mathscr{D}_X$ the sheaf of differential operators on X. An important invariant of a filtered $\mathscr{D}_X$-module $(\mathcal{M}, F)$ of geometric origin is the complexity of its filtration, namely how many steps are required to fully determine it. Concretely, the filtration F is generated at level q if

$$F_\ell \mathscr{D}_X \cdot F_q \mathcal{M} = F_{q+\ell}\mathcal{M} \quad \text{for all} \quad \ell \geq 0.$$

Here $F_\bullet \mathscr{D}_X$ denotes the standard filtration by the order of differential operators.

In this paper we give a bound for the generation level of the Hodge filtration on $\mathscr{D}_X$-modules naturally associated to rational multiples of a reduced effective divisor D on X, in terms of data provided by the Bernstein-Sato polynomial of D. This study was initiated by Saito [Sai09], who provided such bounds for special types of singularities. Some general results were later found in [MP16], [MP19]. We improve them here, using the main result of [MP18], and also exploit the fact that they are, somewhat surprisingly, related to local vanishing theorems for sheaves of forms with log poles in birational geometry.

Reduced divisors. To highlight the main points with a minimum amount of technicalities, we first restrict our discussion to the case when we simply deal with a reduced effective divisor D. The corresponding $\mathscr{D}_X$-module is the localization $\mathscr{O}_X(*D)$, that is, the sheaf of functions with poles of arbitrary order along D. It is well known that $\mathscr{O}_X(*D)$ is regular holonomic, and underlies a mixed Hodge module on X; therefore it comes endowed with a *Hodge filtration* $F_p\mathscr{O}_X(*D)$, with $p \geq 0$. See e.g. [MP16] for an in-depth study of this filtration. If D is smooth, then the filtration is generated at level 0, hence from now on we focus on the case when D is singular. We prove:

Theorem A. *For every singular divisor D, the Hodge filtration on $\mathscr{O}_X(*D)$ is generated at level $n - 1 - \lceil \widetilde{\alpha}_D \rceil$.*

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Here $\widetilde{\alpha}_D$ is the *minimal exponent* of D, a positive rational number which is defined as the negative of the largest root of the *reduced* Bernstein-Sato polynomial $\tilde{b}_D(s)$; see e.g. [Sai93]. It is a refined version of the log canonical threshold of the pair (X, D), which is equal to $\min\{\widetilde{\alpha}_D, 1\}$. See §1 for further details and references. It was Saito who first pointed out in [Sai09] the relevance of the invariant $n-1-\lceil\widetilde{\alpha}_D\rceil$, proving the bound in Theorem A for isolated semi-quasihomogeneous singularities (when $\widetilde{\alpha}_D$ can be computed explicitly).

Since $\widetilde{\alpha}_D > 0$, Theorem A recovers in particular the fact that $F_\bullet\mathscr{O}_X(*D)$ is always generated at level $n-2$, proved in [MP16, Theorem B]. Note also that it is possible to do better than Theorem A: as an extreme case, if D is a singular simple normal crossing divisor, then $F_\bullet\mathscr{O}_X(*D)$ is generated at level 0, but $\widetilde{\alpha}_D = 1$. The bound is nevertheless sometimes optimal; for instance, this is the case when D has an isolated quasihomogeneous singularity by [Sai09, Theorem 0.7].

Moreover, Saito [Sai93, Theorem 0.4] showed that $\widetilde{\alpha}_D > 1$ is equivalent to D having rational singularities, and therefore:

Corollary B. *If $n \geq 3$ and the divisor D has rational singularities, then the Hodge filtration on $\mathscr{O}_X(*D)$ is generated at level $n-3$.*[1]

This was proved when D has isolated singularities, and conjectured to be true in general, in [MOP17]. The general conjecture was already verified recently by Kebekus-Schnell [KS18, §1.3], as a consequence of a local vanishing conjecture; more on this below. Note that $\widetilde{\alpha}_D$ could however be much larger than 1, and is in fact optimally bounded above by $n/2$ in [Sai94] (see also [MP18, Theorem E]).

It turns out that the generation level of the Hodge filtration on $\mathscr{O}_X(*D)$ is intimately linked to a result in birational geometry, namely to local vanishing for pushforwards of bundles of forms with log poles. Consider a log resolution $\mu\colon Y \to X$ of the pair (X, D), which is an isomorphism over $U = X \smallsetminus D$, and denote $E = (\mu^*D)_{\rm red}$. We showed in [MP16, Theorem 17.1] that $F_\bullet\mathscr{O}_X(*D)$ is generated at level q if and only if $R^i\mu_*\Omega_Y^{n-i}(\log E) = 0$ for $i > q$, so consequently we obtain:

Corollary C. *With the above notation, we have*

$$R^i\mu_*\Omega_Y^{n-i}(\log E) = 0 \quad \textit{for} \quad i > n-1-\lceil\widetilde{\alpha}_D\rceil.$$

When $i \geq n-1$ this is shown by elementary methods in [MP16, Theorem B], leading to the coarse bound $n-2$ for the generation level of the Hodge filtration mentioned above. When D has rational singularities and $i = n-2$, it is proved in [MOP17] in the isolated singularities case, and can be deduced in general from a vanishing statement obtained by Kebekus-Schnell [KS18, Theorem 1.9], which answers [MOP17, Conjecture A]. Using Corollary C, we can in fact obtain a strengthening of this conjecture/statement in the absolute case of a *reduced singular hypersurface*: by this here we mean a singular complex scheme D, reduced but not necessarily irreducible, that can be embedded as a hypersurface in a smooth variety. In this case D has an associated minimal exponent $\widetilde{\alpha}_D$, independent of the embedding (since this is the case already for the Bernstein-Sato polynomial). We consider a resolution of singularities $\mu\colon \widetilde{D} \to D$, given by the disjoint union of resolutions of the irreducible components of

[1] As mentioned above, for $n = 2$ the filtration is always generated at level 0.

D. We further assume that μ is an isomorphism over the smooth locus of D and the reduced inverse image of the singular locus of D is a simple normal crossing divisor E on $\widetilde{D}$. We then have [2]

Theorem D. *With the above notation, if* $\dim(D) = n-1$, *then*

$$R^i\mu_*\Omega^{n-1-i}_{\widetilde{D}}(\log E) = 0 \quad \text{for all} \quad i > n-1-\lceil\widetilde{\alpha}_D\rceil.$$

We emphasize that here the overall strategy is reversed: we first show the generation bound in Theorem A using methods from the theory of (Hodge) $\mathscr{D}$-modules, and then deduce the birational Corollary C, which in turn is used to prove Theorem D. At the moment we do not know how to approach the latter vanishing results via more standard methods in birational geometry.

Rational multiples. Following [MP19], [MP18], we also consider a multiple αD, where α is a positive rational number and D is a reduced effective divisor on X, as above. The set-up is local: assuming that D is defined by a regular function f, the natural replacement for the localization $\mathscr{O}_X[1/f]$ is the $\mathscr{D}_X$-module

$$\mathcal{M}(f^{-\alpha}) := \mathscr{O}_X[1/f]f^{-\alpha},$$

the free rank 1 module over $\mathscr{O}_X[1/f]$ generated by the formal symbol $f^{-\alpha}$; see §1. This is a direct summand of a mixed Hodge module, and so analogously it comes endowed with a Hodge filtration $F_p\mathcal{M}(f^{-\alpha})$, with $p \geq 0$. Again, if D is smooth, then this filtration is generated at level 0, hence from now on we focus on the case when f defines a singular hypersurface.

Theorem A and Corollary C above are then special cases (when $\alpha = 1$) of the following two statements that will be the focus of the paper.

Theorem E. *If f defines a singular reduced hypersurface, then the Hodge filtration on $\mathcal{M}(f^{-\alpha})$ is generated at level $n - \lceil\widetilde{\alpha}_f + \alpha\rceil$.*

In the special case when D has an isolated quasihomogeneous singularity, by analogy with the reduced case in [Sai09], this result was conjectured in [Pop18] and proved in [Zha18]. Note also that Theorem E recovers the second statement of [MP19, Theorem 10.1], namely that the filtration on $\mathcal{M}(f^{-\alpha})$ is always generated at level $n-1$.

Consider now a log resolution $\mu\colon Y \to X$ of the pair (X, D) as above, and $E = (\mu^*D)_{\mathrm{red}}$. According to [MP19, Theorem 10.1], the statement of Theorem E is equivalent to the following general form of local vanishing:

Corollary F. *With the above notation, we have*

$$R^i\mu_*\big(\Omega^{n-i}_Y(\log E)\otimes_{\mathscr{O}_Y}\mathscr{O}_Y(-\lceil\mu^*\alpha D\rceil)\big) = 0 \quad \textit{for} \quad i > n - \lceil\widetilde{\alpha}_f + \alpha\rceil.$$

Recall for completeness that it is always the case that

$$R^i\mu_*\big(\Omega^{j}_Y(\log E)\otimes_{\mathscr{O}_Y}\mathscr{O}_Y(-\lceil\mu^*\alpha D\rceil)\big) = 0 \quad \text{for} \quad i+j > n.$$

This is proved in [MP19, Corollary C], still using methods from the theory of mixed Hodge modules, but of a different flavor.

