FILTERED $\mathcal{D}$-MODULES AND HODGE $\mathcal{D}$-MODULES

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The aim of this chapter is to introduce some basic notions in the study of filtered $\mathcal{D}$-modules, focus on the interaction between the $F$-filtration and the $V$-filtration, and provide a first definition of Hodge modules. To get to this notion as quickly as possible, at first I will not develop things in the most systematic possible way. Later on I will rearrange the notes in a better order.

1. The de Rham complex. Let $X$ be a smooth complex variety of dimension $n$. Recall that a left $\mathcal{D}_X$-module $\mathcal{M}$ can be thought of as an $\mathcal{O}_X$-module on which we have a $\mathbb{C}$-linear operation

$\nabla : \mathcal{M} \to \Omega^1_X \otimes \mathcal{M}$

satisfying the properties of an integrable connection. We say that $\mathcal{M}$ is filtered if there exists an increasing filtration $F = F_* \mathcal{M}$ by coherent $\mathcal{O}_X$-modules, bounded from below and satisfying

$F_k \mathcal{D}_X \cdot F_\ell \mathcal{M} \subseteq F_{k+\ell} \mathcal{M}$ for all $k, \ell \in \mathbb{Z}$,

where $F_k \mathcal{D}_X$ is the sheaf of differential operators on $X$ of order $\leq k$. We use the notation $(\mathcal{M}, F)$ for this data. The filtration is called good if the inclusions above are equalities for $\ell \gg 0$, which is in turn equivalent to the fact that the total associated graded object

$\text{gr}_F^\bullet \mathcal{M} = \bigoplus_k \text{gr}_k^F \mathcal{M} = \bigoplus_k F_k \mathcal{M} / F_{k-1} \mathcal{M}$
is finitely generated over $\text{gr}^* \mathcal{D}_X \simeq \text{Sym} T_X$.

**Definition 1.1.** The *de Rham complex* of $\mathcal{M}$ is the complex

$$\text{DR}(\mathcal{M}) = \left[ \mathcal{M} \to \Omega^1_X \otimes \mathcal{M} \to \cdots \to \Omega^n_X \otimes \mathcal{M} \right],$$

with $\mathbb{C}$-linear differentials induced by iterating $\nabla$. We consider it to be placed in degrees $-n, \ldots, 0$. (Strictly speaking, as such it is normally considered to be the de Rham complex associated to the corresponding right $\mathcal{D}$-module.)

Assume now that $\mathcal{M}$ has a good filtration $F^*_\mathcal{M}$. The compatibility of this filtration with the standard filtration on $\mathcal{D}_X$, meaning in particular that $F^1 \mathcal{D}_X \cdot F^p \mathcal{M} \subseteq F^{p+1} \mathcal{M}$ for all $p$ means that this induces a filtration on the de Rham complex of $\mathcal{M}$ by the formula

$$F^k \text{DR}(\mathcal{M}) = \left[ F^k \mathcal{M} \to \Omega^1_X \otimes F^k+1 \mathcal{M} \to \cdots \to \Omega^n_X \otimes F^{k+n} \mathcal{M} \right].$$

For any integer $k$, the associated graded complex for this filtration is

$$\text{gr}^F_k \text{DR}(\mathcal{M}) = \left[ \text{gr}^F_k \mathcal{M} \to \Omega^1_X \otimes \text{gr}^F_k+1 \mathcal{M} \to \cdots \to \Omega^n_X \otimes \text{gr}^F_{k+n} \mathcal{M} \right].$$

The well-known behavior of variations of Hodge structure remains valid here:

**Exercise 1.1.** Show that the differentials of $\text{gr}^F_k \text{DR}(\mathcal{M})$ are $\mathcal{O}_X$-linear. Hence this is now a complex of coherent $\mathcal{O}_X$-modules in degrees $-n, \ldots, 0$, providing an object in $\text{D}^b(\text{Coh}(X))$.

