BRODY HYPERBOLICITY OF BASE SPACES OF CERTAIN FAMILIES OF VARIETIES

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Abstract. We prove that quasi-projective base spaces of smooth families of minimal varieties of general type with maximal variation do not admit Zariski dense entire curves. We deduce the fact that moduli stacks of polarized varieties of this sort are Brody hyperbolic, answering a special case of a question of Viehweg and Zuo. For two-dimensional bases, we show analogous results in the more general case of families of varieties admitting a good minimal model.

1. Introduction

The purpose of this paper is to establish a few results related to the hyperbolicity of base spaces of families of smooth complex varieties having maximal variation. Our study is motivated by the conjecturally degenerate behavior of entire curves inside the moduli P_h of polarized manifolds, corresponding to the moduli functor $\mathscr{P}_h$ which associates to a variety V the set $\mathscr{P}_h(V)$ of pairs $(f\colon U \to V, \mathscr{H})$, where f is a smooth projective morphism whose fibers have semiample canonical bundle and $\mathscr{H}$ is an f-ample line bundle with Hilbert polynomial h, up to isomorphisms and fiberwise numerical equivalence. The coarse moduli spaces P_h were shown to be quasi-projective schemes by Viehweg, cf. [Vie95].

1.1. Families of minimal varieties of general type. The first result partially answers a question of Viehweg and Zuo, cf. [VZ03, Quest. 0.2], who established in *loc. cit.* the analogous result in the case of moduli of canonically polarized manifolds (i.e. those whose canonical bundle is ample), cf. [VZ03, Thm. 0.1].

Theorem 1.1. *Let $f_U\colon U \to V$ be a smooth family of polarized manifolds of* general type *in $\mathscr{P}_h(V)$, with V quasi-projective, such that the induced morphism $\sigma\colon V \to P_h$ is quasi-finite onto its image. Then V is Brody hyperbolic, that is any holomorphic map $\gamma\colon \mathbb{C} \to V$ is constant.*

The question in [VZ03] asks whether the same holds for moduli of arbitrary polarized varieties, i.e. not necessarily of general type. While this was our original goal, in the general case we have not been able to overcome difficulties related to vanishing theorems. We do however give a positive answer to an even more general version of this question when V is a surface; see Corollary 1.5. Note that the more restrictive property of algebraic hyperbolicity, involving algebraic maps from curves and abelian varieties, has been known in great generality. It was established by Kovács [Kov00] for moduli of canonically polarized manifolds, and then by a combination of Viehweg-Zuo [VZ01] and Popa-Schnell [PS17] for families admitting good minimal models. See also Migliorini [Mig95] for families of surfaces.

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Theorem 1.1 is a direct consequence of the following result regarding the base spaces of smooth families of minimal manifolds of general type that have maximal variation. Recall first that the exceptional locus of V is defined as

$$\mathrm{Exc}(V) := \overline{\Big(\bigcup_{\gamma} \gamma(\mathbb{C})\Big)},$$

where the union is taken over all non-constant holomorphic maps $\gamma\colon \mathbb{C} \to V$, and the closure is in the Zariski topology.

Theorem 1.2. *Let $f_U\colon U \to V$ be a smooth projective morphism of smooth, quasi-projective varieties. Assume that f_U has maximal variation, and that its fibers are minimal manifolds of general type. Then the exceptional locus $\mathrm{Exc}(V)$ is a proper subset of V. In particular, every holomorphic map $\gamma : \mathbb{C} \to V$ is algebraically degenerate, that is the image of γ is not Zariski dense.*

For the general definition of the variation $\mathrm{Var}(f)$ of a family, we refer to [Vie83]. We are only concerned with maximal variation, $\mathrm{Var}(f) = \dim V$, which means that the very general fiber can only be birational to countably many other fibers; cf. also Lemma 3.11. For families coming from maps to moduli schemes, maximal variation simply means that the moduli map $V \to M$ is generically finite.

The theorem above is of course especially relevant for families of surfaces, where the minimality assumption becomes unnecessary, as one can pass to smooth minimal models in families. Recall that Giesker [Gie77] has constructed a coarse moduli space M parametrizing birational isomorphism classes of surfaces of general type.

Corollary 1.3. *Let $f_U\colon U \to V$ be a smooth projective family of surfaces of general type with maximal variation. Then $\mathrm{Exc}(V)$ is a proper subset of V. If moreover the family comes from a quasi-finite map $V \to M$ to the moduli space of surfaces of general type, then V is Brody hyperbolic.*

Statements as in Theorem 1.1 and 1.2 are conjecturally expected to be consequences of a different property of a more algebraic flavor, which is the subject of Viehweg's hyperbolicity conjecture; itself a generalization of a conjecture of Shafaravich. Roughly speaking, Viehweg predicted that for families with maximal variation, a log smooth compactification (Y, D) of V is of log general type. The proof of the original statement of the conjecture, in the canonically polarized case, was established in important special cases in [VZ02], [KK08a], [KK08b], [KK10], [Pat12], and was recently completed by Campana and Păun [CP15, Thm. 8.1]; for a more detailed overview of this body of work and for further references, please see [PS17, §1.2]. The statement was subsequently extended to families whose geometric generic fiber admits a good minimal model, so in particular to families of varieties of general type, by the first author and Schnell [PS17, Thm. A]. On the other hand, the conjecture of Green-Griffiths-Lang, [GG80] and [Lan86], predicts that for a pair (Y, D) of log general type, the image of any entire curve $\gamma\colon \mathbb{C} \to V$ is algebraically degenerate, where $V = Y \smallsetminus D$.

In the canonically polarized case, the problem of hyperbolicity of moduli stacks has a rich history from the purely analytic point of view. For the moduli stack $\mathcal{M}_g$ of compact Riemann surfaces of genus g, results of Ahlfors [Ahl61], Royden [Roy74] and Wolpert [Wol86] show that the holomorphic sectional curvature of the Weil-Petersson metric on the base of a family admitting a quasi-finite map to $\mathcal{M}_g$, with $g \geq 2$, is negative and bounded away from zero. In particular, such base spaces are Brody hyperbolic. In higher dimensions, thanks to Aubin-Yau's solution to Calabi's conjecture, one studies equivalently families of compact complex manifolds admitting a smooth Kähler-Einstein metric with negative Ricci curvature. The first breakthrough in this direction was achieved by Siu's computation [Siu86] of the curvature of the Weil-Petersson metric on the moduli via the Kähler-Einstein metric of the fibers of the family (see also [Sch12]). To and Yeung [TY15] built upon Siu's work to prove the Kobayashi hyperbolicity of moduli stacks of canonically polarized manifolds and thus gave a new proof of the Brody hyperbolicity of such moduli stacks (see also [TY16] for the Ricci-flat case). A different proof of this result has recently been established by Berndtsson, Păun and Wang [BPW17].

Once the canonically polarized condition is relaxed, such smooth canonical metrics do not necessarily exist, and it is not yet clear if the analytic methods discussed above can be adapted to tackle the hyperbolicity problem. In this paper we take a different path based on the approach of Viehweg and Zuo, where the key first step is to refine the Hodge theoretic constructions of [VZ03] and [PS17], with the ultimate goal of "generically" endowing any complex line $\mathbb{C}$ in V with a metric with sufficiently negative curvature; this is the content of §2. The next step, presented in §3, is to extend this metric to a singular metric on $\mathbb{C}$ whose curvature current violates the singular Ahlfors-Schwarz inequality. A review of the line of work that has inspired this approach to hyperbolicity can be found at the end of [VZ03, §1].

1.2. Two-dimensional parameter spaces in the general case. As mentioned at the outset, the results in Theorem 1.1 and Theorem 1.2 are expected to hold for families of manifolds of lower Kodaira dimension as well, assuming that they have semiample canonical bundle or, more generally, admit a good minimal model (which also includes the case of arbitrary fibers of general type).

On a related note, in [PS17, Thm. A] it is shown that the base V of any smooth family whose geometric generic fiber admits a good minimal model, and which has maximal variation, is of log general type. Thus the Green-Griffiths-Lang conjecture again predicts hyperbolicity properties for V. Note that when $\dim V = 1$, the two properties are equivalent, and had already been established in [VZ01]. We finish the paper by establishing such results in the case when V is two-dimensional.

Theorem 1.4. *Let $f_U : U \to V$ be a smooth family of projective manifolds, with maximal variation. Assume that V is a quasi-projective surface.*

(1.4.1) If the geometric generic fiber of f has a good minimal model, then every entire curve $\gamma : \mathbb{C} \to V$ is algebraically degenerate.
(1.4.2) Moreover, if the fibers are of general type, then the exceptional locus $\mathrm{Exc}(V)$ is a proper subset of V.

As a consequence of Theorem 1.4, we can extend Theorem 1.1 to the case of moduli of polarized manifolds, not necessarily of general type, as long as V is two-dimensional.

Corollary 1.5. *Let V be a quasi-projective surface admitting a morphism $\sigma : V \to P_h$ induced by a smooth family $f_U : U \to V$ in $\mathscr{P}_h(V)$. If σ is quasi-finite, then V is Brody hyperbolic.*

1.3. Outline of the argument. The paper follows the rough strategy towards proving hyperbolicity for parameter spaces that was developed in the series of works of Viehweg-Zuo [VZ01], [VZ02], [VZ03], and relies heavily on the improvements to this approach provided in [PS17], based on the theory of Hodge modules. Further technical refinements are necessary, as explained in the outline below. Here are the key steps:

(1) First, one constructs a special Hodge theoretic object on a compactification Y of (a birational model of) the base V, namely a graded subsheaf $(\mathscr{F}_\bullet, \theta_\bullet)$ of a Higgs bundle $(\mathscr{E}_\bullet, \theta_\bullet)$ associated to a Deligne canonical extension of a variation of Hodge structure (VHS) supported outside of a simple normal crossing divisor $D + S$, where $D = Y \smallsetminus V$. The system $\mathscr{F}_\bullet$ encodes the data of maximal variation and has positivity properties due to general Hodge theory.

A large part of the construction follows ideas from [PS17]; a key ingredient is the use of Hodge module extensions of VHS, necessary when the fibers are no longer canonically polarized. A detailed discussion of the construction can be found in [PS17, Introduction and §2] (see also [Pop16] for an overview). However, we make some modifications that lead to an a priori slightly different Higgs sheaf $(\mathscr{F}_\bullet, \theta_\bullet)$; the reason is that we crucially need the induced map $\mathscr{T}_Y \to \mathscr{F}_0^\vee \otimes \mathscr{F}_1$ to coincide generically with the Kodaira-Spencer map of the original family. This can be accomplished when the fibers are minimal of general type by appealing to a vanishing theorem due to Bogomolov and Sommese, combined with the basepoint-free theorem. We note that this is the only point in the argument where it is necessary to work with minimal varieties of general type, and which needs to be overcome in order to answer to the Viehweg-Zuo question in the arbitrary polarized case.

Given a holomorphic map $\gamma\colon \mathbb{C} \to V$, this construction eventually allows us to produce, for each $m \geq 0$, morphisms

$$\tau_m\colon \mathscr{T}_{\mathbb{C}}^{\otimes m} \longrightarrow \gamma^*(\mathscr{L}^{-1} \otimes \mathscr{E}_m),$$

where $\mathscr{L}$ is a big and nef line bundle on Y and $\mathscr{E}_\bullet$ is the Higgs bundle mentioned above. This is all done in §2.

(2) For the next step, in the case of Viehweg's hyperbolicity conjecture the point was to apply a powerful criterion detecting the log general type property, due to Campana-Păun [CP15]. In the present case of Brody hyperbolicity, this step is by contrast of an analytic, and in some sense more elementary flavor. Using the relationship with the Kodaira-Spencer map mentioned above, we show that for some $m \geq 1$ the map τ_m factors through

$$\tau_m\colon \mathscr{T}_{\mathbb{C}}^{\otimes m} \longrightarrow \gamma^*\mathscr{L}^{-1} \otimes \mathscr{N}_{(\gamma,m)},$$

where $\mathscr{N}_{(\gamma,m)}$ is defined as the kernel of the generalized Kodaira-Spencer map

$$\gamma^*\mathscr{E}_m \longrightarrow \gamma^*\mathscr{E}_{m+1} \otimes \Omega^1_{\mathbb{C}}(P),$$

with $P = \gamma^{-1}(S)$. As in [VZ03], one uses this, together with results about the curvature of Hodge metrics, in order to construct a sufficiently negative singular metric on $\mathbb{C}$ which violates the Ahlfors-Schwarz inequality.

The relaxation of the assumption on the fibers of the family again creates technical difficulties compared to the situation in [VZ03], where one could work with Hodge theoretic objects with finite monodromy around the components of S. We consider instead a further perturbation along S, which allows us to construct the singular metric we need using only the well-known growth estimates for Hodge metrics at the boundary given in [Sch73] and [CKS86]. This does not require any further knowledge about the monodromy, and so has the advantage of giving a simplified argument, in a more general situation. All of this is discussed in §3.2–§3.4.

(3) When the base V of the family is a surface, one does not need to appeal to the connection with the Kodaira-Spencer map mentioned in (1), and consequently the requirement that the fibers be minimal of general type can be dropped. Instead, we use the map τ_1 in order to produce a foliation on V such that $\gamma(\mathbb{C})$ is contained in one of its leaves. Given that by [PS17] we know that V is of log general type, we can then appeal to a result of McQuillan [McQ98] on the degeneracy of such entire curves, and to an extension to the logarithmic case in [EG03], in order to obtain a contradiction. This is the subject of §3.5.