[2] Note that D has rational singularities if and only if $\widetilde{\alpha}_D > 1$, so the case $i = n-2$ corresponds to the statements in *loc. cit.*

Hodge ideals. The Hodge filtration on $\mathcal{M}(f^{-\alpha})$ is best expressed and studied in terms of the *Hodge ideals* of αD. According to [MP19, §4], for each $p \geq 0$ there is a coherent sheaf of ideals $I_p(\alpha D)$ on X such that

$$F_p\mathcal{M}(f^{-\alpha}) = I_p(\alpha D) \otimes \mathscr{O}_X(pD) f^{-\alpha}.$$

Therefore Theorem E provides an effective bound describing which higher Hodge ideals of αD are fully determined by lower ones. This type of result is very useful for concrete calculations of Hodge ideals, see [MP16] and [MP19].

Corollary G. *For every nonnegative integers ℓ and p, with $p \geq n - \lceil \widetilde{\alpha}_f + \alpha \rceil$, we have*

$$F_\ell \mathscr{D}_X \cdot \big(I_p(\alpha D) \otimes \mathscr{O}_X(pD) f^{-\alpha}\big) = I_{p+\ell}(\alpha D) \otimes \mathscr{O}_X\big((p+\ell)D\big) f^{-\alpha}.$$

Nearby and vanishing cycles. All the above results are consequences of a statement of independent interest regarding the generation level of the Hodge filtration on the graded quotients of the V-filtration associated to the regular function $f \in \mathscr{O}_X(X)$. Concretely, the V-filtration is defined on the the left $\mathscr{D}_{X\times \mathbf{C}}$-module $\iota_+\mathscr{O}_X$, the push-forward of $\mathscr{O}_X$ via the graph embedding

$$\iota\colon X \hookrightarrow X \times \mathbf{C}, \quad x \mapsto \big(x, f(x)\big),$$

with respect to the hypersurface $\{t = 0\}$, where t is the coordinate on $\mathbf{C}$. Recalling that this is a (discrete) decreasing filtration, we consider $\mathrm{Gr}_V^\alpha(\iota_+\mathscr{O}_X) := V^\alpha\iota_+\mathscr{O}_X / V^{>\alpha}\iota_+\mathscr{O}_X$. These are $\mathscr{D}_X$-modules that underlie Hodge modules supported on the graph embedding of X; in particular they come endowed with a Hodge filtration $F_\bullet \mathrm{Gr}_V^\alpha(\iota_+\mathscr{O}_X)$ induced by that on $\iota_+\mathscr{O}_X$. The cases $\alpha = 0$ and $\alpha \in (0,1]$ are intimately related to the vanishing, respectively nearby, cycles of f. For details see §1 and §2. The main result we prove is:

Theorem H. *If f defines a singular, reduced hypersurface, and $\alpha \in [0,1]$ is a rational number, then the Hodge filtration on $\mathrm{Gr}_V^\alpha(\iota_+\mathscr{O}_X)$ is generated at level $n - \lceil \widetilde{\alpha}_f + \alpha \rceil + 1$.*

The proof of this theorem is the technical core of the paper. More precisely, we describe concretely the associated graded quotients of the Hodge filtration on these $\mathscr{D}_X$-modules in the range below the minimal exponent of f; see Proposition 4.5. Using this, we apply a homological criterion for the generation level of the filtration on special filtered $\mathscr{D}_X$-modules $(\mathcal{M}, F)$ via the duality functor. This is proved in Proposition 3.3, and is inspired by a duality approach to generation in [Sai94]. In order to deduce Theorem E from Theorem H, the key tool is to reinterpret the main result of [MP18] as a connection between the Hodge filtration on $\mathcal{M}(f^{-\alpha})$ and the induced Hodge filtration on V^α; see Proposition 5.4.

Bounds in terms of singularity invariants in birational geometry. We conclude by noting that the minimal exponent $\widetilde{\alpha}_f$ can be bounded below in terms of basic invariants of the singularity, or in terms of discrepancies on a log resolution. This can be translated into bounds of a somewhat different flavor in the statements above.

Consider a log resolution $\mu\colon Y \to X$ of the pair (X, D) as above, in the neighborhood of a (singular) point $x \in D$. Assuming in addition that the strict transform $\widetilde{D}$ of D is smooth, we define integers a_i and b_i by the expressions

$$\mu^*D = \widetilde{D} + \sum_{i=1}^m a_i F_i \quad \text{and} \quad K_{Y/X} = \sum_{i=1}^m b_i F_i,$$

where $F_1, \ldots, F_m$ are the prime exceptional divisors, and set

$$\gamma := \min_{i=1,\ldots,m} \left\{ \frac{b_i + 1}{a_i} \right\}.$$

Denote also by $d \geq 2$ the multiplicity of D at x, and by r the dimension of the singular locus of the projectivized tangent cone $\mathbf{P}(C_x D)$ (declaring that $r = -1$ if $\mathbf{P}(C_x D)$ is smooth). We then have the following lower bounds in a neighborhood of x:

- $\widetilde{\alpha}_f \geq \gamma$.
- $\widetilde{\alpha}_f \geq \frac{n-r-1}{d}$.

The first is [MP18, Corollary D] and the second is [MP18, Theorem E(3)]. Note that, unlike $\widetilde{\alpha}_f$, γ depends on the choice of log resolution. Finally, we also have:

- $k_0 := \lfloor \widetilde{\alpha}_f - \alpha \rfloor$ is the k-log canonicity level of the pair $(X, \alpha D)$, according to [MP18, Corollary C].

We recall that $(X, \alpha D)$ is 0-log canonical if it is log canonical, while being k-log canonical for $k \geq 1$ is a refinement of the statement that D has rational singularities. It essentially means that the Hodge filtration on $\mathcal{M}(f^{-\alpha})$ is as simple as possible up to level k, namely equal to the pole order filtration; the upshot of this paper is that this condition also imposes a bound on the generation level of this Hodge filtration.

Further general properties of the minimal exponent $\widetilde{\alpha}_f$, and open problems, can be found in [MP18, §6].

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B. Preliminaries

1. Hodge filtration, V-filtration, and minimal exponent. Let X be a smooth n-dimensional complex algebraic variety and $f \in \mathscr{O}_X(X)$ a nonzero regular function. Consider the graph embedding

$$\iota \colon X \hookrightarrow X \times \mathbf{C}, \quad x \mapsto \big(x, f(x)\big)$$

and the left $\mathscr{D}_{X \times \mathbf{C}}$-module $\iota_+ \mathscr{O}_X$, as well as the corresponding right $\mathscr{D}_{X \times \mathbf{C}}$-module $\iota_+ \omega_X$. A detailed discussion of the material in the paragraph below can be found for instance in [MP18, §2]. We denote by t the coordinate on $\mathbf{C}$. Recall that we have

$$\iota_+ \mathscr{O}_X \simeq \mathscr{O}_X[t]_{f-t} / \mathscr{O}_X[t],$$

with the obvious $\mathscr{D}_{X \times \mathbf{C}}$-module structure. Denoting by δ the class of $\frac{1}{f-t}$, every element in $\iota_+ \mathscr{O}_X$ can be written uniquely as

$$\sum_{i \geq 0} v_i \partial_t^i \delta,$$

with $v_i \in \mathscr{O}_X$, only finitely many nontrivial. We clearly have the relation $t\delta = f\delta$. With this description, multiplication by t is given by

$$t(v\partial_t^i \delta) = fv\partial_t^i \delta - iv\partial_t^{i-1} \delta$$

and the action of a derivation $P \in \mathrm{Der}_{\mathbf{C}}(\mathscr{O}_X)$ is given by

$$P(v\partial_t^i \delta) = P(v)\partial_t^i \delta - P(f)v\partial_t^{i+1} \delta.$$

Recall also that the (trivial) Hodge filtration on $\mathscr{O}_X$ induces a Hodge filtration on $\iota_+\mathscr{O}_X$ given by

$$F_{p+1}(\iota_+\mathscr{O}_X) = \sum_{i=0}^{p} \mathscr{O}_X \partial_t^i \delta \tag{1.1}$$

(see, for example, [Sai93, (1.8.6)]). We note that the shift by 1 is needed in order to ensure compatibility when applying the convention for shifting filtrations as we pass from left to right filtered $\mathscr{D}$-modules on X and $X \times \mathbf{C}$ respectively; see §2.

We next consider the rational V-filtration on $\iota_+\mathscr{O}_X$ with respect to t. Recall that this is an exhaustive, decreasing, discrete, and left continuous filtration $(V^\alpha\iota_+\mathscr{O}_X)_{\alpha\in\mathbf{Q}}$. It is defined uniquely by a number of properties listed for instance in [MP18, §2]. The Hodge filtration on $\iota_+\mathscr{O}_X$ induces a filtration on each $V^\alpha\iota_+\mathscr{O}_X$ and thus the Hodge filtration on $\mathrm{Gr}_V^\alpha(\iota_+\mathscr{O}_X) = V^\alpha\iota_+\mathscr{O}_X/V^{>\alpha}\iota_+\mathscr{O}_X$.

As is standard, we denote by $b_f(s)$ the Bernstein-Sato polynomial of f. Assuming that $D := \mathrm{div}(f) \neq 0$, the polynomial $(s+1)$ divides $b_f(s)$, and $\widetilde{b}_f(s) = b_f(s)/(s+1)$ is the reduced Bernstein-Sato polynomial of f. Following [Sai16], we denote by $\widetilde{\alpha}_f$ the negative of the largest root of $\widetilde{b}_f(s)$. This is a positive rational number, and we use the convention that $\widetilde{\alpha}_f = \infty$ if $b_f(s) = s+1$, which happens precisely when D is smooth. This invariant is called the *minimal exponent* of f, see [Sai93], and is a refined version of the log canonical threshold of f, which is equal to $\min\{\widetilde{\alpha}_f, 1\}$. See [MP18, §6] for a detailed discussion.