**Example 1.2** (The trivial filtered $\mathcal{D}$-module). Consider $\mathcal{M} = \mathcal{O}_X$ with the natural left $\mathcal{D}_X$-module structure, and $F^k \mathcal{O}_X = \mathcal{O}_X$ for $k \geq 0$, while $F^k \mathcal{O}_X = 0$ for $k < 0$. The de Rham complex of $\mathcal{M}$ is

$$\text{DR}(\mathcal{O}_X) = \left[ \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^n_X \right][n],$$

quasi-isomorphic to $\mathbb{C}[n]$ by the holomorphic Poincaré Lemma. With the induced filtration $F^*_\text{DR}(\mathcal{O}_X)$, note that

$$\text{gr}^F_{-k} \text{DR}(\mathcal{O}_X) = \Omega^k_X[n-k] \quad \text{for all } k.$$

**Example 1.3** (Variations of Hodge structure). The previous example corresponds to the trivial variation of Hodge structure (VHS) on $X$, and can be extended to arbitrary ones. Recall that a $\mathbb{Q}$-VHS of weight $\ell$ on $X$ is the data

$$V = (\mathcal{V}, F^*, V_\mathbb{Q})$$

where:

- $V_\mathbb{Q}$ is a $\mathbb{Q}$-local system on $X$.
- $\mathcal{V} = V_\mathbb{Q} \otimes \mathcal{O}_X$ is a vector bundle with flat connection $\nabla$, endowed with a decreasing filtration with subbundles $F^p = F^p \mathcal{V}$ satisfying the following two properties:
  - for all $x \in X$, the data $V_x = (\mathcal{V}_x, F^*_x, V_{\mathbb{Q},x})$ is a Hodge structure of weight $\ell$. 

• Griffiths transversality: for each \( p \), \( \nabla \) induces a morphism
\[
\nabla : F^p \rightarrow F^{p-1} \otimes \Omega^1_X.
\]

Recall now that we can think of \( \mathcal{M} = \mathcal{V} \) as a left \( \mathcal{D}_X \)-module (what we called an integrable connection). We reindex the filtration as \( F_p \mathcal{M} = F^{-p} \mathcal{V} \); this is a good filtration on \( \mathcal{M} \). It is also well known that
\[
\text{DR}(\mathcal{M}) \simeq \mathcal{V}_C[n],
\]
where \( \mathcal{V}_C = \mathcal{V} \otimes \mathbb{Q} \). By construction the graded pieces \( \text{gr}_F^{k} \mathcal{M} \) are locally free, and are sometimes known as the Hodge bundles of the VHS.

2. \( F \)-filtration and \( V \)-filtration. Let \( (\mathcal{M}, F) \) be a coherent \( \mathcal{D}_X \)-module endowed with a good filtration. We will see here that the nearby and vanishing cycles, and so implicitly the \( V \)-filtration, are useful for imposing restrictions on the filtration \( F \).

Assume first that \( D \) is a smooth divisor, given by \( t = 0 \), and consider the \( V \)-filtration on \( \mathcal{M} \) induced by \( t \). Assume that \( \mathcal{M} \) has strict support \( Z \) which is not contained in \( D \).

For every \( p \in \mathbb{Z} \) and \( \alpha \in \mathbb{Q} \) we define
\[
F_p V^\alpha \mathcal{M} := F_p \mathcal{M} \cap V^\alpha \mathcal{M}
\]
and
\[
F_p \text{gr}_V^{\alpha} \mathcal{M} := \frac{F_p \mathcal{M} \cap V^\alpha \mathcal{M}}{F_p \mathcal{M} \cap V^{\alpha+1} \mathcal{M}}.
\]
Recall that in the proof of Proposition 0.36 in the notes on the \( V \)-filtration we saw that
\[
\mathcal{M} = \sum_{i \geq 0} \partial^i \cdot (V^{>0} \mathcal{M}).
\]

On the other hand, we know that \( V^{>0} \mathcal{M} \) is determined by \( \mathcal{M}|_U \) by Lemma 0.6 in the same notes. A situation in which we can also recover \( F_p V^{>0} \mathcal{M} \) from its restriction to \( U \) is provided by the following:

**Lemma 2.1.** Denoting by \( j : U = X \setminus D \hookrightarrow X \) the inclusion map, we have the identity
\[
F_p V^{>0} \mathcal{M} = V^{>0} \mathcal{M} \cap j_* j^* F_p \mathcal{M}
\]
if and only if
\[
t : F_p V^\alpha \mathcal{M} \rightarrow F_p V^{\alpha+1} \mathcal{M}
\]
is surjective for all \( \alpha > 0 \).