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2. Hodge-theoretic constructions

2.1. Relative (graded) Higgs sheaves. We start with a brief discussion of Higgs sheaves with logarithmic poles. We consider the relative setting, which will be necessary for technical reasons later on, though most of the time the constructions are needed in the absolute setting. Suppose X and Y are smooth quasi-projective varieties, and $f\colon X \to Y$ is a smooth morphism of relative dimension d, with D a reduced relative normal crossing divisor over Y.

Recall that an f-relative graded Higgs sheaf with log poles along D is a pair $(\mathscr{E}_\bullet, \theta_\bullet)$ such that

(2.0.1) $\mathscr{E}_\bullet$ is a $\mathbb{Z}$-graded $\mathscr{O}_X$-module, with grading bounded from below.

(2.0.2) $\theta_\bullet$ is a grading-preserving $\mathscr{O}_X$-linear morphism

$$\theta_\bullet : \mathscr{E}_\bullet \longrightarrow \Omega^1_{X/Y}(\log D) \otimes \mathscr{E}_{\bullet+1}$$

satisfying $\theta_\bullet \wedge \theta_\bullet = 0$, where $\Omega^1_{X/Y}(\log D)$ is the sheaf of relative 1-forms with logarithmic poles along D; it is called the Higgs field of the sheaf.

A (relative) Higgs sheaf is called a (relative) Higgs bundle if it consists of $\mathscr{O}_X$-modules that are locally free of finite rank. If f is trivial, then we get the usual notions of a Higgs sheaf or Higgs bundle. The standard example is the Hodge bundle associated to a variation of Hodge structure (VHS). More generally, for a VHS $\mathbb{V}$ on $Y \smallsetminus D$, with quasi-unipotent monodromy along the components of D, the Deligne extension of the VHS across D with eigenvalues in $[0,1)$ is a logarithmic VHS, i.e. the extension of the flat bundle is locally free with a flat logarithmic connection, and the extension of the filtration is a filtration by subbundles; see [Del70, Prop. I.5.4] and [Sai90, (3.10.5)]; see also [Kol86, 2.5]. Hence its generalized Hodge bundle $(\mathscr{E}_\bullet, \theta_\bullet)$ is a logarithmic Higgs bundle.

We denote by $\mathscr{T}_{X/Y}(-\log D)$ the sheaf of relative vector fields with logarithmic zeros along D, and consider its symmetric algebra

$$\mathscr{A}_{X/Y}(-\log D) := \operatorname{Sym} \mathscr{T}_{X/Y}(-\log D)$$

(or $\mathscr{A}^\bullet_{X/Y}(-\log D)$ if we want to emphasize its grading). When $D = 0$ and f is trivial, we have $\mathscr{A}_X = \operatorname{gr}^F_\bullet \mathscr{D}_X$, where $\mathscr{D}_X$ is the sheaf of holomorphic differential operators with the order filtration. We have inclusions of graded $\mathscr{O}_X$-algebras

$$\mathscr{A}_{X/Y}(-\log D) \hookrightarrow \mathscr{A}_{X/Y} \hookrightarrow \mathscr{A}_X \quad \text{and} \quad \mathscr{A}_{X/Y}(-\log D) \hookrightarrow \mathscr{A}_X(-\log D) \hookrightarrow \mathscr{A}_X.$$

We will consider graded modules over these sheaves of rings. For instance, the associated graded of a filtered $\mathscr{D}_X$-module (resp. of a filtered vector bundle with flat connection with log poles along D) is an $\mathscr{A}_X^\bullet$ (resp. $\mathscr{A}_X^\bullet(-\log D)$)-module. The following reinterpretation of the definitions allows us to use relative Higgs sheafs and graded $\mathscr{A}_{X/Y}(-\log D)$-modules interchangeably.

Lemma 2.1. *The data of a relative Higgs sheaf $(\mathscr{E}_\bullet, \theta_\bullet)$ with log poles along D is equivalent to that of a graded $\mathscr{A}^\bullet_{X/Y}(-\log D)$-module structure on $\mathscr{E}_\bullet$, extending the $\mathscr{O}_X$-module structure.*

The Higgs field $\theta_\bullet$ induces a complex of graded $\mathscr{O}_X$-modules, de Rham complex

$$\mathrm{DR}^D_{X/Y}(\mathscr{E}_\bullet) := \big[\mathscr{E}_\bullet \to \Omega^1_{X/Y}(\log D)\otimes \mathscr{E}_{\bullet+1} \to \cdots \to \Omega^d_{X/Y}(\log D)\otimes \mathscr{E}_{\bullet+d}\big]$$

and we have

$$\mathrm{DR}^D_{X/Y}(\mathscr{E}_\bullet) \simeq \mathrm{DR}^D_{X/Y}\big(\mathscr{A}^\bullet_{X/Y}(-\log D)\big) \otimes_{\mathscr{A}^\bullet_{X/Y}(-\log D)} \mathscr{E}_\bullet.$$

Definition 2.2 (Pull-back of Higgs bundles)**.** Let $\mathscr{E}_\bullet$ be a relative Higgs bundle on X, and $\gamma\colon B \to X$ a holomorphic map from a complex manifold B, such that the support E of $\gamma^{-1}(D)$ is relative normal crossing over Y with respect to the induced map $B \to Y$. Then the natural $\mathscr{O}_X$-linear morphism $\mathscr{T}_{B/Y}(-\log E) \to \gamma^*\mathscr{T}_{X/Y}(-\log D)$ induces a morphism

$$\mathscr{A}_{B/Y}(-\log E) \to \gamma^*\mathscr{A}_{X/Y}(-\log D)$$

of graded $\mathscr{O}_B$-algebras. Therefore, $\gamma^*\mathscr{E}_\bullet$ is a graded $\mathscr{A}_B(-\log E)$-module, and in particular a relative Higgs bundle on B with Higgs field induced by that of $\mathscr{E}_\bullet$.

2.2. Hodge modules for rank 1 unitary representations on quasi-projective varieties. We discuss Hodge modules for rank 1 unitary representations, needed in what follows. We fix a line bundle $\mathscr{B}$ on a smooth quasi-projective variety X, and assume that

$$\mathscr{B}^m \simeq \mathscr{O}_X(E),$$

for some $m \in \mathbb{N}$ and an effective divisor $E = \sum_i a_i D_i$ with simple normal crossing support. We denote $D = E_{\mathrm{red}}$. It is well known that, for every $0 < i < m$ and every divisor E' supported on D, the line bundle $\mathscr{B}^{-i}(E')$ admits a flat connection with logarithmic poles along D. As in [EV92, §3], we set

$$\mathscr{B}^{(-i)} = \mathscr{B}^{-i}\,\big(\sum_i \lfloor \frac{a_i}{m} \rfloor \cdot D_i\big)$$

the Deligne canonical extension of $\mathscr{B}^{-i}|_{X\smallsetminus D}$, which is a flat unitary line bundle on $X \smallsetminus D$ coming from a unitary representation of the fundamental group. We also use the notation

$$\mathscr{B}^{-i}(*D) = \bigcup_{k\geq 0} \mathscr{B}^{-i}(kD)$$

for the sheaf of sections of $\mathscr{B}^{-i}$ with poles of arbitrary order along D. We define filtrations on $\mathscr{B}^{(-i)}$, $\mathscr{B}^{(-i)}(D)$ and $\mathscr{B}^{-i}(*D)$ by:

$$F_p\mathscr{B}^{(-i)}(C) = \begin{cases} 0 & \text{if } p<0 \\ \mathscr{B}^{(-i)}(C) & \text{if } p \geq 0, \end{cases}$$

where C is either 0 or D, and

$$F_p\mathscr{B}^{-i}(*D) = \begin{cases} 0 & \text{if } p<0 \\ \mathscr{B}^{(-i)}\big((p+1)D\big) & \text{if } p \geq 0. \end{cases} \tag{2.2.1}$$

With these filtrations, $\mathscr{B}^{(-i)}(D)$ is a filtered line bundle with a flat connection with log poles along D, and $\mathscr{B}^{-i}(*D)$ is a filtered $\mathscr{D}_X$-module. Note that in particular we will always consider $\mathscr{O}_X$ with the trivial filtration $F_k\mathscr{O}_X = \mathscr{O}_X$ for $k \geq 0$, and 0 otherwise, so that $\mathrm{gr}_\bullet^F \mathscr{O}_X \simeq \mathscr{O}_X$.

By [Sai90, (3.10.3) and (3.10.8)], we know that $(\mathscr{B}^{-i}(*D), F_\bullet)$ is a direct summand of the filtered $\mathscr{D}_X$-module underlying $\pi_*\mathbb{Q}_Z^H[\dim Z]$, the direct image of the trivial Hodge module on Z, where $\pi\colon Z \to X$ is the m-th cyclic cover branched along the divisor E.

Note that $\mathrm{gr}_\bullet^F \mathscr{B}^{(-i)}(D)$ is a graded $\mathscr{A}_X^\bullet(-\log D)$-module, while $\mathrm{gr}_\bullet^F \mathscr{B}^{-i}(*D)$ is a graded $\mathscr{A}_X^\bullet$-module. Moreover, the natural inclusions

$$\mathrm{gr}_\bullet^F \mathscr{B}^{(-i)} \hookrightarrow \mathrm{gr}_\bullet^F \left(\mathscr{B}^{(-i)}(D)\right) \hookrightarrow \mathrm{gr}_\bullet^F \mathscr{B}^{-i}(*D)$$

preserve the Higgs structure. We have the following comparison result:

Proposition 2.3. *Assume that $f\colon X \to Y$ is a smooth projective morphism of relative dimension d between smooth quasi-projective varieties, and D is a divisor on X which is relatively normal crossing over Y. Then the natural morphism*

$$\mathrm{DR}_{X/Y}^D\left(\mathrm{gr}_\bullet^F \mathscr{B}^{(-i)}\right) \longrightarrow \mathrm{DR}_{X/Y}\left(\mathrm{gr}_\bullet^F \mathscr{B}^{(-i)}(*D)\right)$$

is a quasi-isomorphism of complexes of graded $\mathscr{O}_X$-modules.

Proof. The absolute case was proved in [Sai90, §3.b] in a more general setting. The relative case is similar; we sketch the proof for completeness.

We define graded sheaves $\mathscr{C}_\bullet$ and $\mathscr{N}_\bullet$ by

$$\mathscr{C}_\bullet := \mathscr{A}_{X/Y}^\bullet \otimes_{\mathrm{gr}_\bullet^F \mathscr{O}_X} \mathrm{gr}_\bullet^F \left(\mathscr{B}^{(-i)}(D)\right)$$

and

$$\mathscr{N}_\bullet := \mathscr{A}_{X/Y}^\bullet \otimes_{\mathscr{A}_{X/Y}^\bullet(-\log D)} \mathrm{gr}_\bullet^F \left(\mathscr{B}^{(-i)}(D)\right).$$

By definition $\mathscr{C}_\bullet$ is a graded $\left(\mathscr{A}_{X/Y}^\bullet\text{-}\mathscr{A}_{X/Y}^\bullet(-\log D)\right)$-bimodule. (The $\mathscr{A}_{X/Y}(-\log D)$-module structure is induced by the product rule; that is, locally $x_i\partial_{x_i} \cdot (\nu \otimes l) = x_i\partial_{x_i} \cdot \nu \otimes l - \nu \otimes x_i\partial_{x_i} \cdot l$, if $\nu \otimes l$ is a section of $\mathscr{C}_\bullet$.) Assume now that $\mathscr{T}_{X/Y}(-\log D)$ is freely generated locally by

$$\partial_{x_1}, \ldots, \partial_{x_i}, x_{i+1}\partial_{x_{i+1}}, \ldots, x_d\partial_{x_d}.$$

The sequence of actions of these elements on $\mathscr{C}_\bullet$ (via the $\mathscr{A}_{X/Y}(-\log D)$-module structure described above) gives rise to a Koszul-type complex. Written in a coordinate free way, this is a complex of $\mathscr{A}_{X/Y}^\bullet$-modules

$$\mathcal{B}_\bullet^\bullet = \left[\mathscr{C}_{\bullet-d} \otimes \bigwedge^d \mathscr{T}_{X/Y}(-\log D) \to \mathscr{C}_{\bullet-d+1} \otimes \bigwedge^{d-1} \mathscr{T}_{X/Y}(-\log D) \to \ldots \to \mathscr{C}_\bullet\right].$$

Using the fact that $\mathrm{gr}_\bullet^F \left(\mathscr{L}^{(-i)}(D)\right)$ is locally free of rank 1 over $\mathrm{gr}_\bullet^F \mathscr{O}_X$, one can check that this sequence is regular; therefore, the natural morphism

$$\mathcal{B}_\bullet^\bullet \longrightarrow H^0\mathcal{B}_\bullet^\bullet = \frac{\mathscr{C}_\bullet}{\sum_{j=1}^i \partial_{x_j}\mathscr{C}_\bullet + \sum_{j=i+1}^d x_j\partial_{x_j}\mathscr{C}_\bullet} \simeq \mathscr{N}_\bullet$$

is a quasi-isomorphism of complexes of graded $\mathscr{A}_{X/Y}^\bullet$-modules. The exactness of the de Rham functor implies that the induced morphism

$$\mathrm{DR}_{X/Y}(\mathcal{B}_\bullet^\bullet) \longrightarrow \mathrm{DR}_{X/Y}(\mathscr{N}_\bullet)$$

is a quasi-isomorphism as well. Moreover, one also sees that the natural morphism

$$\mathrm{DR}_{X/Y}\big(\mathscr{C}_{\bullet-d+p}\otimes\bigwedge^{d-p}\mathscr{T}_{X/Y}(-\log D)\big)\longrightarrow \mathrm{gr}^F_{\bullet+p}\mathscr{B}^{(-i)}\otimes\Omega^p_{X/Y}(\log\ D)$$

is a quasi-isomorphism, thanks to the natural isomorphism given by contraction

$$\omega_{X/Y}(D)\otimes\bigwedge^{d-p}\mathscr{T}_{X/Y}(-\log D)\simeq\Omega^p_{X/Y}(\log\ D),$$

and the fact that $\mathrm{DR}_{X/Y}\big(\mathscr{A}^\bullet_{X/Y}\big)$ is quasi-isomorphic to $\omega_{X/Y}$. Therefore, we find that $\mathrm{DR}_{X/Y}(\mathcal{B}^\bullet_\bullet)$ and $\mathrm{DR}^D_{X/Y}(\mathrm{gr}^F_\bullet\mathscr{B}^{(-i)})$ are quasi-isomorphic. We now conclude by noting that there is an isomorphism of $\mathscr{A}^\bullet_{X/Y}$-modules

$$\mathscr{N}_\bullet\simeq\mathrm{gr}^F_\bullet\mathscr{L}^{-i}(*D);$$

see for instance [Bjö93, Prop. 4.2.18] (where it is stated locally, for more general $\mathscr{D}$-modules). ☐

2.3. Basic set-up. We consider a smooth family $f_U\colon U\to V$ of projective varieties, whose geometric generic fiber admits a good minimal model. (This includes for instance families of varieties of general type, or of varieties whose canonical bundle is semiample.) We assume that the family has maximal variation; following the strategy in [VZ03] and the technical extensions in [PS17], our aim in the next few sections is to endow entire curves inside (a birational model of) V with Hodge theoretic objects that will be used in the next section in order to conclude hyperbolicity.