A crucial point is the following link between the minimal exponent and the V-filtration, combining the statements of [MP18, Lemma 5.3] and [MP18, Corollary 6.1].

Lemma 1.2. *For an integer $p \geq 0$ and $\alpha \in (0,1]$, we have*

$$\partial_t^p\delta \in V^\alpha\iota_+\mathscr{O}_X \iff \widetilde{\alpha}_f \geq p+\alpha.$$

For a $\mathbf{Q}$-divisor E on C, we denote by $\mathcal{I}(E)$ its multiplier ideal; see [Laz04, Chapter 9]. If $D = \mathrm{div}(f)$, $\gamma > 0$ is a rational number, and $E = \gamma D$, we will also use the notation $\mathcal{I}(f^\gamma)$ for $\mathcal{I}(E)$. The main result of [BS05] states that for every $\alpha > 0$, we have

$$\mathcal{I}\big(f^{\alpha-\epsilon}\big) = V^\alpha\iota_+\mathscr{O}_X \quad \text{for} \ \ 0 < \epsilon \ll 1. \tag{1.3}$$

In order to define and study Hodge ideals for $\mathbf{Q}$-divisors, in [MP19] and [MP18] we considered for each $\alpha > 0$ the twisted localization $\mathscr{D}_X$-module

$$\mathcal{M}(f^{-\alpha}) := \mathscr{O}_X(*D)f^{-\alpha},$$

with $D = \mathrm{div}(f)$, i.e. the free $\mathscr{O}_X(*D)$-module of rank 1 with generator the symbol $f^{-\alpha}$, with the action of derivations of $\mathscr{O}_X$ given by

$$P(wf^{-\alpha}) := \left(P(w) - \alpha w\frac{P(f)}{f}\right) f^{-\alpha}.$$

The $\mathscr{D}_X$-module $\mathcal{M}(f^{-\alpha})$ is a filtered direct summand of a $\mathscr{D}_X$-module underlying a mixed Hodge module; see [MP19, §2]. In particular, it is regular holonomic, with quasi-unipotent monodromy, and admits a Hodge filtration $F_p\mathcal{M}(f^{-\alpha})$, with $p \geq 0$. It is shown in [MP19, §4] that if Z is the support of D, then we can write

$$F_p\mathcal{M}(f^{-\alpha}) = I_p(\alpha D) \otimes \mathscr{O}_X(pZ)f^{-\alpha},$$

for an ideal $I_p(\alpha D)$, the *p-th Hodge ideal of* αD.

For every $\alpha \in \mathbf{Q}$, we have an isomorphism of $\mathscr{D}_X$-modules

$$\mathcal{M}(f^{-\alpha}) \to \mathcal{M}(f^{-\alpha-1}), \quad wf^{-\alpha} \mapsto (wf)f^{-\alpha-1}, \tag{1.4}$$

which preserves the Hodge filtration; see [MP19, §2]. As a special case, we naturally identify $\mathcal{M}(f^{-1})$ with the usual localization $\mathscr{O}_X(*D)$. In particular, when D is reduced and $\alpha = 1$, this gives the Hodge ideals considered in [MP16].

An important input for this paper is the main result of [MP18], comparing the Hodge ideals and the V-filtration. We only state the case when $D = \mathrm{div}(f)$ is reduced. We use the notation $Q_i(x) = \prod_{j=0}^{i-1}(x+j)$, with the convention that $Q_0 = 1$.

Theorem 1.5 ([MP18, Theorem A′])**.** *If f defines a reduced divisor D and α is a positive rational number, then for every $p \geq 0$ we have*

$$I_p(\alpha D) = \left\{ \sum_{j=0}^{p} Q_j(\alpha) f^{p-j} v_j \;\middle|\; \sum_{j=0}^{p} v_j \partial_t^j \delta \in V^{\alpha}\iota_+\mathscr{O}_X \right\}.$$

2. Nearby and vanishing cycles. Later on we will need bounds for the generation level of the Hodge filtration on nearby and vanishing cycles. To this end we will make use of the duality functor $\mathbf{D}$ on filtered $\mathscr{D}$-modules [Sai88, §2.4]. In order to apply duality, we will pass to the corresponding right $\mathscr{D}_X$-modules.

We recall that there is an equivalence of categories between filtered left and right $\mathscr{D}_X$-modules. Given a filtered left $\mathscr{D}_X$-modules $(\mathcal{M}, F)$, we denote by $(\mathcal{M}^r, F)$ the corresponding filtered right $\mathscr{D}_X$-module. At the level of $\mathscr{O}_X$-modules we have $\mathcal{M}^r = \omega_X \otimes_{\mathscr{O}_X} \mathcal{M}$, while the filtration on $\mathcal{M}^r$ is given by

$$F_{p-n}\mathcal{M}^r = \omega_X \otimes_{\mathscr{O}_X} F_p\mathcal{M} \quad \text{for all} \quad p \in \mathbf{Z},$$

where $n = \dim(X)$.

For right $\mathscr{D}_X$-modules it is customary to use the increasing V-filtration. This is related to the V-filtration on the corresponding left $\mathscr{D}_X$-module as follows. If $\mathcal{M}$ is a left $\mathscr{D}_{X\times\mathbf{C}}$-module and we consider the V-filtrations with respect to the coordinate t on $\mathbf{C}$, then

$$V_\alpha\mathcal{M}^r = \omega_{X\times\mathbf{C}} \otimes_{\mathscr{O}_{X\times\mathbf{C}}} V^{-\alpha}\mathcal{M} = \omega_X \otimes_{\mathscr{O}_X} V^{-\alpha}\mathcal{M},$$

where we identify in the obvious way $\omega_{X\times\mathbf{C}}$ with the pull-back of ω_X.

It is also customary, for a filtered $\mathscr{D}_X$-module $(\mathcal{M}, F)$ and an integer q, to denote $(\mathcal{M}, F)(q) = \big(\mathcal{M}, F[q]\big)$, with

$$F[q]_p\mathcal{M} = F_{p-q}\mathcal{M} \quad \text{for all} \quad p \in \mathbf{Z}.$$

Let now $(\mathcal{M}, F)$ be the filtered right $\mathscr{D}_{X\times\mathbf{C}}$-module underlying a pure polarizable Hodge module of weight d. Recall that the polarization induces an isomorphism

$\mathbf{D}(\mathcal{M},F)\simeq(\mathcal{M},F)(d)$. The nearby and vanishing cycles of $(\mathcal{M},F)$ with respect to t are given, respectively, by

$$\Psi_t(\mathcal{M},F)=\bigoplus_{-1\leq\beta<0}\big(\mathrm{Gr}^V_\beta(\mathcal{M}),F\big)(1)\quad\text{and}\quad\Phi_{t,1}(M,F)=\big(\mathrm{Gr}^V_0(\mathcal{M}),F\big).$$

We also use the notation $\Psi_{t,\beta}(\mathcal{M},F)$ for $\big(\mathrm{Gr}^V_\beta(\mathcal{M}),F\big)(1)$, when $\beta\in(-1,0)$, but $\Psi_{t,1}(\mathcal{M},F)$ for $\big(\mathrm{Gr}^V_{-1}(\mathcal{M}),F\big)(1)$.

It is a general fact that the duality functor commutes with nearby and vanishing cycles. The results that follow can be found in [Sai89, Theorem 1.6]. Concretely, we have canonical isomorphisms

$$\mathbf{D}\Psi_t(\mathcal{M},F)(1)\simeq\Psi_t\mathbf{D}(\mathcal{M},F)\quad\text{and}\quad\mathbf{D}\Phi_{t,1}(\mathcal{M},F)\simeq\Phi_{t,1}\mathbf{D}(\mathcal{M},F).$$

Using the fact that $\mathbf{D}(\mathcal{M},F)\simeq\big(\mathcal{M},F\big)(d)$, we obtain isomorphisms

$$\mathbf{D}\Psi_t(\mathcal{M},F)\simeq\Psi_t(\mathcal{M},F)(d-1)\quad\text{and}\quad\mathbf{D}\Phi_{t,1}(\mathcal{M},F)\simeq\Phi_{t,1}(\mathcal{M},F)(d).$$

We can in fact be more precise about the first of these isomorphisms; there is a canonical isomorphism

$$\mathbf{D}\Psi_{t,1}(\mathcal{M},F)\simeq\Psi_{t,1}(\mathcal{M},F)(d-1)\tag{2.1}$$

and for every $\beta\in(-1,0)$, there is a canonical isomorphism

$$\mathbf{D}\Psi_{t,\beta}(\mathcal{M},F)\simeq\Psi_{t,-\beta-1}(\mathcal{M},F)(d-1).\tag{2.2}$$

In what follows, we will only be interested in the case when $(\mathcal{M},F)$ is the filtered right $\mathscr{D}_{X\times\mathbf{C}}$-module $(\iota_+\omega_X,F)$ corresponding to $(\iota_+\mathscr{O}_X,F)$. Note that in this case we have $d=n$, hence the isomorphism (2.1) gives

$$\mathbf{D}\mathrm{Gr}^V_{-1}(\iota_+\omega_X)\simeq\mathrm{Gr}^V_{-1}(\iota_+\omega_X)(1+n)\tag{2.3}$$

while the isomorphism (2.2) gives

$$\mathbf{D}\mathrm{Gr}^V_\beta(\iota_+\omega_X)\simeq\mathrm{Gr}^V_{-1-\beta}(\iota_+\omega_X)(1+n)\quad\text{for every}\quad\beta\in(-1,0).\tag{2.4}$$