**Proof.** First assume the identity. Consider \( y \in F_p V^{\alpha+1} \mathcal{M} \), for \( \alpha > 0 \). In particular we have \( y \in F_p V^{>0} \mathcal{M} \), and so by hypothesis \( y \in V^{\alpha+1} \mathcal{M} \cap j_* j^* F_p \mathcal{M} \). We conclude the surjectivity of \( t \) due to the fact that \( t : V^\alpha \mathcal{M} \rightarrow V^{\alpha+1} \mathcal{M} \) is an isomorphism, combined with the fact that the action of \( t \) on \( j_* j^* F_p \mathcal{M} \) is bijective.

Assume now that \( t \) acts surjectively on \( F_p V^\alpha \mathcal{M} \) for every \( \alpha > 0 \). Note that the inclusion from left to right in the identity is clear. Consider now
\[
x \in V^\alpha \mathcal{M} \cap j_* j^* F_p \mathcal{M} = \{ x \in V^\alpha \mathcal{M} \mid x|_U \in F_p \mathcal{M}|_U \},
\]
with $\alpha > 0$. Thus locally there exists $i \geq 0$ such that $t^i x \in F_pM$. Since $x \in V^\alpha M$, we have $t^i x \in V^{\alpha+i} M$, so $t^i x \in F_p V^{\alpha+i} M$. By hypothesis and the injectivity of the action of $t$, we then have that $x \in F_p V^\alpha M \subseteq F_p V^{>0} M$. \hfill \Box

**Remark 2.2.** Note that since $t : V^\alpha M \rightarrow V^{\alpha+1} M$ is an isomorphism for $\alpha > 0$, we have that in this range

$$t : F_p V^\alpha M \rightarrow F_p V^{\alpha+1} M$$

is in any case injective, so the condition in the Lemma is that it’s in fact an isomorphism.

Lemma 2.1 can be combined with the following result in order to obtain a criterion for recovering the full $F_pM$ from its restriction to $U$. Note first that we always have

$$\sum_{i \geq 0} \partial^i_t \cdot (F_p V^{>0} M) \subseteq F_p M.$$  

**Lemma 2.3.** Assume that $\partial^i_t : gr^1 V M \rightarrow gr^0 V M$ is surjective.\(^1\) Then the following are equivalent:

1. $F_p M = \sum_{i \geq 0} \partial^i_t \cdot (F_p V^{>0} M)$.  
2. $\partial^i_t : F_p gr^\alpha V M \rightarrow F_{p+1} gr^{\alpha-1} V M$ is surjective for all $\alpha \leq 1$.

**Proof.** We define a new filtration $F'_p M$ by

$$F'_p M = \sum_{i \geq 0} \partial^i_t \cdot (F_p V^{>0} M).$$

We consider the following two claims:

**Claim 1.** For all $j \geq 0$ we have

$$F'_p V^{>j} M = \sum_{0 \leq i \leq j} \partial^i_t \cdot (F_p V^{>0} M).$$

**Claim 2.** For all $\alpha \leq 0$ we have

$$F'_p V^\alpha M = \sum_{0 \leq i \leq [-\alpha]} \partial^i_t \cdot (F_p V^\alpha M) + \partial^i_t [-\alpha+1] \cdot (F_p V^{[-\alpha] \rightarrow [-\alpha]+1} M).$$

Let’s first conclude the argument assuming these two claims. Note that the first implies

$$F'_p V^{>0} M = F_p V^{>0} M,$$

while the second implies

$$F'_p gr^\alpha V M = \partial^i_t [-\alpha+1] \cdot (F_p V^{[-\alpha] \rightarrow [-\alpha]+1} M).$$