We denote by $f'\colon X'\to Y$ a compactification of f_U, with X' and Y smooth and projective, and by $D_{f'}$ the singular locus of f' (i.e. the locus over which f' is not smooth), which after a birational modification can be assumed to be a divisor. We will make use of the following statements, proved using Viehweg's fiber product trick and weak positivity.

Proposition 2.4 (Generic global generation)**.** *After possibly replacing Y by a birational model,*[1] *there exists a morphism $f\colon X\to Y$ of smooth projective varieties, whose singular locus is $D_f=D_{f'}$, and an integer $m>0$, such that the following properties hold:*

(2.4.1) [PS17, Proof of Thm. B, Step 1] *There exists a line bundle $\mathscr{A}$ on Y such that*

$$H^0\big(X,(\omega_{X/Y}\otimes f^*\mathscr{A}^{-1})^m\big)\neq 0,$$

which can be taken to be of the form $\mathscr{A}=\mathscr{L}(D_Y)$, with $\mathscr{L}$ ample and D_Y an effective divisor such that $D_Y\geq D_f$.

(2.4.2) [VZ03, Cor. 4.3] *If we assume in addition that the fibers of $f_U\colon U\to V$ have semi-ample canonical bundle, then so do the fibers of f over V, and there exists a Zariski open subset $W\subset V$ such that $(\omega_{X/Y}\otimes f^*\mathscr{A}^{-1})^m$ is generated by global sections over $f^{-1}(W)$, where $\mathscr{A}$ is as in (2.4.1).*

2.4. Hodge modules and branched coverings. This section is essentially a review of the constructions in [PS17, §2.3 and 2.4], but with a twist which is important for the applications in this paper. We assume that we have a morphism of smooth projective varieties $f\colon X\to Y$, with connected fibers, and with $\dim Y=n$ and $\dim X=n+d$. Let $\mathscr{A}$ be a line bundle on Y, and define

$$\mathscr{B}:=\omega_{X/Y}\otimes f^*\mathscr{A}^{-1}.$$

[1] More precisely by a smooth $\widetilde{Y}$ with a birational morphism $\widetilde{Y}\to Y$.

We make the following assumption:

$$\text{There exists } 0 \neq s \in H^0(X, \mathscr{B}^m) \text{ for some } m > 0. \tag{2.4.3}$$

The section s defines a branched cover $\psi \colon X_m \to X$ of degree m. Let $\delta \colon Z \longrightarrow X_m$ be a desingularization of the normalization of X_m, which is irreducible if m is chosen to be minimal, and set $\pi = \psi \circ \delta$ and $h = f \circ \pi$, as in the diagram

$$\begin{tikzcd}
Z \arrow[r, "\delta"] \arrow[rr, bend left, "\pi"] \arrow[rrd, "h"'] & X_m \arrow[r, "\psi"] \arrow[rd] & X \arrow[d, "f"] \\
 & & Y
\end{tikzcd}$$

Let $\mathscr{A}_Y = \operatorname{Sym} \mathscr{T}_Y$, with the natural grading, and similarly for $\mathscr{A}_X$. A morphism of graded $\mathscr{A}_Y$-modules

$$\mathbf{R}f_*(\omega_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{B}^{-1} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y) \longrightarrow \mathbf{R}h_*(\omega_{Z/Y} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_Z} h^*\mathscr{A}_Y). \tag{2.4.4}$$

is constructed in [PS17, §2.4]. (We use the notation $\mathscr{B}^{-1} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y$ as shorthand for $\mathscr{B}^{-1} \otimes_{\mathscr{O}_X} \operatorname{gr}^F_\bullet \mathscr{O}_X \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y$, where we make use of the $\mathscr{A}_X$-module structure on $\operatorname{gr}^F_\bullet \mathscr{O}_X \simeq \mathscr{O}_X$.) Here we will construct a similar but slightly different morphism, that a priori coincides with the one in (2.4.4) only generically. The reason for this different construction will become clear in §2.6, where we need to compare Hodge sheafs constructed out of branched coverings with others that are naturally related to Kodaira-Spencer maps.

Before starting the construction, recall from [Sai90, §3.b] that $\mathscr{B}$ uniquely determines a filtered $\mathscr{D}_X$-module $(\mathscr{B}_*^{-1}, F_\bullet)$ with strict support X, which extends $(\mathscr{B}^{-1}|_{X \smallsetminus \operatorname{div}(s)}, F_\bullet)$, where the filtration on the latter is the trivial filtration; notice that the filtered $\mathscr{D}_X$-module is exactly $(\mathscr{B}^{-1}(*D), F_\bullet)$, when $D = \operatorname{div}(s)_{\mathrm{red}}$ is normal crossing. Moreover, $(\mathscr{B}_*^{-1}, F_\bullet)$ is a direct summand of the filtered $\mathscr{D}_X$-module $\mathcal{H}^0\pi_+(\mathscr{O}_Z, F_\bullet)$.

Lemma 2.5. *We have a natural inclusion*

$$\mathscr{B}^{-1} \hookrightarrow F_0\, \mathscr{B}_*^{-1}.$$

Proof. Let $\mu \colon X' \to X$ be a log resolution of the divisor $\operatorname{div}(s)$ which is an isomorphism on its complement. Define $D' = (\mu^*\operatorname{div}(s))_{\mathrm{red}}$ and $\mathscr{B}' = \mu^*\mathscr{B}$. Then, according to the discussion in §2.2, $(\mathscr{B}'^{-1}(*D'), F_\bullet)$ defined as in (2.2.1) is a direct summand of a $\mathscr{D}_{X'}$-module underlying a Hodge module. By the strictness of the direct image functor for Hodge modules, we have

$$\mu_+(\mathscr{B}'^{-1}(*D'), F_\bullet) = (\mathscr{B}_*^{-1}, F_\bullet)$$

and

$$\mu_* F_0 \mathscr{B}'^{-1}(*D') = F_0 \mathscr{B}_*^{-1}.$$

On the other hand, by construction we have the injection

$$\mathscr{B}'^{-1} \subset F_0 \mathscr{B}'^{-1}(*D'),$$

and so the statement follows from the projection formula. $\square$

We now proceed with our construction. The inclusion in Lemma 2.5 induces a morphism of graded $\mathscr{A}_X$-modules

$$\mathscr{B}^{-1} \longrightarrow \mathrm{gr}^F_\bullet \mathscr{B}^{-1}_*,$$

with the trivial graded $\mathscr{A}_X$-module structure on $\mathscr{B}^{-1}$. This in turn induces a morphism

$$\mathbf{R}f_*(\omega_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{B}^{-1} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y) \longrightarrow \mathbf{R}f_*(\omega_{X/Y} \otimes_{\mathscr{O}_X} \mathrm{gr}^F_\bullet \mathscr{B}^{-1}_* \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y).$$

Now the right hand side is a direct summand of the object $\mathbf{R}h_*(\omega_{Z/Y} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_Z} h^*\mathscr{A}_Y)$; indeed, using [PS13, Theorem 2.9], we have an isomorphism

$$\mathrm{gr}^F_\bullet h_+(\mathscr{O}_Z, F_\bullet) \simeq \mathbf{R}h_*(\omega_{Z/Y} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_Z} h^*\mathscr{A}_Y),$$

and we combine this with the filtered direct summand inclusion of $(\mathscr{B}^{-1}_*, F_\bullet)$ in $\mathcal{H}^0\pi_+(\mathscr{O}_Z, F_\bullet)$. Therefore we get an induced morphism

$$\mathbf{R}f_*(\omega_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{B}^{-1} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y) \longrightarrow \mathbf{R}h_*(\omega_{Z/Y} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_Z} h^*\mathscr{A}_Y), \tag{2.5.1}$$

which factors through $\mathbf{R}f_*(\omega_{X/Y} \otimes \mathrm{gr}^F_\bullet \mathscr{B}^{-1}_* \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y)$. One can check that the morphisms (2.4.4) and (2.5.1) coincide over the locus where h is smooth; they are however not necessarily the same globally.

Let now $(\mathcal{M}, F_\bullet)$ be the filtered $\mathscr{D}_Y$-module underlying the Tate twist $M(d)$ of the pure polarizable Hodge module M which is the direct summand of $\mathcal{H}^0 h_* \mathbf{Q}^H_Z[n+d]$ strictly supported on Y. By [PS17, Prop. 2.4], we then have that $\mathrm{gr}^F_\bullet \mathcal{M}$ is a direct summand of $R^0h_*(\omega_{Z/Y} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_Z} h^*\mathscr{A}_Y)$.

Definition 2.6. We define a graded $\mathscr{A}_Y$-module $\mathscr{G}_\bullet$ as the image of the composition

$$R^0f_*(\omega_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{B}^{-1} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y) \to R^0h_*(\omega_{Z/Y} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_Z} h^*\mathscr{A}_Y) \to \mathrm{gr}^F_\bullet \mathcal{M},$$

where the second morphism is given by projection.

Recall that D_f denotes the singular locus of f. We gather the constructions above and further properties in the following result, which is essentially [PS17, Thm. 2.2]; although as pointed out above the new morphism (2.5.1) is constructed slightly differently, the proof is identical.

Theorem 2.7. *With the above notation, assuming* (2.4.3)*, the coherent graded $\mathscr{A}_Y$-module $\mathscr{G}_\bullet$ satisfies the following properties:*

(2.7.1) There is an isomorphism $\mathscr{G}_0 \simeq \mathscr{A}$.
(2.7.2) Each $\mathscr{G}_k$ is torsion-free on $X \smallsetminus D_f$.
(2.7.3) There is an inclusion of graded $\mathscr{A}_Y$-modules $\mathscr{G}_\bullet \subseteq \mathrm{gr}^F_\bullet \mathcal{M}$.

2.5. Main construction on $\mathbb{C}$. Our aim in this section is to use the above constructions in order to produce interesting Hodge-theoretic sheaves on $\mathbb{C}$, assuming the existence of a holomorphic mapping $\gamma\colon \mathbb{C} \to V$.

Assumption: all VHS appearing in this paper are assumed to be polarizable, and all local monodromies to be quasi-unipotent; see for instance [Sch73] for the definitions. This is of course the case for any geometric VHS, i.e. the Gauss-Manin connection of a smooth family of projective manifolds, thanks to the monodromy theorem (see for instance [Sch73, Lem. 4.5]). In general, fixing a polarization induces the Hodge metric on the associated Higgs bundle, its singularities at the boundary will play a crucial role in §3.2.

We now return to the set-up in §2.3. The following is the key output of the Hodge theoretic constructions above, based on arguments in [PS17]. According to the strategy in [VZ03], it will later be combined with analytic arguments in order to conclude the non-existence of dense entire curves.