Similarly, we have

$$\mathbf{D}\mathrm{Gr}^V_0(\iota_+\omega_X)\simeq\mathrm{Gr}^V_0(\iota_+\omega_X)(n).\tag{2.5}$$

Finally, we note that since the Hodge filtration on $\mathrm{Gr}^V_\beta(\iota_+\omega_X)$ is induced by that on $\iota_+\omega_X$, which is the filtered right $\mathscr{D}_{X\times\mathbf{C}}$-module corresponding to $\iota_+\mathscr{O}_X$, using the convention above on upper and lower indexed V-filtrations we have

$$F_{p-n-1}\mathrm{Gr}^V_\beta(\iota_+\omega_X)=\omega_X\otimes_{\mathscr{O}_X}F_p\mathrm{Gr}_V^{-\beta}(\iota_+\mathscr{O}_X).\tag{2.6}$$

3. Generation level. Let $(\mathcal{M},F)$ be a right $\mathscr{D}_X$-module with a good filtration. The filtration F is generated at level q if

$$F_q\mathcal{M}\cdot F_\ell\mathscr{D}_X=F_{q+\ell}\mathcal{M}\quad\text{for all}\quad\ell\geq 0,$$

or equivalently

$$F_p\mathcal{M}\cdot F_1\mathscr{D}_X=F_{p+1}\mathcal{M}\quad\text{for all}\quad p\geq q.$$

A similar definition holds for left $\mathscr{D}_X$-modules, as in the introduction. Note that such q always exists by the definition of a good filtration. Another interpretation is that the filtration is generated at level q if and only if $\mathrm{Gr}^F_\bullet \mathcal{M}$ is generated in degrees $\leq q$ as a graded module over

$$\mathcal{A}_X := \mathrm{Gr}^F_\bullet \mathscr{D}_X \simeq \mathrm{Sym}^\bullet_{\mathscr{O}_X} \mathscr{T}_X,$$

where $\mathscr{T}_X$ is the tangent sheaf of X.

A generation criterion using the duality functor is given by the following result; see [Sai94, Lemma 2.5] and its proof.

Proposition 3.1. *If $(\mathcal{M}, F)$ is a filtered right $\mathscr{D}_X$-module underlying a mixed Hodge module such that $F_{-q-1}\mathbf{D}(\mathcal{M}) = 0$, then the filtration on $\mathcal{M}$ is generated at level q.*

We will also need a refinement of this criterion for (essentially) self-dual $(\mathcal{M}, F)$, and for this we formulate more precisely the setup provided by duality. The 0-section of the cotangent bundle corresponds to a surjective morphism $\mathcal{A}_X \to \mathscr{O}_X$. We denote by $K^\bullet$ the corresponding Koszul complex

$$0 \to K^{-n} \to \cdots \to K^{-1} \to K^0 = \mathscr{O}_X \to 0$$

placed in degrees $-n, \ldots, 0$, where $K^{-i} = \wedge^i \mathscr{T}_X \otimes_{\mathscr{O}_X} \mathcal{A}_X(i)$. Note that we use the opposite of the standard convention for degree-shift, namely $\mathcal{P}(i)_m = \mathcal{P}_{m-i}$. This is a complex of graded free $\mathcal{A}_X$-modules, which gives a free resolution of $\mathscr{O}_X$ as an $\mathcal{A}_X$-module.

Suppose now that $(\mathcal{M}, F)$ is a filtered *right* $\mathscr{D}_X$-module that underlies a mixed Hodge module. In this case we have that $\mathrm{Gr}^F_\bullet \mathcal{M}$ is a Cohen-Macaulay $\mathcal{A}_X$-module by [Sai88, Lemme 5.1.13] (and, more generally, one can consider filtered $\mathscr{D}_X$-modules with this property). Recall from [Sai88, §2.2] that $\widetilde{\mathrm{DR}}(\mathcal{M}, F)$ is the filtered differential complex

$$0 \to \mathcal{M} \otimes_{\mathscr{O}_X} \wedge^n \mathscr{T}_X \to \cdots \to \mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{T}_X \to \mathcal{M} \to 0,$$

placed in degrees $-n, \ldots, 0$, such that the level p part is given by

$$0 \to F_{p-n}\mathcal{M} \otimes_{\mathscr{O}_X} \wedge^n \mathscr{T}_X \to \cdots \to F_{p-1}\mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{T}_X \to F_p\mathcal{M} \to 0.$$

The maps are *not* $\mathscr{O}_X$-linear, but by taking the associated graded objects, we obtain complexes of $\mathscr{O}_X$-modules. More precisely, we have

$$\mathrm{Gr}^F_p \widetilde{\mathrm{DR}}(\mathcal{M}, F) \simeq (P \otimes_{\mathcal{A}_X} K^\bullet)_p,$$

where $P = \mathrm{Gr}^F_\bullet \mathcal{M}$. Note that $P \otimes_{\mathcal{A}_X} K^\bullet$ represents the object $P \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_X} \mathscr{O}_X$ in the derived category of graded $\mathscr{O}_X$-modules.

An important feature of the duality functor is the following isomorphism in the derived category of filtered differential complexes of $\mathscr{O}_X$-modules:

$$\mathbf{D}\big(\widetilde{\mathrm{DR}}(\mathcal{M}, F)\big) \simeq \widetilde{\mathrm{DR}}\big(\mathbf{D}(\mathcal{M}, F)\big),$$

having the property:

$$\mathrm{Gr}^F_p \mathbf{D}\big(\widetilde{\mathrm{DR}}(\mathcal{M}, F)\big) \simeq \mathbf{R}\mathscr{H}om_{\mathscr{O}_X}\big(\mathrm{Gr}^F_{-p}\widetilde{\mathrm{DR}}(\mathcal{M}, F), \omega_X[n]\big) \quad \text{for all} \quad p \in \mathbf{Z}.$$

See [Sai88, §2.4], and also [Sai94, Remark 2.6].

Suppose now that $(\mathcal{M}, F)$ satisfies $\mathbf{D}(\mathcal{M}, F) \simeq (\mathcal{M}, F)(d)$ for some $d \in \mathbf{Z}$; this is for instance the case for the nearby and vanishing cycle modules in the previous section.

By combining the above facts, we see that for every $p \in \mathbf{Z}$ we have an isomorphism in the derived category of $\mathscr{O}_X$-modules:

$$(P \otimes_{\mathcal{A}_X} K^\bullet)_{p-d} \simeq \mathbf{R}\mathscr{H}om_{\mathscr{O}_X}\big((P \otimes_{\mathcal{A}_X} K^\bullet)_{-p}, \omega_X\big)[n]. \tag{3.2}$$

Denoting $A^\bullet := P \otimes_{\mathcal{A}_X} K^\bullet$, using the discussion at the beginning of the section we see that the filtration on $\mathcal{M}$ is generated at level q if and only if $\mathcal{H}^0(A^\bullet)_p = 0$ for every $p > q$. The isomorphism (3.2) gives

$$\mathcal{H}^0(A^\bullet)_p \simeq \mathscr{E}xt^n_{\mathscr{O}_X}(A^\bullet_{-p-d}, \omega_X).$$

On the other hand, we have the first-quadrant spectral sequence

$$E_1^{i,j} = \mathscr{E}xt^j_{\mathscr{O}_X}(A^{-i}_{-p-d}, \omega_X) \Rightarrow \mathscr{E}xt^{i+j}_{\mathscr{O}_X}(A^\bullet_{-p-d}, \omega_X).$$

Recall also that by definition, we have

$$A^{-i}_{-p-d} = \mathrm{Gr}^F_{-p-d-i}\mathcal{M} \otimes_{\mathscr{O}_X} \wedge^i \mathscr{T}_X.$$

Thus for such filtered $\mathscr{D}_X$-modules we obtain the following refinement of the criterion in Proposition 3.1:

Proposition 3.3. *If $(\mathcal{M}, F)$ underlies a mixed Hodge module and $\mathbf{D}(\mathcal{M}, F) \simeq (\mathcal{M}, F)(d)$, then the filtration on $\mathcal{M}$ is generated at level q if*

$$\mathscr{E}xt^j_{\mathscr{O}_X}(\mathrm{Gr}^F_{-p-d-n+j}\mathcal{M} \otimes_{\mathscr{O}_X} \wedge^i \mathscr{T}_X, \omega_X) = 0 \quad \text{for all} \quad 0 \le j \le n,$$

for every $p > q$.

C. Main results

We continue to work on a smooth complex variety X, endowed with a nonzero regular function $f \in \mathscr{O}_X(X)$. We use the notation of the previous section.

4. Generation level for $\mathrm{Gr}_V^\alpha(\iota_+\mathscr{O}_X)$. We start by proving the key Theorem H; this is split here into Propositions 4.1, 4.2 and 4.7, the last being the most involved. We begin with a generation bound for $\mathrm{Gr}_V^\alpha(\iota_+\mathscr{O}_X)$ with $\alpha \in (0,1)$. This case only needs the criterion in Proposition 3.1.