We clearly have $F'_p M \subseteq F_p M$. Since the $V$-filtration is discrete, descending gradually through the values of $\alpha$ for which there are jumps, we therefore conclude that $F'_p M = F_p M$ if and only if

$$F_p gr^\alpha V M = \partial^i_t [-\alpha+1] \cdot (F_p V^{[-\alpha] \rightarrow [-\alpha]+1} M),$$

\(^1\)By Proposition 0.36 in the notes on the $V$-filtration, this is equivalent to $M = D_X \cdot V^{>0} M$, so to the fact that there are no proper submodules of $M$ that agree with $M$ on $U = X \setminus D$.  

which is equivalent to the condition in (2).

It remains to prove the two claims. Note first that Claim 2 reduces to Claim 1. Indeed the mapping
\[ \partial_j^t : \text{gr}^{\alpha+j} \mathcal{M} \to \text{gr}^{\alpha} \mathcal{M} \]
is injective for \( \alpha + j < 1 \), and so
\[ \partial_j^{[-\alpha]+1} \cdot (F_{p-[-\alpha]-1} V^0 \mathcal{M}) \cap V^\alpha \mathcal{M} = \partial_j^{[-\alpha]+1} \cdot (F_{p-[-\alpha]-1} V^{\alpha+[-\alpha]+1} \mathcal{M}). \]
for \( \alpha \leq 0 \).

Finally, to prove Claim 1, it suffices to show that
\[ \partial_j^{i+1} \cdot (F_{p-j} V^0 \mathcal{M}) \cap V^{>-j} \mathcal{M} \subseteq \partial_j^i \cdot (F_{p-j} V^0 \mathcal{M}). \]
Let \( u \in F_{p-j} V^0 \mathcal{M} \) such that \( \partial_j^{i+1} u \in V^{>-j} \mathcal{M} \). Since
\[ \partial_j^i : V^{>-1} \mathcal{M}/V^0 \mathcal{M} \to V^{>-j-1} \mathcal{M}/V^{>-j} \mathcal{M} \]
in injective, it follows that \( \partial_j u \in V^0 \mathcal{M} \). This in turn implies \( \partial_j^{i+1} u \in \partial_j^i \cdot (F_{p-j} V^0 \mathcal{M}) \), which is what we want. □

**Lemma 2.4.** If \( \text{Supp} \mathcal{M} \subseteq D \), define \( \mathcal{M}_0 = \text{Ker}(t : \mathcal{M} \to \mathcal{M}) \), and \( F_p \mathcal{M}_0 = F_p \mathcal{M} \cap \mathcal{M}_0 \). Then the following are equivalent:

1. \( \mathcal{M} = \mathcal{M}_0 \otimes \mathbb{C} [\partial_t] \) and \( F_p \mathcal{M} = \sum_{i \geq 0} F_{p-i} \mathcal{M}_0 \otimes \partial_t^i \).
2. \( \partial_t : F_p \text{gr}_V^\alpha \mathcal{M} \to F_{p+1} \text{gr}_V^{\alpha-1} \mathcal{M} \) is surjective for all \( \alpha < 1 \).

**Proof.** The hypothesis means of course that \( \mathcal{M} = \iota_* \mathcal{M}_0 \), where \( \iota : D \to X \). Moreover, as in Example 0.3(3) in the notes on the \( V \)-filtration, based on Kashiwara’s equivalence, we know that for all \( \alpha \leq 0 \) we have
\[ V^\alpha \mathcal{M} = \sum_{0 \leq i \leq [-\alpha]} \mathcal{M}_0 \otimes \partial_t^i. \]
In particular it suffices to focus on integral \( \alpha \).