Proposition 2.8. *Let $f_U : U \to V$ be a smooth family of projective varieties, with maximal variation, and whose geometric generic fiber has a good minimal model. Then, after possibly replacing V by a birational model, there exists a smooth projective compactification Y of V, with $D = Y \smallsetminus V$ a simple normal crossing divisor, together with a big and nef line bundle $\mathscr{L}$ and an inclusion of graded $\mathscr{A}_Y^\bullet(-\log D)$-modules*

$$(\mathscr{F}_\bullet, \theta_\bullet) \subseteq (\mathscr{E}_\bullet, \theta_\bullet),$$

on Y, that verify the following properties:

(2.8.1) $(\mathscr{E}_\bullet, \theta_\bullet)$ is the Higgs bundle underlying the Deligne extension with eigenvalues in $[0, 1)$ of a VHS defined outside of a simple normal crossing divisor $D + S$.
(2.8.2) $\mathscr{F}_0$ is a line bundle, and we have an inclusion $\mathscr{L} \subseteq \mathscr{F}_0$ which is an isomorphism on V.
(2.8.3) If $\gamma : \mathbb{C} \to V \subseteq Y$ is a holomorphic map, then for each $k \geq 0$ there exists a morphism

$$\tau_{(\gamma,k)} : \mathscr{T}_{\mathbb{C}}^{\otimes k} \to \gamma^*\big(\bigotimes^k \mathscr{T}_Y(-\log D)\big) \to \gamma^*({\mathscr{F}_0}^{-1} \otimes \mathscr{E}_k) \to \gamma^*(\mathscr{L}^{-1} \otimes \mathscr{E}_k).$$

Proof. We first fix a compactification $f' : X' \to Y$ of f_U, with $D = D_{f'}$. Let X and $f : X \to Y$ be as in Proposition 2.4. According to Item (2.4.1), there exists an integer $m > 0$ and a line bundle $\mathscr{A}$ on Y, which we can take to be of the form $\mathscr{A} = \mathscr{L}(D_Y)$ with $\mathscr{L}$ ample and $D_Y \geq D$, such that

$$H^0\big(X, (\omega_{X/Y} \otimes f^*\mathscr{A}^{-1})^m\big) \neq 0.$$

This means that we can apply the constructions in §2.4; we set

$$\mathscr{B} = \omega_{X/Y} \otimes f^*\mathscr{A}^{-1}$$

and pick $0 \neq s \in H^0(X', \mathscr{B}^m)$. Associated to this section, by applying Theorem 2.7, we obtain a Hodge sheaf $\mathscr{G}_\bullet$ and a Hodge module M on Y. By taking a further log resolution, we can assume that there is an effective divisor S on Y such that the singular locus of M is (contained in) $D + S$, and has simple normal crossings; this has the effect of changing V to a birational model, and consequently turning $\mathscr{L}$ into a big and nef line bundle.

Recall now that M is the direct summand of $\mathcal{H}^0 h_* \mathbf{Q}_Z^H[n+d]$ which is strictly supported on Y. We take $(\mathscr{E}_\bullet, \theta_\bullet)$ to be the Higgs bundle underlying the Deligne extension with eigenvalues in $[0, 1)$ of the VHS that coincides with M outside of $D + S$. Following [PS17, §2.7 and §2.8], we define a subsheaf $(\mathscr{F}_\bullet, \theta_\bullet)$ of $(\mathscr{E}_\bullet, \theta_\bullet)$ by

$$\mathscr{F}_\bullet = \big(\mathscr{G}_\bullet \cap \mathscr{E}_\bullet\big)^{\vee\vee}.$$

Note that the intersection makes sense, since both $\mathscr{G}_\bullet$ and $\mathscr{E}_\bullet$ are contained in $\mathrm{gr}_\bullet^F \mathcal{M}$. Precisely as in [PS17, Prop. 2.14 and Prop. 2.15], one has the following properties for $\mathscr{F}_\bullet$:

(2.8.4) We have $\mathscr{A}(-D) \subseteq \mathscr{F}_0 \subseteq \mathscr{A}$, for some integer $l > 0$.
(2.8.5) The Higgs field θ maps $\mathscr{F}_k$ into $\Omega_Y^1(\log D) \otimes \mathscr{F}_{k+1}$.

Note that $\mathscr{F}_0$ is a reflexive sheaf of rank 1 on the smooth variety Y, and hence is a line bundle. Thus (2.8.4) shows Item (2.8.2), while (2.8.5) leads to Item (2.8.3) by the following construction. Note that (2.8.5) means $\mathscr{F}_\bullet$ is an $\mathscr{A}_Y(-\log D)$-module. The $\mathscr{A}_Y(-\log D)$-module structure induces a map

$$\rho_k : \bigotimes^k \mathscr{T}_Y(-\log D) \to \operatorname{Sym}^k \mathscr{T}_Y(-\log D) \to {\mathscr{F}_0}^{-1} \otimes \mathscr{F}_k \to {\mathscr{F}_0}^{-1} \otimes \mathscr{E}_k.$$

By composing ρ_k with the k-th tensor power of the differential

$$d\gamma : \mathscr{T}_{\mathbb{C}} \to \gamma^* \mathscr{T}_Y(-\log D),$$

we obtain

$$\tau_{(\gamma,k)} : \mathscr{T}_{\mathbb{C}}^{\otimes k} \xrightarrow{d\gamma^{\otimes k}} \gamma^*\big(\bigotimes^k \mathscr{T}_Y(-\log D)\big) \xrightarrow{\gamma^*\rho_k} \gamma^*({\mathscr{F}_0}^{-1} \otimes \mathscr{E}_k) \hookrightarrow \gamma^*(\mathscr{L}^{-1} \otimes \mathscr{E}_k),$$

where the last morphism is induced by the inclusion of $\mathscr{L}$ into $\mathscr{F}_0$. □

Remark 2.9. If $f_U : U \to V$ has fibers with semi-ample canonical bundle, then we may as well assume in Proposition 2.8 that the line bundle $\mathscr{A}$ is as in Proposition 2.4, Item (2.4.2). This will be used in the next section.

Finally, we record a fact that will be of use later on.

Lemma 2.10. *In the notation of Proposition 2.8, the Higgs map*

$$\theta_0 \colon \mathscr{F}_0 \longrightarrow \mathscr{F}_1 \otimes \Omega^1_Y(\log D)$$

is injective.

Proof. It suffices to show that θ_0 is not the zero map, since $\mathscr{F}_0$ is a line bundle and $\mathscr{F}_1$ is torsion free. By Item (2.8.2), we know that $\mathscr{F}_0$ is a big line bundle. On the other hand, if θ_0 were identically zero, then we would have that $\mathscr{F}_0 \subseteq K_0$, where

$$K_0 := \ker\big(\theta_0 \colon \mathscr{E}_0 \to \mathscr{E}_1 \otimes \Omega^1_Y(\log D + S)\big).$$

Now $K_0^\vee$ is a weakly positive sheaf by [PW16, Theorem 4.8] (an easy consequence of the results of [Zuo00] and [Bru15] in the unipotent case), so this would imply that $\mathscr{F}_0^{-1}$ is also a pseudoeffective line bundle, a contradiction. □

2.6. Further refinements for families of minimal manifolds of general type. In the current section, assuming that the members of the family are minimal and of general type, we will establish a connection between the sheaf $(\mathscr{F}, \theta)$ defined in Proposition 2.8 and the Kodaira-Spencer map of f. In the canonically polarized case treated in [VZ03], an analogous statement is proved as an application of the Akizuki-Nakano vanishing theorem, which in the present context is not available any more; we will be able to achieve this using a different argument based on transversality and a more restrictive vanishing theorem due to Bogomolov and Sommese.

We continue to be in the set-up of §2.3, and we fix the morphism $f \colon X \to Y$ as in the proof of Proposition 2.8. We define a new graded $\mathscr{A}_Y$-module $\widetilde{\mathscr{F}_\bullet}$ by

$$\widetilde{\mathscr{F}_\bullet} = R^0 f_*\big(\omega_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{B}^{-1} \otimes \operatorname{gr}^F_\bullet \mathscr{O}_X \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y\big), \tag{2.10.1}$$

i.e. the left hand side of (2.5.1), where the $\mathscr{A}_Y$-module structure is induced by the $f^*\mathscr{A}_Y$-module structure on $\omega_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{B}^{-1} \otimes \mathrm{gr}^F_\bullet \mathscr{O}_X \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y$. This structure induces a morphism

$$\mathscr{T}_Y \longrightarrow \widetilde{\mathscr{F}}_0^{-1} \otimes \widetilde{\mathscr{F}}_1. \tag{2.10.2}$$

Also, by the projection formula, we have $\widetilde{\mathscr{F}}_0 = \mathscr{A}$.

On the other hand, over the locus where f is smooth, using the fact that the natural morphism

$$[\mathscr{A}_X^{\bullet-d} \otimes \bigwedge^d \mathscr{T}_{X/Y} \longrightarrow \mathscr{A}_X^{\bullet-d+1} \otimes \bigwedge^{d-1} \mathscr{T}_{X/Y} \longrightarrow \cdots \longrightarrow \mathscr{A}_X^\bullet] \longrightarrow f^*\mathscr{A}_Y^\bullet \tag{2.10.3}$$

induced by the natural mapping $\mathscr{T}_X \to f^*\mathscr{T}_Y$ is a quasi-isomorphism of complexes of graded $\mathscr{A}_X$-modules (see for example [Pha79, Lem. 14.3.5][2]), we know that

$$\widetilde{\mathscr{F}}_\bullet \simeq R^0 f_*\big(f^*\mathscr{A} \otimes \mathrm{gr}^F_\bullet \mathscr{O}_X \otimes_{\mathscr{A}_X} \mathrm{DR}_{X/Y}(\mathscr{A}_X^\bullet)\big).$$

In particular, over this locus we have

$$\widetilde{\mathscr{F}}_1 \simeq \mathscr{A} \otimes R^1 f_* \mathscr{T}_{X/Y}.$$

Therefore, by construction we obtain:

Lemma 2.11. *Over $V = Y \smallsetminus D$, the morphism (2.10.2) is precisely the Kodaira–Spencer map*

$$\mathscr{T}_Y \longrightarrow R^1 f_* \mathscr{T}_{X/Y}.$$

Consequently, in order to establish a connection between the Kodaira-Spencer map and $(\mathscr{F}_\bullet, \theta_\bullet)$ in Proposition 2.8, it suffices to establish one between $(\widetilde{\mathscr{F}}_\bullet, \widetilde{\theta}_\bullet)$ and $(\mathscr{F}_\bullet, \theta_\bullet)$. This follows immediately from the next result.

Proposition 2.12. *For $k \leq 1$, the natural morphism*

$$\widetilde{\mathscr{F}}_k \longrightarrow \mathscr{G}_k$$

is generically an isomorphism.

Proof. For $k = 0$, the statement follows from by Theorem 2.7, Item (2.7.1). We now focus on the $k = 1$ case. By the basepoint-free theorem, the fibers have semiample canonical bundle, hence Proposition 2.4, Item (2.4.2) applies. Let W be the open subset of V defined there. Replacing Y by W, after shrinking it further if necessary, and X by $\mu^{-1}(W)$, we can assume that $f|_H \colon H \longrightarrow Y$, f and h are smooth and $\mathscr{B}^m \simeq \mathscr{O}_X(H)$ is globally generated, where H is a smooth divisor transversal to the fibers. Here h is the morphism defined in §2.4 by the resolution of the branched covering associated to the global section defining H. Since h is smooth, we have $\mathcal{H}^0 h_* \mathbf{Q}^H_Z[n+d] = M(-d)$ and so it is enough to show that the morphism

$$\widetilde{\mathscr{F}}_1 \longrightarrow R^0 f_*\big(\omega_{X/Y} \otimes \mathrm{gr}^F_\bullet \mathscr{B}_*^{-1} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y\big)_1 \tag{2.12.1}$$

defined in §2.4 is injective.

On the other hand, as f is smooth, as we have seen above we have

$$\widetilde{\mathscr{F}}_\bullet \simeq R^0 f_*\big(\mathscr{B}^{-1} \otimes \mathrm{gr}^F_\bullet \mathscr{O}_X \otimes_{\mathscr{A}_X} \mathrm{DR}_{X/Y}(\mathscr{A}_X^\bullet)\big)$$

[2] In *loc. cit.* it is stated for $\mathscr{D}_X$ and $\mathscr{D}_Y$ respectively, as opposed to their associated graded objects.

In particular, since $\mathscr{B}^{-1} = \mathscr{B}^{(-1)}$ we have

$$\widetilde{\mathscr{F}}_1 \simeq R^0 f_*\big(\mathscr{B}^{-1} \otimes \mathrm{gr}^F_\bullet \mathscr{O}_X \otimes_{\mathscr{A}_X} \mathrm{DR}_{X/Y}(\mathscr{A}^\bullet_X)\big)_1 \simeq R^1 f_*\big(\mathscr{B}^{-1} \otimes \Omega^{d-1}_{X/Y}\big).$$

Moreover, since H is smooth (so that $\mathscr{B}^{-1}_*$ is the same as $\mathscr{B}^{-1}(*D)$) and transversal to the fibers, according to (2.10.3) and Proposition 2.3, we also have

$$R^0 f_*\big(\omega_{X/Y} \otimes \mathrm{gr}^F_\bullet \mathscr{B}^{-1}_* \overset{\mathbf{L}}{\otimes}_{\mathscr{A}_X} f^*\mathscr{A}_Y\big)_1 \simeq R^1 f_*\big(\mathscr{B}^{-1} \otimes \Omega^{d-1}_{X/Y}(\log H)\big).$$

It follows that the morphism in (2.12.1) is induced by the first map of the following short exact sequence

$$0 \longrightarrow \Omega^{d-1}_{X/Y} \longrightarrow \Omega^{d-1}_{X/Y}(\log H) \longrightarrow \Omega^{d-2}_{H/Y} \longrightarrow 0.$$

Claim 2.13. We can choose $H \in |\mathscr{B}^m|$ such that $\mathscr{L}|_{F_H}$ is big on the general fiber F_H of $f|_H$.

Let us for the moment assume that Claim 2.13 holds. Then, according to the Bogomolov-Sommese vanishing theorem (see for instance [EV92, Cor. 6.9]), we know that

$$f|_{H*}(\mathscr{B}^{-1} \otimes \Omega^{d-2}_{H/Y}) = 0$$

generically, and hence everywhere since it is torsion-free. Therefore, we get the desired injectivity for the morphism in (2.12.1), and this finishes the proof of the proposition.