Proposition 4.1. *For $\alpha \in (0,1)$ and $q \ge 1$, the Hodge filtration on $\mathrm{Gr}_V^\alpha(\iota_+\mathscr{O}_X)$ is generated at level q if $F_{n-q}\mathrm{Gr}_V^{1-\alpha}(\iota_+\mathscr{O}_X) = 0$. In particular, if f defines a singular hypersurface, then the Hodge filtration on $\mathrm{Gr}_V^\alpha(\iota_+\mathscr{O}_X)$ is generated at level $n - \lceil \widetilde{\alpha}_f + \alpha \rceil + 1$.*

Proof. It follows from (2.6) that the filtration on $\mathrm{Gr}_V^\alpha(\iota_+\mathscr{O}_X)$ is generated at level q if and only if the filtration on $\mathrm{Gr}^V_{-\alpha}(\iota_+\omega_X)$ is generated at level $q - n - 1$. Using the isomorphism (2.4), we deduce in turn from Proposition 3.1 that this is the case if

$$F_{n-q}\mathrm{Gr}^V_{\alpha-1}(\iota_+\omega_X)(n+1) = F_{-q-1}\mathrm{Gr}^V_{\alpha-1}(\iota_+\omega_X)$$

is 0. The latter condition is equivalent with $F_{n-q}\mathrm{Gr}_V^{1-\alpha}(\iota_+\mathscr{O}_X) = 0$ by another application of (2.6), giving the first assertion in the proposition.

For the second assertion, note that by Lemma 1.2, for every $j \ge 0$ and every $\beta \in (0,1)$ we have the equivalence

$$\partial_t^j \delta \in V^\beta \iff \widetilde{\alpha}_f \ge j + \beta.$$

In particular, if this holds for $j\geq 1$, it also holds for $j-1$. If $q=n-\lceil\widetilde{\alpha}_f+\alpha\rceil+1$, then $q>n-\widetilde{\alpha}_f-\alpha$, and we conclude that there is β with $1-\alpha<\beta<1$, such that $\partial_t^{n-q-1}\delta\in V^\beta\iota_+\mathscr{O}_X$. In this case we have $F_{n-q}V^\beta\iota_+\mathscr{O}_X=F_{n-q}\iota_+\mathscr{O}_X$, hence clearly $F_{n-q}\mathrm{Gr}_V^{1-\alpha}(\iota_+\mathscr{O}_X)=0$. □

A similar proof works for $\alpha=0$; we include it for completeness, even though this is not relevant for the rest of the paper.

Proposition 4.2. *If $F_{n-q+1}\mathrm{Gr}_V^0(\iota_+\mathscr{O}_X)=0$ for some $q\geq 1$, then the Hodge filtration on $\mathrm{Gr}_V^0(\iota_+\mathscr{O}_X)$ is generated at level q. In particular, if f defines a singular hypersurface, then the Hodge filtration on $\mathrm{Gr}_V^0(\iota_+\mathscr{O}_X)$ is generated at level $n-\lceil\widetilde{\alpha}_f\rceil+1$.*

Proof. Arguing as above, using (2.5) and Proposition 3.1 we see that the Hodge filtration on $\mathrm{Gr}_V^0(\iota_+\mathscr{O}_X)$ is generated at level q if $F_{n-q+1}\mathrm{Gr}_V^0(\iota_+\mathscr{O}_X)=0$. This in turn holds if $q=n-\lceil\widetilde{\alpha}_f\rceil+1$, since Lemma 1.2 implies that there exists $\beta>0$ such that $\partial_t^{\lceil\widetilde{\alpha}_f\rceil-1}\delta\in V^\beta$. □

For $\mathrm{Gr}_V^1(\iota_+\mathscr{O}_X)$ we need to use a more refined argument. We start by specializing the criterion in Proposition 3.3 to the $\mathscr{D}_{X\times\mathbf{C}}$-module $\mathcal{M}=\mathrm{Gr}^V_{-1}(\iota_+\omega_X)$, in which case we have $d=n+1$ by (2.3), so that the vanishing in the proposition concerns

$$\mathscr{E}xt^j_{\mathscr{O}_X}\big(\mathrm{Gr}^F_{j-p-2n-1}\mathrm{Gr}^V_{-1}(\iota_+\omega_X)\otimes_{\mathscr{O}_X}\wedge^i\mathscr{T}_X,\omega_X\big)$$
$$\simeq\mathscr{E}xt^j_{\mathscr{O}_X}\big(\mathrm{Gr}^F_{j-p-n}\mathrm{Gr}_V^1(\iota_+\mathscr{O}_X),\mathscr{O}_X\big)\otimes_{\mathscr{O}_X}\Omega_X^i.$$

Furthermore, the filtration on $\mathrm{Gr}_V^1(\iota_+\mathscr{O}_X)$ is generated at level q if and only if the filtration on $\mathrm{Gr}^V_{-1}(\iota_+\omega_X)$ is generated at level $q-n-1$. We thus obtain

Corollary 4.3. *The Hodge filtration on $\mathrm{Gr}_V^1(\iota_+\mathscr{O}_X)$ is generated at level q if*

$$\mathscr{E}xt^j_{\mathscr{O}_X}\big(\mathrm{Gr}^F_{j-p}\mathrm{Gr}_V^1(\iota_+\mathscr{O}_X),\mathscr{O}_X\big)=0\quad\text{for all}\ \ 0\leq j\leq n\ \ \text{and}\ \ p>q-1.$$

To apply this criterion, we need a better understanding of the terms $\mathrm{Gr}_k^F\mathrm{Gr}_V^1(\iota_+\mathscr{O}_X)$. To this end, for every $k\geq 0$ we introduce the following coherent ideals of $\mathscr{O}_X$:

$$J_k=\{h\in\mathscr{O}_X\mid h\partial_t^k\delta\in V^1\iota_+\mathscr{O}_X\}\quad\text{and}\quad J'_k=\{h\in\mathscr{O}_X\mid h\partial_t^k\delta\in V^{>1}\iota_+\mathscr{O}_X\}.$$

From now on, we will only deal with the V-filtration on $\iota_+\mathscr{O}_X$, hence in order to simplify the notation we often denote $V^\alpha=V^\alpha\iota_+\mathscr{O}_X$ and $\mathrm{Gr}_V^\alpha=\mathrm{Gr}_V^\alpha(\iota_+\mathscr{O}_X)$.

We will make use of the fact that $J'_k\subseteq(f)$ for all $k\geq 0$. In fact, we prove the following more precise result:

Lemma 4.4. *If f defines a reduced hypersurface, then for every $k\geq 0$, we have $J'_k=(f^{k+1})$.*

Proof. It is well known that $\mathcal{I}(f^{k+1})=(f^{k+1})$, and so by (1.3) it follows that $f^{k+1}\delta\in V^{>(k+1)}$. We thus have $f^{k+1}\partial_t^k\delta\in V^{>1}$, hence $f^{k+1}\in J'_k$.

It suffices to prove the reverse inclusion $J'_k\subseteq(f^{k+1})$ on an open subset U of X such that $\mathrm{codim}_X(X\smallsetminus U)\geq 2$. Since f defines a reduced hypersurface, we can find such a subset U on which f is smooth. We will therefore assume from now on that $\mathrm{div}(f)$ is smooth. After passing to a suitable open cover of X, we may further assume that we have an algebraic system of coordinates $x_1, \ldots, x_n$ such that $f = x_1$.

Recall that in this case the V-filtration on $\iota_+\mathscr{O}_X$ only jumps at integers (hence $V^{>1} = V^2$) and for every $m \geq 1$, V^m is generated over $\mathscr{D}_X$ by x_1^{m-1}. This follows easily by checking that this definition satisfies the defining properties of the V-filtration. (For a more general statement valid for arbitrary simple normal crossing divisors, see [Sai90, Theorem 3.4].) In particular, we see that V^2 is generated as an $\mathscr{O}_X$-module by $\partial_{x_1}^i x_1\delta$, for $i \geq 0$. Since $\partial_{x_1}^i\delta = (-1)^i\partial_t^i\delta$, we have

$$\partial_{x_1}^i x_1\delta = x_1\partial_{x_1}^i\delta + [\partial_{x_1}^i, x_1]\delta = (-1)^i x_1\partial_t^i\delta + (-1)^{i-1}i\partial_t^{i-1}\delta.$$

We conclude that given a regular function h, we have $h\partial_t^k\delta \in V^2$ if and only if there are regular functions $g_0, \ldots, g_p$ such that

$$h\partial_t^k\delta = \sum_{i=0}^p g_i\partial_{x_1}^i x_1\delta = g_0x_1\delta + \sum_{i=1}^p (-1)^i g_i(x_1\partial_t^i\delta - i\partial_t^{i-1}\delta).$$

This equality holds if and only if $g_i = 0$ for $i > k$, $h = (-1)^k x_1 g_k$, and

$$x_1g_i + (i+1)g_{i+1} = 0 \quad \text{for} \quad 0 \leq i \leq k-1.$$

This clearly implies that $h \in (x_1^{k+1})$, completing the proof of the lemma. □

We are now able to establish the connection between the Hodge filtration on Gr_V^1 and the minimal exponent $\widetilde{\alpha}_f$.

Proposition 4.5. *If f defines a reduced hypersurface and $p \geq 0$ is an integer such that $\widetilde{\alpha}_f > p$, then*

$$\mathrm{Gr}_{p+1}^F\mathrm{Gr}_V^1(\iota_+\mathscr{O}_X) \simeq J_p/(f) \quad \text{and} \quad \mathrm{Gr}_{i+1}^F\mathrm{Gr}_V^1(\iota_+\mathscr{O}_X) \simeq \mathscr{O}_X/(f) \ \ \text{for} \ \ 0 \leq i \leq p-1$$

(*note that the second statement is vacuous for $p = 0$*).