We define a new filtration \( F'_\bullet \mathcal{M} \) by
\[ F'_p \mathcal{M} = \sum_{i \geq 0} F_{p-i} \mathcal{M}_0 \otimes \partial_t^i. \]
Note that we clearly have \( F'_p \mathcal{M} \subseteq F_p \mathcal{M} \), and moreover by definition
\[ F'_p V^0 \mathcal{M} = F_p V^0 \mathcal{M} (= F_p \mathcal{M}_0) \quad \text{and} \quad F'_p \text{gr}_V^{-i} \mathcal{M} = \partial_t^i \cdot (F_{p-i} \text{gr}_V^0 \mathcal{M}) \]
for all \( p \in \mathbb{Z} \) and \( i \geq 0 \). This implies (proceeding inductively on \( i \)) that we have \( F'_p \mathcal{M} = F_p \mathcal{M} \) if and only if
\[ F_p \text{gr}_V^{-i} \mathcal{M} = \partial_t^i \cdot (F_{p-i} \text{gr}_V^0 \mathcal{M}) \]
for all such \( p \) and \( i \). This is easily seen to be equivalent to the condition in (2). □
We now move to the case of hypersurfaces defined by arbitrary functions \( f \in \mathcal{O}_X(X) \). We are led to collecting the conditions in the lemmas above into the following definition proposed by Saito [Sa1, 3.2.1]. To simplify the notation, we write

\[ \mathcal{M}_f := \iota_+ \mathcal{M}, \]

where \( \iota \) stands as always for the graph embedding of \( X \) along \( f \).

**Definition 2.5** (Regular and quasi-unipotent condition). We say that \((\mathcal{M}, F)\) is *quasi-unipotent along \( f \)* if \( \mathcal{M}_f \) admits a rational \( V \)-filtration along the coordinate \( t \) on \( X \times \mathbb{C} \), and the following conditions are satisfied:

1. \( t \cdot (F_p V^\alpha \mathcal{M}_f) = F_p V^{\alpha+1} \mathcal{M}_f \) for \( \alpha > 0 \).
2. \( \partial_t \cdot (F_p \text{gr}_V^\alpha \mathcal{M}_f) = F_{p+1} \text{gr}_V^{\alpha-1} \mathcal{M}_f \) for \( \alpha < 1 \).

Moreover, \((\mathcal{M}, F)\) is called *regular along \( f \)* if in addition the filtration \( F_* \text{gr}_V^\alpha \mathcal{M}_f \) is a good filtration for \( 0 \leq \alpha \leq 1 \).

**Remark 2.6.** (1) Given the properties of the \( V \)-filtration discussed before, we have in fact that the actions of \( t \) and \( \partial_t \), in the ranges in (1) and (2) respectively, are bijective.

(2) We do not include \( \alpha = 1 \) in (2) in order for this notion to make sense even in the case when the morphism \( \partial_t : \text{gr}_V^1 \mathcal{M} \to \text{gr}_V^0 \mathcal{M} \) is not surjective, for more flexibility. In the case of Hodge modules however, this condition will be automatically satisfied, and therefore Lemma 2.3 will apply.

It is not so hard, but it is very useful in practice, to recognize when a filtered \( \mathcal{D} \)-module with support contained in \( D \) is regular and quasi-unipotent with respect to a defining equation:

**Proposition 2.7.** Let \((\mathcal{M}, F)\) be a filtered coherent \( \mathcal{D} \)-module with support contained in \( D = (f = 0) \), such that \( \mathcal{M}_f \) admits a rational \( V \)-filtration along \( t \). Denote by \( \iota : X \hookrightarrow X \times \mathbb{C} \) the graph embedding of \( X \) along \( f \), and by \( \iota_0 : X = X \times X \times \{0\} \hookrightarrow X \times \mathbb{C} \) the natural embedding. Then the following are equivalent:

1. \( \mathcal{M} \) is regular and quasi-unipotent along \( f \).
2. \( F_k \mathcal{M} \subseteq f \cdot F_{k-1} \mathcal{M} \) for all \( k \).
3. There is a canonical isomorphism \((\mathcal{M}_f, F) = \iota_+ (\mathcal{M}, F) \simeq \iota_0+ (\mathcal{M}, F) \).

**Proof.** This is [Sa1, Lemma 3.2.6]. I will include a proof after more discussion of pushforward of filtered \( \mathcal{D} \)-modules.