It now remains to show Claim 2.13. To this end, we first note that

$$\mathscr{B}|_F \simeq \omega_F$$

on each fiber F of f. Since F is minimal of general type, after perhaps shrinking Y even further, by the relative basepoint-free theorem (see for instance [KM98, Thm. 3.24]) we can assume that the relative Iitaka map

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & \mathbb{P}(f_*\omega^m_{X/Y}) \arrow[d] \\ Y \arrow[r, "="] & Y \end{tikzcd}$$

for some $m \gg 0$, is a birational morphism. Let E be its exceptional locus. For each fiber F, it is clear that every subvariety $Z \subseteq F$ such that $\omega_F|_Z$ is not big has to be contained in $E \cap F$. But we are free to choose an H that avoids the generic points of all components of E, and in this case for general F we have that $F_H = F \cap H$ is not contained in $E \cap F$. In this case $\mathscr{L}|_{F_H}$ is big. □

Corollary 2.14. *In the situation of Proposition 2.8, if we further assume that the fibers of f_U are minimal and of general type, then the natural morphism induced by the $\mathscr{A}_Y(-\log D)$-module structure*

$$\mathscr{T}_Y(-\log D) \longrightarrow \mathscr{F}_0{}^{-1} \otimes \mathscr{F}_1$$

coincides with the Kodaira-Spencer map of f over a Zariski open subset of V.

Proof. Thanks to Proposition 2.12, we know that the sheaves $\widetilde{\mathscr{F}}_k$ and $\mathscr{G}_k$ are generically isomorphic for $k = 0, 1$. On the other hand, $\mathscr{F}_\bullet$ and $\mathscr{G}_\bullet$ are generically the same by construction. Therefore, $\mathscr{F}_k$ and $\widetilde{\mathscr{F}}_k$ are generically isomorphic for $k = 0, 1$. But Lemma 2.11 says that the morphism $\mathscr{T}_Y \to \widetilde{\mathscr{F}}_0^{-1} \otimes \widetilde{\mathscr{F}}_1$ coincides with the Kodaira-Spencer map of f over V, which proves the claim. □

3. Hyperbolicity properties of base spaces of families

In this final part we establish the two main results of this paper, Theorem 1.2 (and implicitly Theorem 1.1) and Theorem 1.4. Besides Proposition 2.8 and Corollary 2.14, the main ingredient in the proofs of these theorems is Proposition 3.5 below.

3.1. Preliminaries on singular metrics on line bundles, and on Hodge metrics. We start with a construction and analysis of particular singular metrics on line bundles that will be of use later on. This follows very closely the material in [VZ03, p.136–139]. Nevertheless we include the details for later reference, and we also make a distinction between the boundary divisors D and S, as the perturbation along S will later allow us to bypass monodromy arguments in [VZ03] in order to extend the range of applicability.

We note to begin with that a priori by a singular metric on a line bundle $\mathscr{L}$ we mean, as in [HPS16, §13], a metric h given by a weight function $e^{-\varphi}$, where φ is taken to only be a measurable function with values in $[-\infty, \infty]$. In this way, the notion is compatible with that of a singular metric on a vector bundle, in the sense of Berndtsson, Păun and Takayama (cf. [HPS16, §17]), which will also make an appearance later on. In the line bundle case, usually it is also required that φ be locally integrable, in which case one can talk about its curvature form as a $(1,1)$-current; for this we use the standard notation

$$F(\mathscr{L}, h) = \frac{\sqrt{-1}}{\pi}\, \partial\bar{\partial}\varphi = -\frac{\sqrt{-1}}{2\pi}\, \partial\bar{\partial} \log \|e\|_h^2,$$

where e is a holomorphic section which trivializes $\mathscr{L}$ locally.

Let $(Y, D+S)$ be a pair consisting of a smooth projective variety Y and simple normal crossings divisors $D = D_1 + \cdots + D_k$ and $S = S_1 + \cdots + S_\ell$. For $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, \ell\}$ pick

$$f_{D_i} \in H^0\big(Y, \mathscr{O}_Y(D_i)\big), \quad f_{S_j} \in H^0\big(Y, \mathscr{O}_Y(S_j)\big)$$

such that $D_i = (f_{D_i} = 0)$ and $S_j = (f_{S_j} = 0)$. For each i, j, let g_{D_i}, g_{S_j} be smooth metrics on $\mathscr{O}_Y(D_i)$ and $\mathscr{O}_Y(S_i)$, respectively; after rescaling, we may assume $\|f_{D_i}\|_{g_{D_i}} < 1$ and $\|f_{S_j}\|_{g_{S_j}} < 1$.

Now, for each i and j, set

$$r_{D_i} = -\log \|f_{D_i}\|^2_{g_{D_i}}, \quad r_{S_j} = -\log \|f_{S_j}\|^2_{g_{S_j}},$$

and define

$$r_D := \prod_i r_{D_i} \quad \text{and} \quad r_S := \prod_j r_{S_j}.$$

The functions r_D^α and $\log r_D$ (resp. r_S^α and $\log r_S$) are locally L^1 on Y for all $\alpha \in \mathbb{Z}$. Indeed, if we write locally $f_{D_i} = z_i \cdot \tilde{s}_i$ (resp. $f_{S_j} = z_j \cdot \tilde{s}_j$), where $\tilde{s}_i$ (resp. $\tilde{s}_j$) trivializes $\mathscr{O}_Y(D_i)$ (resp. $\mathscr{O}_Y(S_j)$) and z_i (resp. z_j) is a coordinate, then locally

$$r_{D_i} = -\log |z_i|^2 - \log \|\tilde{s}_i\|^2_{g_{D_i}},$$

and similarly for r_{S_j}. When $\alpha < 0$, r_D^α and r_S^α are bounded and hence continuous on Y (and smooth outside of D, resp. S).

We now fix an ample line bundle $\mathscr{L}$ on Y with a smooth hermitian metric g, so that its curvature $F(\mathscr{L}, g)$ is positive. The metric g induces a hermitian metric g^{-1} on $\mathscr{L}^{-1}$. For $\alpha \in \mathbb{N}$, we define

$$g_\alpha = g \cdot (r_D \cdot r_S)^\alpha$$

to be a singular metric on $\mathscr{L}$. There is an induced singular metric $g_\alpha^{-1} = g^{-1} \cdot (r_D \cdot r_S)^{-\alpha}$ on $\mathscr{L}^{-1}$. With this notation, we have

$$
\begin{aligned}
F(\mathscr{L}, g_\alpha) = F(\mathscr{L}, g) &- \alpha \cdot \sum_i r_{D_i}^{-1} \cdot F(\mathscr{O}_Y(D_i), g_{D_i}) \\
&- \alpha \cdot \sum_j r_{S_j}^{-1} \cdot F(\mathscr{O}_Y(S_j), g_{S_j}) \\
&+ \alpha \frac{\sqrt{-1}}{2\pi} \sum_i r_{D_i}^{-2} \cdot \partial r_{D_i} \wedge \bar{\partial} r_{D_i} \\
&+ \alpha \frac{\sqrt{-1}}{2\pi} \sum_j r_{S_j}^{-2} \cdot \partial r_{S_j} \wedge \bar{\partial} r_{S_j}.
\end{aligned} \tag{3.0.1}
$$

Next we define a continuous $(1,1)$-form η_α on Y by the formula

$$
\eta_\alpha := F(\mathscr{L}, g) - \alpha \cdot \sum_i r_{D_i}^{-1} \cdot F(\mathscr{O}_Y(D_i), g_{D_i}) - \alpha \cdot \sum_j r_{S_j}^{-1} \cdot F(\mathscr{O}_Y(S_j), g_{S_j}),
$$

where we use the fact that r_D^{-1} and r_S^{-1} are continuous on Y. As Y is compact, after rescaling f_{D_i} and f_{S_j}, we can arrange for the contributions of the last two terms in η_α to be sufficiently small for η_α to be a continuous and positive definite $(1,1)$-form on Y. On the other hand, one can easily check that

$$
\frac{\sqrt{-1}}{2\pi} \partial r_{D_i} \wedge \bar{\partial} r_{D_i} \quad \text{and} \quad \frac{\sqrt{-1}}{2\pi} \partial r_{S_j} \wedge \bar{\partial} r_{S_j}
$$

are smooth and semi-positive $(1,1)$-forms on $Y \smallsetminus D_i$ (resp. $Y \smallsetminus S_j$) for all i and j.

Lemma 3.1. *In the above setting, for each $\alpha \in \mathbb{N}$, after rescaling f_{D_i} and f_{S_i} there is a continuous, positive definite, hermitian form w_α on $\mathscr{T}_Y(-\log D)$ such that*

$$
F(\mathscr{L}, g_\alpha)|_{Y \smallsetminus (D+S)} \geq r_D^{-2} \cdot w_\alpha|_{Y \smallsetminus (D+S)}.
$$

Proof. After suitable rescaling, we may assume that $r_{D_i} \geq 1$ for all i. Since η_α is positive on Y, and $\frac{\sqrt{-1}}{2\pi} \partial r_{D_i} \wedge \bar{\partial} r_{D_i}$ (resp. $\frac{\sqrt{-1}}{2\pi} \partial r_{S_j} \wedge \bar{\partial} r_{S_j}$) are semi-positive on $Y \smallsetminus D_i$ (resp. $Y \smallsetminus S_j$), using (3.0.1) we obtain

$$
\begin{aligned}
F(\mathscr{L}, g_\alpha)|_{Y \smallsetminus (D+S)} &\geq \big(\eta_\alpha + \alpha \frac{\sqrt{-1}}{2\pi} \sum r_{D_i}^{-2} \cdot \partial r_{D_i} \wedge \bar{\partial} r_{D_i}\big)|_{Y \smallsetminus (D+S)} \\
&\geq r_D^{-2} \cdot \underbrace{\big(\eta_\alpha + \alpha \frac{\sqrt{-1}}{2\pi} \sum \partial r_{D_i} \wedge \bar{\partial} r_{D_i}\big)}_{w_\alpha}|_{Y \smallsetminus (D+S)}.
\end{aligned}
$$

Now, the claim that w_α is positive-definite on $\mathscr{T}_Y(-\log D)$ follows from the fact that η_α is positive on $\mathscr{T}_Y$ and that $\sum \partial r_{D_i} \wedge \bar{\partial} r_{D_i}$ is positive definite along the vector fields that are tangent to D, cf. [VZ03, Claim. 7.2]. ☐

We now switch focus to Hodge metrics. Recall that we are always dealing with polarizable VHS with quasi-unipotent monodromy along simple normal crossings boundary. The following lemma translates the results of [CKS86] on the singularities of Hodge metrics in the unipotent case to this setting, and will be important for the proof of Proposition 3.5.

Lemma 3.2 (Estimates for Hodge metrics; the quasi-unipotent case)**.** *Suppose Δ^n is a polydisk with coordinates $(z_1, \dots, z_n)$. Let $\mathbb{V}$ be a polarized VHS on the open set $U = \Delta^n \smallsetminus (\{(z_1, \dots, z_k) | \prod_{i=1}^k z_i = 0\})$, $k \leq n$, with quasi-unipotent monodromies along each $\{z_i = 0\}$, and denote by $\mathscr{E}_\bullet$ the Higgs bundle associated to the Deligne extension of $\mathbb{V}$ with eigenvalues in $[0, 1)$. Then the Hodge metric induced by the polarization has at most logarithmic singularities along each z_i, for $i = 1, \dots, k$; that is, there exists an integer $d > 0$ such that for any section e of $\mathscr{E}_\bullet$ locally we have*

$$\|e\|_h^2 \leq C \cdot \prod_{i=1}^k \big(- \log |z_i| \big)^d \tag{3.2.1}$$

for some constant $C = C(e) \in \mathbb{R}_{>0}$.

Proof. Let L be the local system underlying $\mathbb{V}$, with monodromy Γ_i along z_i, for $i = 1, \dots, k$. Since the Γ_i commute pairwise, we have

$$L = \bigoplus_\alpha L_\alpha.$$

as the simultaneous (generalized) eigenspace decomposition with respect to the monodromy actions. Thus the monodromy action Γ_i on $L_{\alpha = (\alpha_1, \dots, \alpha_k)}$ has a unique eigenvalue $e^{-2\pi\sqrt{-1}\alpha_i}$. By the quasi-unipotent assumption, we can assume all α_i are rational numbers contained in $[0, 1)$. By the lower semicontinuity of rank functions of matrices, the above decomposition induces a decomposition of polarized variations of Hodge structure

$$\mathbb{V} = \bigoplus \mathbb{V}_\alpha,$$

and hence a decomposition of Higgs bundles

$$\mathscr{E}_\bullet = \bigoplus \mathscr{E}_\bullet^\alpha,$$

where $\mathscr{E}_\bullet^\alpha$ is the Higgs bundle associated to the Deligne extension of $\mathbb{V}_\alpha$ with eigenvalues in $[0, 1)$. (Note that the extension of $\mathbb{V}_\alpha$ has only one eigenvalue along each z_i.)

If $\alpha = (\alpha_1, \dots, \alpha_k) \neq 0$, then we can write $\alpha_i = \frac{p_i}{q_i}$ for some non-negative integers $p_i < q_i$. Now, let $g : \Delta^n \longrightarrow \Delta^n$ be the branched covering given by

$$g^* z_i = \begin{cases} w_i^{q_i} & \text{if} \quad i = 1, \dots, k \\ w_i & \qquad \text{otherwise,} \end{cases} \tag{3.2.2}$$

where $(w_1, \dots, w_n)$ define a coordinate system on the domain of g. It follows that the monodromies of $g^*\mathbb{V}_\alpha$ along w_i are unipotent. By comparing the eigenvalues of the residues upstairs, we have

$$g^* \mathscr{E}_\bullet^\alpha = \prod_{i=1}^k w_i^{p_i} \cdot \mathscr{E}_{g,\bullet}^\alpha,$$

where $\mathscr{E}_{g,\bullet}^\alpha$ is the Higgs bundle associated to the Deligne canonical extension of $g^*\mathbb{V}_\alpha$. Since the Hodge metric on $\mathscr{E}_{g,\bullet}^\alpha$ has logarithmic singularities (see [CKS86, §5.21]), for a section e of $\mathscr{E}_\bullet^\alpha$ we know that

$$\|e\|_h^2 \leq C \cdot \prod_{i=1}^k \big(|z_i|^{\alpha_i} \cdot (-\log |z_i|)^{d_i} \big) \leq C \cdot \prod_{\alpha_i = 0} (-\log |z_i|)^{d_i},$$

for some positive integers $d_i > 0$. The Hodge metric has logarithmic singularities along the z_i whenever $\alpha_i = 0$. In particular, we get the inequality (3.2.1) when $\alpha = (\alpha_1, \dots, \alpha_k) \neq 0$.