Proof. Fix $0 \leq k \leq p$. Since $k < \widetilde{\alpha}_f$, it follows from Lemma 1.2 that $\partial_t^i\delta \in V^{>0}$ for $0 \leq i \leq k$. This implies that for every such i, we have $t\partial_t^i\delta \in V^{>1}$. Note that

$$t\partial_t^i\delta = f\partial_t^i\delta - i\partial_t^{i-1}\delta \quad \text{for} \quad 1 \leq i \leq k,$$

hence $t\partial_t\delta, \ldots, t\partial_t^k, \partial_t^k\delta$ give a basis of $F_{k+1}\iota_+\mathscr{O}_X$ over $\mathscr{O}_X$. Since all but the last one of these elements lie in $F_{k+1}V^{>1}$, we have a canonical isomorphism

$$F_{k+1}\mathrm{Gr}_V^1 = F_{k+1}V^1/F_{k+1}V^{>1} \simeq J_k/J_k'. \tag{4.6}$$

If $k \leq p-1$, then $\partial_t^k\delta \in V^1$ by Lemma 1.2, hence $J_k = \mathscr{O}_X$. Moreover, via the isomorphisms (4.6), the inclusion

$$F_{k+1}\mathrm{Gr}_V^1 \to F_{k+2}\mathrm{Gr}_V^1$$

maps the class of 1 in $\mathscr{O}_X/J_k'$ to the class of $\frac{1}{k+1}f$ in J_{k+1}/J_{k+1}'. Indeed, this follows from the fact that

$$\partial_t^k\delta = \frac{1}{k+1}f\partial_t^{k+1}\delta - \frac{1}{k+1}t\partial_t^{k+1}\delta.$$

We thus conclude that

$$\mathrm{Gr}_{k+2}^F\mathrm{Gr}_V^1 \simeq J_{k+1}/\big(J_{k+1}' + (f)\big) = J_{k+1}/(f),$$

where the equality follows from Lemma 4.4. Furthermore, as we have already mentioned, if $k \leq p-2$, then $J_{k+1} = \mathscr{O}_X$, hence $\mathrm{Gr}^F_{k+2}\mathrm{Gr}^1_V \simeq \mathscr{O}_X/(f)$.

On the other hand, note that we always have

$$\mathrm{Gr}^F_1\mathrm{Gr}^1_V = F_1\mathrm{Gr}^1_V \simeq J_0/J_0' = J_0/(f),$$

where the last equality holds by Lemma 4.4. Furthermore, $J_0 = \mathscr{O}_X$ if $p \geq 1$. This completes the proof of the proposition. □

Proposition 4.7. *If f defines a singular, reduced hypersurface, then the Hodge filtration on $\mathrm{Gr}^1_V(\iota_+\mathscr{O}_X)$ is generated at level $n - \lceil \widetilde{\alpha}_f \rceil$.*

Proof. Equivalently, we need to check that if p is a nonnegative integer such that $\widetilde{\alpha}_f > p$, then the filtration on Gr^1_V is generated at level $n-1-p$. (Note that since f defines a singular hypersurface, we have $\widetilde{\alpha}_f \leq \frac{n}{2}$ as mentioned in the introduction, hence our assumption on p implies $n-1-p \geq 1$.) It follows then from Corollary 4.3 that it is enough to show:

$$\mathscr{E}xt^j_{\mathscr{O}_X}(\mathrm{Gr}^F_{j-i}\mathrm{Gr}^1_V, \mathscr{O}_X) = 0 \quad \text{for} \quad 0 \leq j \leq n \quad \text{and} \quad i > n-2-p. \tag{4.8}$$

Note that we only need to consider i and j such that $0 \leq j-i-1 \leq n-i-1 \leq p$.

To see this, we use the isomorphisms in Proposition 4.5. First, the short exact sequence

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\cdot f} \mathscr{O}_X \longrightarrow \mathscr{O}_X/(f) \longrightarrow 0$$

gives $\mathscr{E}xt^m_{\mathscr{O}_X}\big(\mathscr{O}_X/(f), \mathscr{O}_X\big) = 0$ for all $m \geq 2$. We thus see that if $0 \leq j-i-1 \leq p-1$, we have

$$\mathscr{E}xt^j_{\mathscr{O}_X}(\mathrm{Gr}^F_{j-i}\mathrm{Gr}^1_V, \mathscr{O}_X) \simeq \mathscr{E}xt^j_{\mathscr{O}_X}\big(\mathscr{O}_X/(f), \mathscr{O}_X\big) = 0,$$

since $j \geq i+1 \geq n-p \geq 2$. On the other hand, if $j-i-1 = p$, then $j = n$, and the short exact sequence

$$0 \to J_p/(f) \to \mathscr{O}_X/(f) \to \mathscr{O}_X/J_p \to 0$$

implies that

$$\mathscr{E}xt^n_{\mathscr{O}_X}\big(\mathrm{Gr}^F_{p+1}\mathrm{Gr}^1_V, \mathscr{O}_X\big) \simeq \mathscr{E}xt^n_{\mathscr{O}_X}\big(J_p/(f), \mathscr{O}_X\big)$$

is a quotient of $\mathscr{E}xt^n_{\mathscr{O}_X}\big(\mathscr{O}_X/(f), \mathscr{O}_X\big) = 0$. This completes the proof of the proposition. □

Remark 4.9. In the statements of Propositions 4.1, 4.2, and 4.7, we assumed that the hypersurface defined by f is singular, in order to avoid the case when $\widetilde{\alpha}_f = \infty$. If f defines a smooth hypersurface, then Gr^α_V is nonzero only when α is an integer and the Hodge filtration on both Gr^0_V and Gr^1_V is generated in level 0.

5. The Hodge filtrations on V^α and $\mathcal{M}(f^{-\alpha})$. Let $\pi\colon X \times \mathbf{C} \to X$ be the projection onto the first component. Given $\alpha \in \mathbf{Q}$, we consider the map

$$\tau_\alpha\colon \pi_*V^\alpha\iota_+\mathscr{O}_X \to \mathcal{M}(f^{-\alpha})$$

given by

$$\tau_\alpha\left(\sum_{i=0}^p v_i\partial_t^i\delta\right) = \left(\sum_{i=0}^p Q_i(\alpha)\frac{v_i}{f^i}\right)f^{-\alpha},$$

where $Q_i(x) = \prod_{j=0}^{i-1}(x+j)$ (with the convention that $Q_0 = 1$). Note that both sides have $\mathscr{D}_X$-module structure; in fact $\pi_*V^\alpha\iota_+\mathscr{O}_X$ is naturally a $\mathscr{D}_X[t, \partial_t t]$-module.

Lemma 5.1. *The map τ_α is a morphism of $\mathscr{D}_X$-modules. Moreover, we have*

$$\tau_{\alpha+1}(tv) = \tau_\alpha(v) \quad \textit{for every} \quad v \in \pi_* V^\alpha \iota_+ \mathscr{O}_X \quad \textit{and} \tag{5.2}$$

$$\tau_\alpha(\partial_t v) = \alpha \cdot \tau_{\alpha+1}(v) \quad \textit{for every} \quad v \in \pi_* V^{\alpha+1} \iota_+ \mathscr{O}_X, \tag{5.3}$$

where the equalities hold via the identification in (1.4).

Proof. We may and will assume that X is affine. The fact that $\tau_\alpha(gu) = g \cdot \tau_\alpha(u)$ for every $g \in \mathscr{O}_X(X)$ and every global section u of V^α is clear. Suppose now that $v = \sum_{i=0}^p v_i \partial_t^i \delta \in V^\alpha$ and P is a $\mathbf{C}$-derivation of $\mathscr{O}_X(X)$. We have

$$Pv = \sum_{i=0}^p P(v_i)\partial_t^i \delta - \sum_{i=0}^p P(f) v_i \partial_t^{i+1}\delta,$$

hence

$$\tau_\alpha(Pv) = \left(\sum_{i=0}^p Q_i(\alpha) \frac{P(v_i)}{f^i} - \sum_{i=0}^p Q_{i+1}(\alpha) \frac{v_i P(f)}{f^{i+1}} \right) f^{-\alpha}$$
$$= P\left(\left(\sum_{i=0}^p Q_i(\alpha) \frac{v_i}{f^i} \right) f^{-\alpha} \right) = P(\tau_\alpha(v)),$$

where we used the fact that $Q_{i+1}(\alpha) = (\alpha+i)Q_i(\alpha)$ and

$$P\left(\frac{h}{f^i} f^{-\alpha} \right) = \frac{P(h)}{f^i} f^{-\alpha} - \frac{(\alpha+i)hP(f)}{f^{i+1}} f^{-\alpha}.$$

By the definition of the V-filtration, if $v \in V^\alpha$, then $tv \in V^{\alpha+1}$ (and for $\alpha > 0$, multiplication by t induces an isomorphism of $\mathscr{D}_X$-modules $V^\alpha \to V^{\alpha+1}$). In order to prove (5.2), note first that if $v = \sum_{i=0}^p v_i \partial_t^i \delta$, then

$$tv = \sum_{i=0}^p f v_i \partial_t^i \delta - \sum_{i=1}^p i v_i \partial_t^{i-1} \delta.$$

We thus have

$$\tau_{\alpha+1}(tv) = \left(\sum_{i=0}^p Q_i(\alpha+1) \frac{f v_i}{f^i} - \sum_{i=1}^p Q_{i-1}(\alpha+1) \frac{i v_i}{f^{i-1}} \right) f^{-\alpha-1}.$$

Since

$$Q_i(\alpha+1) - iQ_{i-1}(\alpha+1) = Q_i(\alpha) \quad \text{for} \quad i \geq 1$$

and $Q_0(\alpha+1) = Q_0(\alpha)$, we conclude that $\tau_{\alpha+1}(tv) = \tau_\alpha(v)$ via (1.4).