**Conclusion.** The main point of this section can roughly be summarized as follows. Assume that \((\mathcal{M}, F)\) is a \( \mathcal{D}_X \)-module admitting a rational \( V \)-filtration, with strict support \( Z \). If it is regular and quasi-unipotent along \( D = (f = 0) \), such that \( f \mid Z \) is not constant, then most conditions in Lemmas 2.1 and 2.3 are satisfied. If we assume that the last needed condition holds as well, namely that

\[ \partial_t : F_p \text{gr}_V^1 \mathcal{M}_f \longrightarrow F_p \text{gr}_V^0 \mathcal{M}_f \]
is surjective, which will always hold in the case of Hodge modules, then we conclude that

\[(2.1) \quad F_p^i M_f = \sum_{i=0}^{\infty} \partial^i \cdot (V^{>0} M_f \cap j_* j^* F_{p-i} M_f),\]

so in particular \((M, F)\) is uniquely determined by its restriction to \(Z \setminus (Z \cap D)\).

3. \(\mathcal{D}\)-modules with \(\mathbb{Q}\)-structure. The following definition underlies the notion of a Hodge module, which will be introduced later. I will therefore start with the following:

**Note.** The definition that follows refers to two notions that we have not focused on in this course, namely that of a regular holonomic \(\mathcal{D}\)-module, and that of a perverse sheaf. If you are unfamiliar with them, the ideal thing to do would be to acquire some familiarity, by reading for example parts of Chapters 6-8 in [HTT]. In the meanwhile however, for temporarily developing the theory, such familiarity is not really crucial. Specifically, on one hand we will not deal with the theory of perverse sheaves at all; on the other hand, in these notes regularity is only used to ensure the existence of a \(V\)-filtration (due to the Kashiwara-Malgrange theorem), and as an input for the Riemann-Hilbert correspondence.

**Definition 3.1.** (1) A filtered regular holonomic \(\mathcal{D}_X\)-module with \(\mathbb{Q}\)-structure is a triple \(M = (M, F, P)\) consisting of:

- A perverse sheaf \(P\) of \(\mathbb{Q}\)-vector spaces on \(X\).
- A regular holonomic \(\mathcal{D}_X\) module such that \(DR(M) \simeq P \otimes \mathbb{Q}\mathbb{C}\).
- A good filtration \(F_p M\) by coherent \(\mathcal{O}_X\)-modules.

(2) Given such a triple, for any \(k \in \mathbb{Z}\), its \(k\)-th Tate twist is defined as

\[M(k) := (M, F_{\bullet - k}, P(k)),\]

where \(P(k) = P \otimes \mathbb{Q} \mathbb{Q}(k)\), with \(\mathbb{Q}(k) = (2\pi i)^k \cdot \mathbb{Q} \subset \mathbb{C}\).

(3) Given a filtered regular holonomic \(\mathcal{D}_X\)-module with \(\mathbb{Q}\)-structure \(M = (M, F, P)\) and a function \(f: X \to \mathbb{C}\), and denoting \(\lambda = e^{2\pi i \alpha}\) for \(\alpha \in \mathbb{Q}\), we define

- \(\psi_f M := \bigoplus_{0 < \alpha < 1} \left( \text{gr}_{V}^{0} M_f, F_{\bullet - 1} \text{gr}_{V}^{0} M_f, \psi_{f,\lambda} P \right)\).
- \(\psi_{f,1} M := \left( \text{gr}_{V}^{1} M_f, F_{\bullet - 1} \text{gr}_{V}^{1} M_f, \psi_{f,1} P \right)\).
- \(\phi_{f,1} M := \left( \text{gr}_{V}^{0} M_f, F_{\bullet} \text{gr}_{V}^{0} M_f, \psi_{f,1} P \right)\).

to be the total and unipotent nearby cycles, and the unipotent vanishing cycles of \(M\) along \(f\), respectively. Here the last term in the parenthesis denotes the corresponding topological construction on the perverse sheaf \(P\). Note that all of these objects have support contained in \(f^{-1}(0)\).