On the other hand, if $\alpha = (\alpha_1, \dots, \alpha_k) = 0$, then we know that the monodromies of $\mathbb{V}_\alpha$ are unipotent. Therefore, again thanks to [CKS86], the Hodge metric on $\mathscr{E}_\bullet^\alpha$ has logarithmic singularities along each z_i, as required. □

Remark 3.3. The above lemma implies that the Hodge metric is a singular metric on the vector bundle $\mathscr{E}_\bullet$.

Let us now return to the setting described at the beginning of this section, and suppose in addition that $\mathscr{E}_\bullet$ is the Higgs bundle associated to the Deligne extension of a VHS on $Y \smallsetminus (D+S)$, with eigenvalues in $[0,1)$. We define a singular metric h_g^α on the vector bundle $\mathscr{L}^{-1} \otimes \mathscr{E}_\bullet$ by

$$h_g^\alpha = g_\alpha^{-1} \otimes h,$$

where h is the Hodge metric on $\mathscr{E}_\bullet$.

Corollary 3.4. *For all $\alpha \gg 0$, the singular metric h_g^α is locally bounded.*

Proof. Assume that in local coordinates $D+S$ is given by $z_1 \cdots z_{k+\ell} = 0$. By construction, the singular metric g_α^{-1} degenerates to 0 at a rate proportional to

$$(r_D \cdot r_S)^{-\alpha} = \prod_{i=1}^{k} (-\log |z_i|^2 - \log \|\tilde{s}_i\|^2_{g_{D_i}})^{-\alpha} \cdot \prod_{i=k+1}^{k+\ell} (-\log |z_i|^2 - \log \|\tilde{s}_i\|^2_{g_{S_i}})^{-\alpha}.$$

On the other hand, the Hodge metric h on $\mathscr{E}_\bullet$ blows up to infinity along $z_i = 0$ bounded by a quantity proportional to $\prod(-\log|z_i|^2)^d$, for some fixed $d > 0$, thanks to Lemma 3.2. Hence, the metric h_g^α is bounded by a quantity proportional to

$$(r_D \cdot r_S)^{-\alpha+d} \cdot \prod_{i=1}^{k} \Big(\frac{-\log |z_i|^2}{-\log |z_i|^2 - \log \|\tilde{s}_i\|^2_{g_{D_i}}}\Big)^d \cdot \prod_{j=k+1}^{k+\ell} \Big(\frac{-\log |z_j|^2}{-\log |z_j|^2 - \log \|\tilde{s}_j\|^2_{g_{S_j}}}\Big)^d.$$

When $\alpha > d$ the above product is bounded. The compactness of Y gives the conclusion. □

3.2. An application of the singular Ahlfors-Schwarz Lemma. In this section we establish the key technical ingredient. This is done by applying the tools discussed in the previous section to the base spaces of families of varieties, via the Hodge-theoretic set-up provided by the constructions in §2, especially those in Proposition 2.8.

Proposition 3.5. *In the situation of Proposition 2.8, the morphism*

$$\tau_{(\gamma,1)} : \mathscr{T}_{\mathbb{C}} \longrightarrow \gamma^*(\mathscr{L}^{-1} \otimes \mathscr{E}_1)$$

induced by the entire curve $\gamma : \mathbb{C} \to Y \smallsetminus D$ is identically zero.

Proof. The proof will be by contradiction. First we note that, assuming that $\tau_{(\gamma,1)}$ is non-trivial, the following claim holds.

Claim 3.6. There exist:

(3.6.1) integers $m > 0$ and $p > 0$,
(3.6.2) an ample line bundle $\mathscr{H}$ on Y, and
(3.6.3) a Higgs bundle $(\mathscr{E}'_\bullet, \theta'_\bullet)$ on Y underlying the Deligne extension with eigenvalues in $[0,1)$ of a VHS defined outside of $D+S$

such that there is a non-trivial (hence injective) morphism $\tau_m \colon \mathscr{T}_{\mathbb{C}}^{\otimes m} \to \gamma^*(\mathscr{H}^{-1} \otimes \mathscr{E}'_p)$ factoring as

$$\tau_m \colon \mathscr{T}_{\mathbb{C}}^{\otimes m} \xrightarrow{d\gamma^{\otimes m}} \gamma^*\big(\bigotimes^m \mathscr{T}_Y(-\log D)\big) \longrightarrow \gamma^*\mathscr{H}^{-1} \otimes \mathscr{N}'_{(\gamma,p)} \hookrightarrow \gamma^*(\mathscr{H}^{-1} \otimes \mathscr{E}'_p), \tag{3.6.4}$$

where $\mathscr{N}'_{(\gamma,\bullet)} = \ker \theta'_{(\gamma,\bullet)}$, with $\theta'_{(\gamma,\bullet)}$ the Higgs field of $\gamma^*\mathscr{E}'_\bullet$ (see Definition 2.2).

Proof of Claim 3.6. By construction, for all sufficiently large k we have $\tau_{(\gamma,k)} = 0$. We set

$$p := \max\{k \mid \tau_{(\gamma,k)} \neq 0\}.$$

By assumption (the injectivity of $\tau_{(\gamma,1)}$), we have $p \geq 1$. On the other hand, we know that $\tau_{(\gamma,p+1)}$ factors as

$$\tau_{(\gamma,p+1)} : \mathscr{T}_{\mathbb{C}}^{\otimes(p+1)} \xrightarrow{\mathrm{Id}\otimes\tau_{(\gamma,p)}} \mathscr{T}_{\mathbb{C}} \otimes \gamma^*\mathscr{L}^{-1} \otimes \gamma^*\mathscr{E}_p \to \gamma^*\mathscr{L}^{-1} \otimes \gamma^*\mathscr{E}_{p+1}(P),$$

where the last map is induced by the $\mathscr{A}_{\mathbb{C}}(-\log P)$-module structure on $\gamma^*\mathscr{E}_\bullet$, with $P = \gamma^{-1}(S)$. (Note that in fact its image lands in $\gamma^*\mathscr{L}^{-1} \otimes \gamma^*\mathscr{E}_{p+1}$ as required, due to the fact that in the definition, see Proposition 2.8, we factor through the Higgs field of $\mathscr{F}_\bullet$, which does not have poles along S.) Since $\tau_{(\gamma,p+1)} = 0$, we obtain that $\tau_{(\gamma,p)}$ injects $\mathscr{T}_{\mathbb{C}}^{\otimes p}$ into $\gamma^*\mathscr{L}^{-1} \otimes \mathscr{N}_{(\gamma,p)}$, where $\mathscr{N}_{(\gamma,\bullet)} = \ker \theta_{(\gamma,\bullet)}$, with $\theta_{(\gamma,\bullet)}$ the induced Higgs field of $\gamma^*\mathscr{E}_\bullet$. Thus we have a nontrivial composition of morphisms

$$\tau_{(\gamma,p)} \colon \mathscr{T}_{\mathbb{C}}^{\otimes p} \xrightarrow{d\gamma^{\otimes p}} \gamma^*\big(\bigotimes^p \mathscr{T}_Y(-\log D)\big) \longrightarrow \gamma^*\mathscr{L}^{-1} \otimes \mathscr{N}_{(\gamma,p)} \hookrightarrow \gamma^*(\mathscr{L}^{-1} \otimes \mathscr{E}_p).$$

Now since $\mathscr{L}$ is big and nef, there exists $q > 0$ and an ample line bundle $\mathscr{H}$ such that $\mathscr{H} \subseteq \mathscr{L}^{\otimes q}$. Similarly to the proof of [VZ03, Lemma 6.5], we consider the Higgs bundle $(\mathscr{E}'_\bullet, \theta'_\bullet)$ on Y given by

$$\mathscr{E}'_\bullet = \mathscr{E}_\bullet^{\otimes q} \quad \text{and} \quad \theta'_\bullet \colon \mathscr{E}_\bullet^{\otimes q} \to \mathscr{E}_{\bullet+1}^{\otimes q} \otimes \Omega^1_Y(D+S), \tag{3.6.5}$$

$$\theta'_\bullet = \theta_\bullet \otimes \mathrm{id}_{\mathscr{E}} \otimes \cdots \otimes \mathrm{id}_{\mathscr{E}} + \mathrm{id}_{\mathscr{E}} \otimes \theta_\bullet \otimes \cdots \otimes \mathrm{id}_{\mathscr{E}} + \cdots + \mathrm{id}_{\mathscr{E}} \otimes \cdots \otimes \mathrm{id}_{\mathscr{E}} \otimes \theta_\bullet.$$

As noted in *loc. cit.*, this Higgs bundle corresponds to the locally free extension V' to Y of the bundle coming from the VHS $\mathbb{V}^{\otimes q}$ on $Y \smallsetminus (D+S)$, where $\mathbb{V}$ is the VHS underlying $\mathscr{E}_\bullet$. The induced connection on V' has residues with eigenvalues in $\mathbb{Q}_{\geq 0}$, and therefore V' is contained in V'', the Deligne extension with eigenvalues in $[0,1)$ (see [PW16, Prop. 4.4]). Therefore, without loss of generality, in the paragraph below we can assume that $(\mathscr{E}'_\bullet, \theta'_\bullet)$ is in fact the Higgs bundle associated to this extension. Note moreover that when pulling back by γ, the above construction implies that we have an inclusion of logarithmic Higgs bundles on $\mathbb{C}$

$$\big((\gamma^*\mathscr{E}_\bullet)', \theta'_\bullet\big) \subseteq \big(\gamma^*\mathscr{E}'_\bullet, \theta'_{(\gamma,\bullet)}\big), \tag{3.6.6}$$

where the Higgs bundle on the left is the analogue for $\gamma^*\mathscr{E}_\bullet$ of the construction in (3.6.5).

Finally, let $m := pq$. Raising $\tau_{(\gamma,p)}$, seen as the composition of morphisms above, to the q-th tensor power, gives rise to a new nontrivial composition of morphisms:

$$\tau_m \colon \mathscr{T}_{\mathbb{C}}^{\otimes m} \xrightarrow{d\gamma^{\otimes m}} \gamma^*\big(\bigotimes^m \mathscr{T}_Y(-\log D)\big) \longrightarrow \gamma^*\mathscr{H}^{-1} \otimes \mathscr{N}_{(\gamma,p)}^{\otimes q} \hookrightarrow \gamma^*\mathscr{H}^{-1} \otimes \gamma^*\mathscr{E}'_p,$$

where we used the inclusion of $\mathscr{L}^{\otimes -q}$ into $\mathscr{H}^{-1}$. In addition, the formula for the Higgs field on the left hand side of (3.6.6) (cf. (3.6.5)) implies immediately that $\mathscr{N}_{(\gamma,p)}^{\otimes q} \subseteq \mathscr{N}'_{(\gamma,p)}$, where we recall that $\mathscr{N}'_{(\gamma,p)} = \ker\theta'_{(\gamma,p)}$, so τ_m does factor as in (3.6.4). This concludes the proof of Claim 3.6. $\square$

We continue with the proof of Proposition 3.5. For simplicity, after renaming again $\mathscr{H}$ and $\mathscr{E}'_\bullet$ in Claim 3.6 by $\mathscr{L}$ and $\mathscr{E}_\bullet$, we assume from now on that $\mathscr{L}$ is ample, and that the morphism τ_m is given as

$$\tau_m\colon \mathscr{T}_{\mathbb{C}}^{\otimes m} \xrightarrow{d\gamma^{\otimes m}} \gamma^*\big(\bigotimes^m \mathscr{T}_Y(-\log D)\big) \longrightarrow \gamma^*\mathscr{L}^{-1}\otimes\mathscr{N}_{(\gamma,p)} \hookrightarrow \gamma^*\mathscr{L}^{-1}\otimes\gamma^*\mathscr{E}_p. \tag{3.6.7}$$

Our aim is now to extract a contradiction from the existence of a non-trivial such morphism, by showing that $\mathbb{C}$ inherits a singular metric $h_{\mathbb{C}}$ satisfying the distance decreasing property for any holomorphic map $g\colon(\mathbb{D},\rho)\to(\mathbb{C},h_{\mathbb{C}})$, that is $d_{h_{\mathbb{C}}}(g(x),g(y))\le A\cdot d_\rho(x,y)$, where ρ is the Poincaré metric on the unit disk, and $A\in\mathbb{R}_{>0}$. Since the Kobayashi pseudometric is larger than any such distance function, this forces it to be non-degenerate, contradicting the fact that on $\mathbb{C}$ it is identically zero. For background on this material, see for instance [Kob05, Chapt. IV, Sect.1].