Suppose now that $v = \sum_{i=0}^p v_i \partial_t^i \delta \in V^{\alpha+1}$, hence $\partial_t v = \sum_{i=0}^p v_i \partial_t^{i+1} \delta \in V^\alpha$. We then have

$$\tau_\alpha(\partial_t v) = \left(\sum_{i=0}^p Q_{i+1}(\alpha) \frac{v_i}{f^{i+1}} \right) f^{-\alpha} = \alpha \cdot \left(\sum_{i=0}^p Q_i(\alpha+1) \frac{v_i}{f^i} \right) f^{-\alpha-1} = \alpha \cdot \tau_{\alpha+1}(v),$$

which proves (5.3). □

Proposition 5.4. *If $D = \mathrm{div}(f)$ is a reduced divisor, then for every $\alpha > 0$ the morphism τ_α is surjective, and the Hodge filtration on the image is, up to a shift by 1, the induced filtration from that on $V^\alpha \iota_+ \mathscr{O}_X$. More precisely, we have*

$$F_p \mathcal{M}(f^{-\alpha}) = \tau_\alpha(F_{p+1} V^\alpha \iota_+ \mathscr{O}_X) \quad \textit{for all} \quad p \geq 0.$$

Proof. Thanks to (1.1), the elements of $F_{p+1}V^{\alpha}$ are the sums $\sum_{i=0}^{p} v_i\partial_t^i\delta$ that belong to V^{α}. The fact that for all $\alpha > 0$ we have

$$F_p\mathcal{M}(f^{-\alpha}) = \tau_\alpha(F_{p+1}V^{\alpha}) \quad \text{for all} \quad p \geq 0$$

is then precisely the content of Theorem 1.5. Since the Hodge filtration on $\mathcal{M}(f^{-\alpha})$ is exhaustive, we deduce that τ_α is surjective. □

Remark 5.5. The same statement holds more generally when $D = \mathrm{div}(f)$ is not necessarily reduced, but $\alpha > 0$ is such that $\lceil \alpha D\rceil$ is reduced. For this one simply needs to refer to [MP18, Theorem A] instead.

6. Proof of the main result. We begin with the following general (and well-known) fact:

Lemma 6.1. *If $u \in \iota_+\mathscr{O}_X$ is such that $\partial_t u \in V^{\alpha}$ for some $\alpha \leq 0$, then $u \in V^{\alpha+1}$.*

Proof. Certainly if $\beta \ll 0$, then $u \in V^{\beta}$. We may assume that $u \neq 0$ and choose β which is largest with this property, so that $u \notin V^{>\beta}$. If $\beta \geq \alpha + 1$, then we are done. Otherwise $\beta - 1 < \alpha \leq 0$, and $\partial_t u$ vanishes in $\mathrm{Gr}_V^{\beta-1}$. Recall however that an easy consequence of the definition of the V-filtration is that for every $\gamma \neq 0$, the map

$$\mathrm{Gr}_V^{\gamma+1} \xrightarrow{\partial_t\cdot} \mathrm{Gr}_V^{\gamma}$$

is bijective. It follows that u vanishes in Gr_V^{β}, a contradiction. □

Next, using the result of the previous section, we show that in order to bound the generation level of $\mathcal{M}(f^{-\alpha})$ for any $\alpha > 0$, it suffices to study the Hodge filtration on the associated graded terms Gr_V^{β}, for special rational β.

Corollary 6.2. *If $\alpha \in (0, 1]$ is a rational number and $q \geq 0$ is such that the Hodge filtration on $\mathrm{Gr}_V^{\beta}(\iota_+\mathscr{O}_X)$ is generated at level $q + 1$ for all $\beta \in [\alpha, 1]$, then the Hodge filtration on $\mathcal{M}(f^{-\alpha})$ is generated at level q.*

Proof. We need to show that $F_p\mathcal{M}(f^{-\alpha}) \subseteq F_1\mathscr{D}_X \cdot F_{p-1}\mathcal{M}(f^{-\alpha})$ for every $p > q$. Given such p and $u \in F_p\mathcal{M}(f^{-\alpha})$, it follows from Proposition 5.4 that we can find $\widetilde{u} \in F_{p+1}V^{\alpha}$ such that $\tau_\alpha(\widetilde{u}) = u$. The V-filtration is discrete, hence after using the hypothesis finitely many times, we obtain

$$F_{p+1}V^{\alpha} \subseteq F_1\mathscr{D}_X \cdot F_pV^{\alpha} + F_{p+1}V^{>1}.$$

Since τ_α maps $F_1\mathscr{D}_X \cdot F_pV^{\alpha}$ to $F_1\mathscr{D}_X \cdot F_{p-1}\mathcal{M}(f^{-\alpha})$, we may clearly assume that $\widetilde{u} \in F_{p+1}V^{>1}$. In this case we can write $\widetilde{u} = tv$ for some $v \in F_{p+1}V^{>0}$; see for instance (the proof of) [MP18, Lemma 4.5]. Furthermore, by the definition of $F_{p+1}\iota_+\mathscr{O}_X$, we can write $v = v_0\delta + \partial_t w$, for some $v_0 \in \mathscr{O}_X$ and $w \in F_p\iota_+\mathscr{O}_X$. Note that $\delta \in V^{>0}$, hence $v_0\delta \in V^{>0}$, and thus $\partial_t w \in V^{>0}$. By Lemma 6.1, we have $w \in F_pV^1$, so in particular $w \in F_pV^{\alpha}$. Since $tv_0\delta = v_0f\delta$, we have

$$u = \tau_\alpha(\widetilde{u}) = \tau_\alpha(tv_0\delta + t\partial_t w) = (v_0f)f^{-\alpha} + \tau_\alpha(t\partial_t w) = (v_0f)f^{-\alpha} + \alpha \cdot \tau_\alpha(w),$$

where the last equality follows from (5.2) and (5.3). But $(v_0f)f^{-\alpha} \in F_0\mathcal{M}(f^{-\alpha})$, which follows for example from Proposition 5.4, since $f\delta \in V^{>1} \subseteq V^{\alpha}$ by (1.3). Also, since $w \in F_pV^{\alpha}$, it follows from Proposition 5.4 that $\tau_\alpha(w) \in F_{p-1}\mathcal{M}(f^{-\alpha})$. We conclude that $u \in F_{p-1}\mathcal{M}(f^{-\alpha})$, completing the proof. □

We are finally able to give the proof of the main result:

Proof of Theorem E. According to Corollary 6.2, it suffices to know that $\mathrm{Gr}_V^{\beta}(\iota_+\mathscr{O}_X)$ is generated at level $n-\lceil\widetilde{\alpha}_f+\alpha\rceil+1$ for all $\beta\in[\alpha,1]$. But this follows from Propositions 4.1 and 4.7, which show that each $\mathrm{Gr}_V^{\beta}(\iota_+\mathscr{O}_X)$ is generated at level $n-\lceil\widetilde{\alpha}_f+\beta\rceil+1$. □

7. Proof of Theorem D. Consider a reduced complex scheme D, which can be embedded as a hypersurface in a smooth variety X, with minimal exponent $\widetilde{\alpha}_D$. We consider a resolution of singularities $\mu\colon \widetilde{D}\to D$. (Recall that by this we mean the disjoint union of resolutions of the irreducible components of D.) We further assume that f is an isomorphism over the smooth locus of D and that the reduced inverse image of the singular locus $D_{\rm sing}$ of D is a simple normal crossing divisor E on $\widetilde{D}$.

We start with the following observation:

Lemma 7.1. *The statement of Theorem D is independent of the choice of such a resolution.*

Proof. A standard argument shows that it is enough to compare the assertion for μ and for another resolution with the same properties of the form $\mu\circ g$, for some morphism $g\colon D'\to\widetilde{D}$. Note that if E' is the reduced inverse image of $D_{\rm sing}$ on D', then $E'=(g^*E)_{\rm red}$ and g is an isomorphism over $\widetilde{D}\smallsetminus \mathrm{Supp}(E)$. In this case, we have for all i

$$g_*\Omega^i_{D'}(\log E')=\Omega^i_{\widetilde{D}}(\log E)\quad\text{and}\quad R^q\Omega^i_{D'}(\log E')=0\quad\text{for all}\quad q>0$$

by [EV82, Lemmas 1.2 and 1.5]; cf. also [MP16, Theorem 31.1(i)]. The assertion in the lemma thus follows via the Leray spectral sequence. □

If D is smooth, then μ is an isomorphism, and we trivially have $R^i\mu_*\Omega^j_Y(\log E)=0$ for all $i>0$ and all j. From now on, we focus on the case when D is singular (in which case recall, as mentioned in the Introduction, that $\widetilde{\alpha}_D\le n/2$, where $\dim(D)=n-1$).