The criterion regarding decomposition by strict support in terms of nearby and vanishing cycles, Corollary 0.41 in the notes on the \(V\)-filtration, has an enhancement to the setting of filtered \(\mathcal{D}\)-modules with \(\mathbb{Q}\)-structure:

**Theorem 3.2.** Let \(M = (M, F, P)\) be a filtered regular holonomic \(\mathcal{D}_X\)-module with \(\mathbb{Q}\)-structure, and suppose that \((M, F)\) is regular and quasi-unipotent along \((f = 0)\) for
all locally defined functions \( f \in \mathcal{O}_X(U) \) on all open sets \( U \subset X \). Then there exists a decomposition

\[
M \simeq \bigoplus_{Z \subseteq X} M_Z,
\]

with each \( M_Z = (M_Z, F, P_Z) \) a filtered regular holonomic \( \mathcal{D}_X \)-module with \( \mathbb{Q} \)-structure and strict support \( Z \), if and only if

\[
\phi_{f,1}M = \text{Ker}(\text{var}: \phi_{f,1}M \to \psi_{f,1}M(-1)) \oplus \text{Im}(\text{can}: \psi_{f,1}M \to \phi_{f,1}M),
\]

where the filtration on \( \psi_{f,1}M \) is induced by that on \( \phi_{f,1}M \).

**Exercise .2.** Prove Theorem 3.2.

**Monodromy weight filtration.** Before defining pure Hodge modules, we need a discussion of the monodromy weight filtration.

First, I review a number of well-known facts regarding nearby and vanishing cycles on perverse sheaves. (It’s ok to skip if you are not familiar with this story. For more details and further references you can consult for instance D. Massey’s notes arXiv:math/9908107; see also [Sch, §8].) Let \( P \) be a perverse sheaf on \( X \), and \( f: X \to \mathbb{C} \) a holomorphic function. We have associated perverse nearby and vanishing cycles \( p_\psi fP \) and \( p_\phi fP \), together with natural morphisms

\[
\text{can}: p_\psi fP \to p_\phi fP \quad \text{and} \quad \text{var}: p_\phi fP \to p_\psi fP(-1),
\]

and an action

\[
T: p_\psi fP \to p_\psi fP
\]

of the monodromy operator.

Recall that perverse sheaves form an abelian category; therefore there is a generalized eigenspace decomposition

\[
p_\psi fP = \bigoplus_{\lambda \in \mathbb{C}} p_\psi f,\lambda P
\]

under the action of \( T \), and similarly for \( p_\phi fP \). Concretely, we have

\[
p_\psi f,\lambda P = \text{Ker}(T - \lambda \cdot \text{Id})^m, \quad \text{for} \ m \gg 0.
\]

The unipotent nearby cycles are \( p_\psi f,1 P = \text{Ker}(\text{Id} - T)^m \), with \( m \gg 0 \).

Consider now the nilpotent operator

\[
N = \frac{1}{2\pi i} \cdot \log T_u,
\]

where \( T_u \) is the unipotent part of the monodromy \( T \). (Recall from linear algebra that there is always a decomposition \( T = T_u \cdot T_s = T_s \cdot T_u \), where \( T_u \) is unipotent and \( T_s \) is semisimple.) It turns out that we have

\[
N = \text{var} \circ \text{can} : p_\psi f,1 P \to p_\psi f,1 P.
\]

This operator, as any nilpotent endomorphism, induces a filtration which in this context is called the **monodromy weight filtration**; this is an increasing filtration \( W_\bullet p_\psi f,1 P \), going from \( W_0 = 0 \) to \( W_{2k} \) if \( N^{k+1} = 0 \). Denoting \( W_\ell = W_\ell p_\psi f,1 P \), it has the properties
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(1) $N(W_\ell) \subseteq W_{\ell-2}$.

(2) $N_p^p : \text{gr}_{\ell+p}^W \longrightarrow \text{gr}_{\ell-p}^W$ is an isomorphism.

Moreover, if we denote $P_\ell = \text{Ker}(N^\ell-k+1) \subseteq \text{gr}_\ell^W$ for $\ell > k$, and $P_\ell = 0$ for $\ell < k$, then $\text{gr}_\ell^W = \oplus_i N^i(P_{\ell+2i})$.