Note first that, according to the Ahlfors-Schwarz lemma for (locally integrable) singular metrics over curves, cf. [Dem97, Lem. 3.2], any singular metric verifying, for some $B\in\mathbb{R}_{>0}$, the inequality

$$F(\mathscr{T}_{\mathbb{C}},h_{\mathbb{C}})\le -B\cdot w_{h_{\mathbb{C}}} \tag{3.6.8}$$

in the sense of currents, satisfies the above distance decreasing property. (Here $\omega_{h_{\mathbb{C}}} = \frac{\sqrt{-1}}{2\pi}\|\partial_t\|^2_{h_{\mathbb{C}}}dt\wedge d\bar t$ denotes the fundamental form of the metric $h_{\mathbb{C}}$, which we have assumed to be a $(1,1)$-current, where t is the coordinate of $\mathbb{C}$.) Therefore, to conclude, it suffices to construct a metric $h_{\mathbb{C}}$ on $\mathbb{C}$ verifying the inequality (3.6.8). We next proceed to construct such a metric.

We first fix a smooth metric g on $\mathscr{L}$, so that the curvature form $F(\mathscr{L},g)$ is positive. Following the notation in §3.1, this induces a singular metric g_α on $\mathscr{L}$, and a singular metric $h_g^\alpha = g_\alpha^{-1}\otimes h$ on $\mathscr{L}^{-1}\otimes\mathscr{E}_\bullet$, where we fix an $\alpha\gg 0$ as in Corollary 3.4. Consequently $\gamma^*h_g^\alpha$ is a singular metric on $\gamma^*(\mathscr{L}^{-1}\otimes\mathscr{E}_\bullet)$, and the m-th root of its pullback,

$$h_{\mathbb{C}} := (\tau_m^*\gamma^*h_g^\alpha)^{\frac{1}{m}},$$

defines a singular metric on (the trivial line bundle) $\mathscr{T}_{\mathbb{C}}$.

Similarly, we have the continuous positive definite hermitian form ω_α on $\mathscr{T}_Y(-\log D)$ as in Lemma 3.1, and so $\gamma^*\omega_\alpha$ induces a singular metric on $\gamma^*\mathscr{T}_Y(-\log D))$, and hence also a singular metric on $\mathscr{T}_{\mathbb{C}}$ through the differential map. For the next claim, recall that $P=\gamma^{-1}(S)\subset\mathbb{C}$.

Claim 3.7. We have $m\cdot F(\mathscr{T}_{\mathbb{C}},h_{\mathbb{C}})|_{\mathbb{C}\smallsetminus P}\le -\gamma^*(r_D^{-2})\cdot\gamma^*\omega_\alpha|_{\mathbb{C}\smallsetminus P}$, in the sense of currents.

Proof of Claim 3.7. Note that $F(\mathscr{T}_{\mathbb{C}},h_{\mathbb{C}})$ makes sense as a current on $\mathbb{C}\smallsetminus P$. The proof of the claim will also imply that it is indeed a current everywhere on $\mathbb{C}$, as we explain afterwards.

Denote by $\mathscr{B}$ the saturation of $\tau_m(\mathscr{T}_{\mathbb{C}}^{\otimes m})$ inside $\gamma^*(\mathscr{L}^{-1}\otimes\mathscr{E}_\bullet)$, so that

$$\mathscr{B}\simeq\mathscr{T}_{\mathbb{C}}^{\otimes m}(G),$$

where $G\ge 0$ is a divisor on $\mathbb{C}$. Since τ_m factors through $\gamma^*\mathscr{L}^{-1}\otimes\mathscr{N}_{(\gamma,p)}$, we know that $\theta_{(\gamma,\bullet)}(\gamma^*\mathscr{L}\otimes\mathscr{B})=0$. Recall that as a consequence of Griffiths' curvature estimates for Hodge metrics, it is well known (see e.g. [VZ01, Lem. 1.1] and the references therein) that the Hodge metric restricted to any subbundle inside the kernel of the Higgs field associated to a VHS has semi-negative curvature. We thus conclude that

$$F(\mathscr{B}, \gamma^* h_g^\alpha|_{\mathscr{B}})|_{\mathbb{C}\smallsetminus P} + \gamma^* F(\mathscr{L}, g_\alpha)|_{\mathbb{C}\smallsetminus P} \leq 0,$$

and since

$$(\mathscr{T}_{\mathbb{C}}^{\otimes m} \otimes \gamma^*\mathscr{L})(G) \simeq \mathscr{B} \otimes \gamma^*\mathscr{L},$$

this implies

$$\begin{aligned} m \cdot F(\mathscr{T}_{\mathbb{C}}, h_{\mathbb{C}})|_{\mathbb{C}\smallsetminus P} + \gamma^* F(\mathscr{L}, g_\alpha)|_{\mathbb{C}\smallsetminus P} &\leq \\ \leq F(\mathscr{B}, \gamma^* h_g^\alpha|_{\mathscr{B}})|_{\mathbb{C}\smallsetminus P} + \gamma^* F(\mathscr{L}, g_\alpha)|_{\mathbb{C}\smallsetminus P} &\leq 0. \end{aligned} \tag{3.7.1}$$

Now the statement follows from Lemma 3.1. □

As mentioned above, the proof of the claim also implies that $F(\mathscr{T}_{\mathbb{C}}, h_{\mathbb{C}})$ is a current on $\mathbb{C}$. Indeed, from construction, we know $F(\mathscr{L}, g_\alpha)$ is a $(1,1)$-current and $F(\mathscr{L}, g_\alpha)|_{Y\smallsetminus(D+S)}$ is positive. Hence, by (3.7.1), we know $F(\mathscr{T}_{\mathbb{C}}, h_{\mathbb{C}})|_{\mathbb{C}\smallsetminus P}$ is negative; or equivalently, $\log\|\partial_t\|^2_{h_{\mathbb{C}}}$ is subharmonic on $\mathbb{C}\smallsetminus P$. Since $h_{\mathbb{C}}$ is locally bounded (see Corollary 3.4), $\log\|\partial_t\|^2_{h_{\mathbb{C}}}$ extends to a subharmonic function on $\mathbb{C}$ (see [Dem09, Thm. 5.23]), and so $F(\mathscr{T}_{\mathbb{C}}, h_{\mathbb{C}})$ is a negative current.

Next we fix a polydisk neighborhood $\Delta^n \subseteq Y$. The continuous metric $\|\cdot\|_{\omega_\alpha}$ on $\mathscr{T}_Y(-\log D)$ given by ω_α induces a metric on $\bigotimes^m \mathscr{T}_Y(-\log D)$. We also fix an orthonormal basis $\{\psi_1, \ldots, \psi_N\}$ of continuous sections of $\bigotimes^m \mathscr{T}_Y(-\log D)|_{\Delta^n}$ with respect to the induced metric. (By abuse of notation, we use $\bigotimes^m \mathscr{T}_Y(-\log D)|_{\Delta^n}$ even when considering the associated sheaf of continuous sections.)

We fix a holomorphic basis $\{e_1, e_2, \ldots, e_M\}$ of $\mathscr{L}^{-1}\otimes\mathscr{E}_p|_{\Delta^n}$ as well. We write

$$\tilde{\tau}_m(\psi_i) = \sum_j b_i^j \cdot e_j \tag{3.7.2}$$

for some continuous functions b_i^j on Δ^n, where

$$\tilde{\tau}_m \colon \bigotimes^m \mathscr{T}_Y(-\log D) \to \mathscr{L}^{-1}\otimes\mathscr{E}_p,$$

and we also write

$$d\gamma^{\otimes m}(\partial_t^m|_{\gamma^{-1}(\Delta^n)}) = \sum_i c_i \cdot \gamma^*\psi_i, \tag{3.7.3}$$

for some continuous (complex valued) functions c_i.

Claim 3.8. We have $\gamma^*(r_D^{-2})\cdot\gamma^*\omega_\alpha \geq B\cdot\omega_{h_{\mathbb{C}}}$ in the sense of currents on $\mathbb{C}$, for some $B>0$.

Proof of Claim 3.8. Since $\mathscr{T}_{\mathbb{C}}$ is trivialized by ∂_t globally, it it enough to show

$$\gamma^*(r_D^{-2m})\cdot\|d\gamma^{\otimes m}(\partial_t^m)\|_{\gamma^*\omega_\alpha} \geq B\cdot\|\tau_m(\partial_t^m)\|_{\gamma^* h_g^\alpha}.$$

By the compactness of Y, it is enough to prove the inequality locally on neighborhoods of the form $\gamma^{-1}(\Delta^n)$, with $\Delta^n \subset Y$ as above.

First, since $\{\psi_1, \ldots, \psi_N\}$ is an orthonormal basis, by (3.7.3) we see that

$$\|d\gamma^{\otimes m}(\partial_t^m|_{\gamma^{-1}(\Delta^n)})\|_{\gamma^*\omega_\alpha} = \big(\sum_i |c_i|^2\big)^{\frac{1}{2}}. \tag{3.8.1}$$

By (3.7.2) and (3.7.3), we also have

$$\|\tau_m(\partial_t^m|_{\gamma^{-1}(\Delta^n)})\|_{\gamma^* h_g^\alpha} = \| \sum_i c_i \sum_j \gamma^*\big(b_i^j \cdot e_j\big)\|_{\gamma^* h_g^\alpha}.$$

On the other hand, by the Cauchy-Schwarz inequality, we have

$$\| \sum_i c_i \sum_j \gamma^*\big(b_i^j \cdot e_j)\big)\|_{\gamma^* h_g^\alpha} \le \big(\sum_i |c_i|^2\big)^{\frac{1}{2}} \cdot \big(\sum_i \gamma^* \| \sum_j \big(b_i^j \cdot e_j\big)\|_{h_g^\alpha}^2\big)^{\frac{1}{2}}.$$

By Corollary 3.4, we know that h_g^α is bounded over Δ^n by a quantity proportional to

$$(r_D \cdot r_S)^{-\alpha+d} \cdot \prod_{i=1}^{k} \big(\frac{-\log|z_i|^2}{-\log|z_i|^2 - \log\|\tilde{s}_i\|_{g_{D_i}}^2}\big)^d \cdot \prod_{j=k+1}^{k+\ell} \big(\frac{-\log|z_j|^2}{-\log|z_j|^2 - \log\|\tilde{s}_j\|_{g_{S_j}}^2}\big)^d,$$

for some fixed $d > 0$. Therefore, we have

$$\|\tau_m(\partial_t^m|_{\gamma^{-1}\Delta^n})\|_{\gamma^* h_g^\alpha} \le \frac{1}{B} \cdot \gamma^*(r_D^{\frac{d-\alpha}{2}}) \cdot \big(\sum |c_i|^2\big)^{\frac{1}{2}}$$

for some $B > 0$ and α sufficiently large (recall that r_S^γ is bounded for $\gamma < 0$). This implies the conclusion, given (3.8.1) and the fact that earlier we have chosen our scaling so that $r_D \ge 1$. □

Finally, the inequality (3.6.8) follows from Claim 3.7, Claim 3.8, and the fact that if the inequality

$$F(\mathscr{T}_{\mathbb{C}}, h_{\mathbb{C}})|_{(\mathbb{C}\smallsetminus P)} \le -B \cdot \big(\omega_{h_{\mathbb{C}}}|_{(\mathbb{C}\smallsetminus P)}\big)$$

holds as currents for some $B > 0$, then we also have

$$F(\mathscr{T}_{\mathbb{C}}, h_{\mathbb{C}}) \le -B \cdot \omega_{h_{\mathbb{C}}},$$

as currents on $\mathbb{C}$. But this is an easy consequence of the negativity of $F(\mathscr{T}_{\mathbb{C}}, h_{\mathbb{C}})$, together with the continuity of $\omega_{h_{\mathbb{C}}}$.

□

3.3. Some further background. In this section we collect a few useful facts regarding entire maps on the one hand, and families with maximal variation on the other.

3.3.1. *Algebraic degeneracy to Brody hyperbolicity.* In §2.3 we observed that the Hodge theoretic constructions of §2.4 are valid as long as we replace the initial family $f_U : U \to V$ by a birational model, compactified by the family $f : X \to Y$ in Proposition 2.8. We recall below, following [VZ03, §1], that the study of the hyperbolicity properties can be reduced to investigating algebraic nondegeneracy on such models.

Lemma 3.9 ([VZ03, Lem. 1.2])**.** *Let $\gamma: \mathbb{C} \to V$ be an entire curve with a Zariski-dense image, and $\mu: \widetilde{V} \to V$ a birational morphism. Then the map $(\mu^{-1} \circ \gamma)$ extends to a holomorphic map $\widetilde{\gamma} : \mathbb{C} \to \widetilde{V}$.*

Proposition 3.10 (Reduction of Brody hyperbolicity to algebraic degeneracy)**.** *Let P_h be a coarse moduli space of polarized manifolds, as in the Introduction, and V and Y as in Proposition 2.8.*

(3.10.1) The image of $\gamma: \mathbb{C} \to V$ is algebraically degenerate if and only if the induced morphism $\widetilde{\gamma}: \mathbb{C} \to \widetilde{V}$ defined in Lemma 3.9 is so.

(3.10.2) To prove the Brody hyperbolicity of P_h, in the sense of Theorem 1.1, it suffices to show that for every smooth quasi-projective variety V with a generically finite morphism $V \to P_h$, every entire curve $\mathbb{C} \to V$ is algebraically degenerate.

Proof. Item (3.10.1) is the direct consequence of Lemma 3.9. For Item (3.10.2), note that given a quasi-finite morphism $W \to P_h$ from a variety W, and $\gamma\colon \mathbb{C} \to W$, the restriction W' of $\operatorname{Im}(\gamma)$ to the Zariski closure W' of $\operatorname{Im}(\gamma)$ is also quasi-finite. Furthermore, we can desingularize W' by $\mu\colon \widetilde{W}' \to W'$, and by (3.10.1), the degeneracy of the induced map $\mathbb{C} \to \widetilde{W}'$ is equivalent to the fact that γ is constant. □

Therefore, to prove Theorem 1.1 on the Brody hyperbolicity of P_h, it suffices to establish Theorem 1.2.