The proof of Theorem D is inspired by the proof of [MOP17, Theorem E], which partly treats the case $k=1$. We begin with an auxiliary result:

Lemma 7.2. *Let $g\colon Y\to X$ be the blow-up of a smooth variety X along a smooth, irreducible subvariety Z, of codimension $r\ge 2$. Let F be a reduced simple normal crossing divisor on X, having simple normal crossings with Z as well, and denote by $\widetilde{F}$ the strict transform of F and by E the exceptional divisor on Y. Then for every $i<r$, the following hold:*

$$g_*\Omega^i_Y\big(\log(E+\widetilde{F})\big)=\Omega^i_X(\log F)\quad\textit{and}\quad R^qg_*\Omega^i_Y\big(\log(E+F)\big)=0\quad\textit{for all}\quad q\ge 1.$$

Proof. For $i=0$ the assertion is clear and for $i=1$ it follows from [MP16, Theorem 31.1(ii)], so from now on we assume $i\ge 2$, hence $r\ge 3$. We argue by induction on r. If $Z\subseteq \mathrm{Supp}(F)$, then the assertion holds for all i, using again [EV82, Lemmas 1.2 and 1.5]. Suppose now that Z is not contained in $\mathrm{Supp}(F)$. Since the assertion is local on X, we may assume that we have algebraic coordinates $x_1,\ldots,x_n$ on X such that Z is defined by $x_1,\ldots,x_r$ and all components of F are defined by some x_k, with $k>r$. Let T be the smooth divisor on X defined by x_1 and consider the induced morphism

$h\colon \widetilde{T} \to T$, where $\widetilde{T}$ is the strict transform of T on Y. Consider the standard residue short exact sequence on Y:

$$0 \to \Omega_Y^i\big(\log(E+\widetilde{F})\big) \to \Omega_Y^i\big(\log(E+\widetilde{F}+\widetilde{T})\big) \to \Omega_{\widetilde{T}}^{i-1}\big(\log(E|_{\widetilde{T}}+\widetilde{F}|_{\widetilde{T}})\big) \to 0. \tag{7.3}$$

Note that h is the blow-up of T along Z, with exceptional divisor $E|_{\widetilde{T}}$. Moreover, the strict transform of $F|_T$ is $\widetilde{F}|_{\widetilde{T}}$. Since $\operatorname{codim}_T(Z) = r-1 \geq 2$, the inductive assumption thus gives

$$h_*\Omega_{\widetilde{T}}^{i-1}\big(\log(E|_{\widetilde{T}}+\widetilde{F}|_{\widetilde{T}})\big) = \Omega_T^{i-1}(\log F|_T) \quad \text{and}$$
$$R^q h_*\Omega_{\widetilde{T}}^{i-1}\big(\log(E|_{\widetilde{T}}+\widetilde{F}|_{\widetilde{T}})\big) = 0 \quad \text{for all} \quad q \geq 1.$$

On the other hand, since $Z \subseteq \operatorname{Supp}(F+T)$ it follows, again from the reference above, that

$$g_*\Omega_Y^i\big(\log(E+\widetilde{F}+\widetilde{T})\big) = \Omega_X^i\big(\log(F+T)\big) \quad \text{and}$$
$$R^q g_*\Omega_Y^i\big(\log(E+\widetilde{F}+\widetilde{T})\big) = 0 \quad \text{for all} \quad q \geq 1.$$

The long exact sequence for higher direct images associated to (7.3) gives

$$R^q g_*\Omega_Y^i\big(\log(E+\widetilde{F})\big) = 0 \quad \text{for all} \quad q \geq 2,$$

together with an exact sequence

$$0 \to g_*\Omega_Y^i\big(\log(E+\widetilde{F})\big) \to \Omega_X^i\big(\log(F+T)\big) \to \Omega_T^{i-1}(\log F|_T)$$
$$\to R^1 g_*\Omega_Y^i\big(\log(E+\widetilde{F})\big) \to 0,$$

which compared to the standard residue sequence gives the assertions in the lemma. □

In order to apply the previous lemma, we will need to control the codimension of the blow-up centers when we have a lower bound on $\widetilde{\alpha}_D$. This is provided by:

Proposition 7.4. *If D is a singular effective divisor on X such that $\widetilde{\alpha}_D > k$ for some nonnegative integer k, then we have the following lower bound for the codimension of the singular locus $D_{\rm sing}$ of D:*

$$\operatorname{codim}_X(D_{\rm sing}) \geq 2k+1.$$

To see this, we first prove a general lemma concerning the behavior of $\widetilde{\alpha}_D$ under restriction to a general hypersurface.

Lemma 7.5. *If D is an effective divisor on X and H is a general smooth hypersurface in X (for example, a general member of a basepoint-free linear system), then*

$$\widetilde{\alpha}_{D|_H} \geq \widetilde{\alpha}_D.$$

Proof. We may assume that D is reduced: otherwise $\operatorname{lct}(X,D) < 1$, hence $\operatorname{lct}(X,D) = \widetilde{\alpha}_D$ and for H general we have

$$\widetilde{\alpha}_{D|_H} \geq \operatorname{lct}(H, D|_H) \geq \operatorname{lct}(X,D),$$

where the second inequality follows, for example, from the Generic Restriction theorem for multiplier ideals, see [Laz04, Theorem 9.5.35]. Supposing now that D is reduced, we appeal to results on Hodge ideals (for **Q**-divisors). If we write $\widetilde{\alpha}_D = p+\alpha$, for some $\alpha \in (0,1]$ and some nonnegative integer p, it follows from [MP18, Corollary C] that $I_p(\alpha D) = \mathscr{O}_X$ and since H is general, according to [MP19, Theorem 13.1] we have

$$I_p(\alpha D|_H) = I_p(\alpha D) \cdot \mathscr{O}_H = \mathscr{O}_H.$$

Another application of [MP18, Corollary C] gives $\widetilde{\alpha}_{D|_H} \geq p + \alpha = \widetilde{\alpha}_D$. □

Proof of Proposition 7.4. We may assume that X is an affine variety. We denote $r = \dim(D_{\rm sing})$. If $r \geq 1$ and H is a general hyperplane section of X, then H is smooth, $D|_H$ is singular, and $\dim\big((D|_H)_{\rm sing}\big) = r-1$. Moreover, it follows from Lemma 7.5 that $\widetilde{\alpha}_{D|_H} > k$. After iterating this r times, we obtain a smooth subvariety Y of X, with $\dim(Y) = n-r$, such that $D|_Y$ is a singular effective divisor and $\widetilde{\alpha}_{D|_Y} > k$. Since $\widetilde{\alpha}_{D|_Y} \leq \frac{1}{2}\dim(Y)$, we conclude that $k < \frac{1}{2}(n-r)$, hence

$$\mathrm{codim}_X(D_{\rm sing}) = n - r \geq 2k+1.$$

□

We can finally approach our main goal for this section.

Proof of Theorem D. Let X be a smooth variety in which D embeds as a hypersurface. We need to show, equivalently, that if k is a nonnegative integer such that $\widetilde{\alpha}_D > k$, then

$$R^{n-1-i}\mu_*\Omega^i_{\widetilde{D}}(\log E) = 0 \quad \text{for all} \quad i \leq k.$$

By Lemma 7.1, the assertion in the theorem is independent of the choice of resolution μ. We thus first construct a log resolution $\mu\colon Y \to X$ of the pair (X, D), as a composition

$$Y = X_N \xrightarrow{\mu_N} X_{N-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\mu_1} X_0 = X,$$

where

i) Each μ_j with $1 \leq j \leq N$ is the blow-up of a smooth, irreducible subvariety Z_{j-1} of X_{j-1} that lies over $D_{\rm sing} \subseteq X$. We denote by F_j the exceptional divisor of $X_j \to X$ and by D_j the strict transform of D on X_j.
ii) Each Z_{j-1} with $1 \leq j \leq N$ has simple normal crossings with $D_{j-1} + F_{j-1}$.

In particular, we see inductively that each X_j is smooth and $F_j + D_j$ is a simple normal crossing divisor. We may assume that $\widetilde{D} = D_N$ is smooth, so that the induced morphism $\varphi\colon \widetilde{D} \to D$ is a resolution of D that is an isomorphism over $D \smallsetminus D_{\rm sing}$. Furthermore, if $F = F_N$, and $E = F|_{\widetilde{D}}$, then $E = \mu^{-1}(D_{\rm sing})_{\rm red}$ and this is a simple normal crossing divisor on $\widetilde{D}$.

Claim. For every $i \leq 2k$, we have

$$\mu_*\Omega^i_Y(\log F) = \Omega^i_X \quad \text{and} \quad R^q\mu_*\Omega^i_Y(\log F) = 0 \ \text{ for all } \ q \geq 1. \tag{7.6}$$

To see this, using the Leray spectral sequence, it is enough to show that for every $1 \leq j \leq N$ we have

$$\mu_{j*}\Omega^i_{X_j}(\log F_j) = \Omega^i_{X_{j-1}}(\log F_{j-1}) \ \text{ and } \ R^q\mu_{j*}\Omega^i_{X_j}(\log F_j) = 0 \ \text{ for all } \ q \geq 1. \tag{7.7}$$

If $Z_{j-1} \subseteq F_{j-1}$, then this follows from [EV82, Lemmas 1.2 and 1.5] (or [MP16, Theorem 31.1(i)]). On the other hand, if $Z_{j-1} \not\subseteq F_{j-1}$, then Z_{j-1} is equal to the strict transform of its image in X. By construction and Proposition 7.4, it follows that $\mathrm{codim}_{X_{j-1}}(Z_{j-1}) \geq 2k+1$, and (7.7) then follows from Lemma 7.2. This proves our claim.

Consider now the residue short exact sequence

$$0 \to \Omega_Y^{i+1}(\log F) \to \Omega_Y^{i+1}\big(\log(\widetilde{D}+F)\big) \to \Omega_{\widetilde{D}}^{i}(\log E) \to 0$$

on Y, and the following piece in the corresponding long exact sequence for higher direct images:

$$R^{n-1-i}\mu_*\Omega_Y^{i+1}\big(\log(\widetilde{D}+F)\big) \to R^{n-1-i}\varphi_*\Omega_{\widetilde{D}}^{i}(\log E) \to R^{n-i}\mu_*\Omega_Y^{i+1}(\log F).$$

Since

$$i \le k < \widetilde{\alpha}_D \le n/2,$$

the first term vanishes because of Corollary C. Since the third term vanishes by the above Claim, we conclude that the middle term vanishes as well. This completes the proof of the theorem. □

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