3.3.2. *More on families with maximal variation.* We recall a few facts about families with maximal variation that were established by Kollár [Kol87]. Here $f\colon U \to V$ is a smooth projective morphism of smooth varieties, with fibers of non-negative Kodaira dimension.

Lemma 3.11 ([Kol87, Cor. 2.9])**.** *If $\operatorname{Var}(f) = \dim V$, and if v is a very general point of V, then for any analytic arc $\gamma\colon \Delta \to V$ passing through v, not all fibers of f over $\gamma(\Delta)$ are birational.*

We will denote by $W \subset V$ the locus of points v satisfying the property in Lemma 3.11. In general we see that W is the complement of a countable union of closed subsets of V. The following result says that when the fibers of f are of general type, it is guaranteed to contain a Zariski open set V_0.

Lemma 3.12 ([Kol87, Thm. 2.5])**.** *If the fibers of f are of general type, then there exists an open subset $V_0 \subseteq V$ and a morphism $g\colon V_0 \to Z$ onto an algebraic variety, such that for $v_1, v_2 \in V_0$ the fibers U_{v_1} and U_{v_2} are birational if and only if $g(v_1) = g(v_2)$.*

Indeed, when $\operatorname{Var}(f) = \dim V$, in the lemma above we have $\dim Z = \dim V$, and the map g is generically finite. Thus there exists a, perhaps smaller, dense open subset $\widetilde{V_0} \subseteq V$, such that $\widetilde{V_0} \subseteq W$ (namely the complement of the positive dimensional fibers of g).

3.4. Algebraic degeneracy for base spaces of families of minimal varieties of general type. We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We first show that every holomorphic curve $\gamma\colon \mathbb{C} \to V$ is algebraically degenerate. According to Proposition 3.10, Item (3.10.1), we can assume that $V = Y \smallsetminus D$ as in Proposition 2.8.

Recall that the mapping appearing in Proposition 3.5 can be written as the composition

$$\tau_{(\gamma,1)}\colon \mathscr{T}_{\mathbb{C}} \to \gamma^* \mathscr{T}_Y(-\log D) \to \gamma^*\big({\mathscr{F}_0}^{-1} \otimes \mathscr{F}_1\big) \hookrightarrow \gamma^*(\mathscr{L}^{-1} \otimes \mathscr{E}_1).$$

Now by Corollary 2.14 we have a generic identification of

$$\tau_1\colon \mathscr{T}_Y(-\log D) \longrightarrow {\mathscr{F}_0}^{-1} \otimes \mathscr{F}_1$$

with the Kodaira-Spencer map of the family $f\colon X \to Y$, and so by base change the composition of the first two maps in the definition of $\tau_{(\gamma,1)}$ can be identified with the Kodaira-Spencer map of the induced family over $\mathbb{C}$. If $\gamma(\mathbb{C})$ were dense, we would obtain a family with maximal variation over $\mathbb{C}$, implying that this Kodaira-Spencer map is injective; indeed, over a curve it can only be injective or 0, the latter case of course implying that the family is locally trivial. But this in turn implies that $\tau_{(\gamma,1)}$ is injective, which contradicts Proposition 3.5.

We now show the stronger statement that $\operatorname{Exc}(V)$ is a proper subset, knowing that the algebraic degeneracy statement we just proved holds for any base of a family as in the theorem. Let V_0 be the Zariski open subset in Lemma 3.12 and $\widetilde{V}_0$ be the subset of V_0 over which the morphism g is finite. We claim that

$$\operatorname{Exc}(V) \subseteq V \smallsetminus \widetilde{V}_0.$$

To see this, assume that there exists an entire curve $\gamma : \mathbb{C} \to V$ with $\gamma(\mathbb{C}) \cap \widetilde{V}_0 \neq \emptyset$, and denote by W the Zariski closure of $\gamma(\mathbb{C})$ in V. If γ is not constant, then by definition the restriction of the family f over W has maximal variation. Furthermore, using again Proposition 3.10, we can assume that W is smooth. We then obtain a contradiction with the algebraic degeneracy of all maps $\mathbb{C} \to W$.

3.5. Algebraic degeneracy for surfaces mapping to moduli stacks of polarized varieties. We now prove the stronger statements in the case when the base of the family is a smooth surface. We start with two basic lemmas about pulling back sheaf morphisms via γ, the first of which is immediate.

Lemma 3.13. *Let $\gamma\colon \mathbb{C} \to V$ be a holomorphic map with Zariski dense image, where V is an algebraic variety. If $\varphi\colon \mathscr{E} \to \mathscr{F}$ is an injective morphism of locally free $\mathscr{O}_V$-modules, then $\gamma^*\varphi\colon \gamma^*\mathscr{E} \to \gamma^*\mathscr{F}$ is also injective.*

Lemma 3.14. *Let $\gamma\colon \mathbb{C} \to V$ be a holomorphic map with Zariski dense image, where V is a smooth algebraic surface. Let Z be a 0-dimensional local complete intersection subscheme of V. Then we have an inclusion $\gamma^* I_Z \hookrightarrow \mathscr{O}_{\mathbb{C}}$.*

Proof. We can cover $\mathbb{C}$ with the preimages of open subsets in V on which Z is given as $f_1 = f_2 = 0$, where f_1 and f_2 are two non-proportional functions. Denoting by D_1 and D_2 the divisors of these two functions, so that Z is the scheme theoretic intersection $D_1 \cap D_2$, we can thus assume that we have a Koszul complex

$$0 \to \mathscr{O}_V(-D_1 - D_2) \to \mathscr{O}_V(-D_1) \oplus \mathscr{O}_V(-D_2) \to I_Z \to 0.$$

Pulling back this sequence by γ, we still have a short exact sequence, as the first map degenerates only at the points of Z. Therefore we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \gamma^*\mathscr{O}_V(-D_1 - D_2) \arrow[r] \arrow[d] & \gamma^*\mathscr{O}_V(-D_1) \oplus \gamma^*\mathscr{O}_V(-D_2) \arrow[r] \arrow[d] & \gamma^* I_Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_{\mathbb{C}} \arrow[r] \arrow[d] & \mathscr{O}_{\mathbb{C}} \oplus \mathscr{O}_{\mathbb{C}} \arrow[r] \arrow[d] & \mathscr{O}_{\mathbb{C}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_{P_1 + P_2} \arrow[r] \arrow[d] & \mathscr{O}_{P_1} \oplus \mathscr{O}_{P_2} \arrow[r] \arrow[d] & \mathscr{O}_{P_1 \cap P_2} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $P_1 = \gamma^* D_1$ and $P_2 = \gamma^* D_2$ are divisors on $\mathbb{C}$, and we used the identification $\gamma^*\mathscr{O}_V = \mathscr{O}_{\mathbb{C}}$. Note that the two left vertical sequences are exact because of the Zariski density of the image of γ, which consequently cannot be contained in any divisor on V (a special example of Lemma 3.13 above). By the Snake Lemma we obtain that the map in the upper right corner is also injective. $\square$

3.5.1. *Proof of Theorem 1.4.* We first prove Item (1.4.1). Aiming for a contradiction, we assume that the image $\gamma(\mathbb{C})$ is Zariski dense in V. We follow the set-up and notation of Proposition 2.8. By Proposition 3.10, we may assume that $V = Y \smallsetminus D$.

We may also assume that the morphism

$$\mathscr{T}_Y(-\log D) \xrightarrow{\psi} \mathscr{F}_0{}^{-1} \otimes \mathscr{F}_1$$

is not injective, as otherwise by Lemma 3.13 it follows that the composition of morphisms

$$\mathscr{T}_{\mathbb{C}} \to \gamma^* \mathscr{T}_Y(-\log D) \to \gamma^*\big(\mathscr{F}_0{}^{-1} \otimes \mathscr{F}_1\big) = \gamma^*\big(\mathscr{L}^{-1} \otimes \mathscr{F}_1\big) \hookrightarrow \gamma^*\big(\mathscr{L}^{-1} \otimes \mathscr{E}_1\big)$$

is also injective, contradicting Proposition 3.5. By Lemma 2.10, we also know that ψ is not the zero map. We define $\mathscr{G} := \operatorname{Im}(\psi)$, which therefore has generic rank one, and leads to a short exact sequence

$$0 \longrightarrow \mathscr{K} \to \mathscr{T}_Y(-\log D) \longrightarrow \mathscr{G} \longrightarrow 0.$$

Since $\mathscr{G}$ injects in a torsion-free sheaf, it is torsion-free itself. Therefore $\mathscr{K}$ is reflexive, hence an invertible sheaf since we are on a smooth surface. Moreover, since it is saturated in $\mathscr{T}_Y(-\log D)$, we must have

$$\mathscr{G} \simeq \mathscr{M} \otimes \mathcal{I}_Z,$$

where $\mathscr{M}$ is a line bundle and Z is a (possibly empty) 0-dimensional subscheme of Y. It is standard that Z is a local complete intersection.

Note that since $\mathscr{L} \subseteq \mathscr{F}_0$, we have an inclusion $\mathscr{G} \subseteq \mathscr{L}^{-1} \otimes \mathscr{E}_1$. We claim that this induces an inclusion

$$\gamma^*\mathscr{G} \subseteq \gamma^*(\mathscr{L}^{-1} \otimes \mathscr{E}_1),$$

which in particular shows that $\gamma^*\mathscr{G}$ it torsion free. To see this, note that the initial inclusion factors as a composition

$$\mathscr{M} \otimes I_Z \hookrightarrow \mathscr{M} \hookrightarrow \mathscr{L}^{-1} \otimes \mathscr{E}_1,$$

and the second map pulls back to an injective map by Lemma 3.13. It suffices then to have that the inclusion $I_Z \hookrightarrow \mathscr{O}_V$ also pulls back to an injective map, and this is precisely the content of Lemma 3.14.

Again by Lemma 3.13, the pullback sequence

$$0 \longrightarrow \gamma^*\mathscr{K} \longrightarrow \gamma^*\mathscr{T}_Y(-\log D) \longrightarrow \gamma^*\mathscr{G} \longrightarrow 0$$

is also exact. Since $\gamma^*\mathscr{G}$ it torsion free, and the image of $\mathscr{T}_{\mathbb{C}}$ inside $\gamma^*(\mathscr{L}^{-1} \otimes \mathscr{E}_1)$ is zero by Proposition 3.5, it follows that the map $\mathscr{T}_{\mathbb{C}} \to \gamma^*\mathscr{T}_Y(-\log D)$ factors through $\gamma^*\mathscr{K}$.

Consider now the saturation $\mathscr{K}'$ of $\mathscr{K}$ in $\mathscr{T}_Y$, which defines a foliation on Y. Since the differential $\mathscr{T}_{\mathbb{C}} \to \gamma^*\mathscr{T}_Y$ clearly factors through $\gamma^*\mathscr{K}'$ as well, the image $\gamma(\mathbb{C})$ sits inside (or equivalently is tangent to) a leaf of this foliation. On the other hand, according to [PS17, Thm. A], the pair (Y, D) is of log general type. But this contradicts McQuillan's result [McQ98] on the degeneracy of entire curves tangent to leaves of non-trivial foliations on surfaces of general type (cf. also [Rou15, Theorem 3.13]), and more precisely its natural extension to the log setting as in El Goul [EG03, Theorem 2.4.2]. This finishes the proof of Item (1.4.1).

To prove Item (1.4.2), just as in the proof of Theorem 1.2 let V_0 be the Zariski open subset in Lemma 3.12 and $\widetilde{V}_0$ be the subset of V_0 over which the morphism g is finite. We again claim that

$$\operatorname{Exc}(V) \subseteq V \smallsetminus \widetilde{V}_0.$$

Assume on the contrary that there exists an entire curve $\gamma : \mathbb{C} \to V$ with $\gamma(\mathbb{C}) \cap \widetilde{V}_0 \neq \emptyset$. Then, by definition, the pull-back of the family f via γ has maximal variation. Since $\gamma(\mathbb{C})$ cannot be Zariski dense in V by Item (1.4.1), it is either a point, or it is dense in a quasi-projective curve C, which by Proposition 3.10 can be assumed to be smooth. In the latter case, we thus obtain a smooth family of varieties of general type over C, with maximal variation. But then by [VZ01, Theorem 0.1] we know that C cannot be $\mathbb{C}^*$, $\mathbb{C}$, $\mathbb{P}^1$ or an elliptic curve, which gives a contradiction.

3.5.2. *Proof of Corollary 1.5.* According to Proposition 3.10 (Item (3.10.2)), it is enough to show that there cannot be algebraically nondegenerate holomorphic maps $\gamma\colon \mathbb{C} \to V$, where V is a smooth quasi-projective variety of dimension 1 or 2 with a generically finite map $V \to P_h$. If $\dim V = 2$, this follows from Theorem 1.4, Item (1.4.1). If $\dim V = 1$, it follows again from [VZ01, Theorem 0.1], as explained at the end of the proof of Theorem 1.4.

Remark 3.15. We note that Proposition 3.5 gives an alternative proof of [VZ01, Theorem 0.1], since it shows that a quasi-projective variety V of dimension one is hyperbolic if it supports a birationally non-isotrivial smooth family of projective varieties whose geometric generic fiber admits a good minimal model. This is because, in this case, the map $\mathscr{T}_V \to (\mathscr{L}^{-1} \otimes \mathscr{E}_1)|_V$ induced by $\mathscr{F}_0 \to \mathscr{F}_1 \otimes \Omega^1_Y(\log D)$ as in Proposition 2.8 is an injection, as the latter map is injective by Lemma 2.10.

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