Going back to filtered $\mathcal{D}$-modules $(\mathcal{M}, F)$ with $V$-filtration along $f$ an important result is that the two notions of nearby and vanishing cycles we have seen are related by the Riemann-Hilbert correspondence. (This is of course the reason for adopting this terminology in the world of $\mathcal{D}$-modules.) The following is a result of Kashiwara and Malgrange, enhanced by Saito to the case of the rational $V$-filtration.

**Theorem 3.3** ([Sa1, Proposition 3.4.12]). If $\mathcal{M}$ is holonomic, and is regular and quasi-unipotent along $f$, then we have

$$\text{DR}(\text{gr}_V^0 \mathcal{M}_f) \simeq \begin{cases} p^\phi_{f,\lambda} \text{DR}(\mathcal{M}) & \text{for } 0 \leq \lambda < 1 \\ p^\psi_{f,\lambda} \text{DR}(\mathcal{M}) & \text{for } 0 < \lambda \leq 1 \end{cases}$$

where $\lambda = e^{-2\pi i \alpha}$. Moreover, via this identification the operators $N$, can and var at the level of perverse sheaves are identified with $t\partial_t$, $\partial_t$ and $t$ at the level of $\mathcal{D}$-modules.

To repeat, on $\psi_{f,1} \mathcal{M} = \text{gr}_V^1 \mathcal{M}_f$, the nilpotent operator $N = \text{var}$ can corresponds to the operator $t\partial_t$. This respects the filtration $F$; thus putting all of this together, if $\mathcal{M} = (\mathcal{M}, F, P)$ is a filtered regular holonomic $\mathcal{D}_X$-module with $\mathbb{Q}$-structure, we get an induced nilpotent operator

$$N : \psi_{f,1} M \longrightarrow \psi_{f,1} M,$$

and therefore a monodromy weight filtration $W$ with associated graded objects $\text{gr}_k^W \psi_{f,1} M$. A similar construction can be made for the full nearby cycle functor $\psi_f M$.

### 4. Pure Hodge modules.

We can now define the category $\text{HM}(X, w)$ of pure Hodge modules of weight $w$ on a smooth complex variety $X$. The definition is inductive on the dimension of the support of the Hodge module, so it makes sense to successively define the categories $\text{HM}_{\leq d}(X, w)$ of pure Hodge modules on $X$ of weight $w$ and support of dimension at most $d$.

In any case, the objects of $\text{HM}(X, w)$ are filtered regular holonomic $\mathcal{D}_X$-modules with $\mathbb{Q}$-structure $M = (\mathcal{M}, F, P)$ which are subject to certain conditions; namely:

(1) $(\mathcal{M}, F)$ is regular and quasi-unipotent along every locally defined holomorphic function $f : U \rightarrow \mathbb{C}$, with $U \subseteq X$ open.

(2) $M$ admits a strict support decomposition in the category of filtered regular holonomic $\mathcal{D}_X$-modules with $\mathbb{Q}$-structure. (Thus the formula in Theorem 3.2 holds.)

Pausing for a second from imposing conditions, note that (2) implies that it is enough to define pure Hodge modules with strict support. Therefore it is enough to define for each irreducible closed subset $Z \subseteq X$ the category $\text{HM}_Z(X, w)$ of pure Hodge modules on $X$ of weight $w$ and strict support $Z$. 

Moreover, because of (1) and the discussion in the previous section, for every $f : U \to \mathbb{C}$ the nearby and vanishing cycles $\psi_f M$, $\psi_f M$, and $\phi_f M$ do make sense as filtered regular holonomic $\mathscr{D}_X$-modules with $\mathbb{Q}$-structure, this time with support contained in $f = 0$. They will be used for the inductive definition.

Continuing with the definition, for the base case we impose the following:

(3) If $Z = \{x\}$, where $x \in X$ is a point, embedded via $i : \{x\} \hookrightarrow X$, then
\[ \text{H}_{f_{\{x\}}} (X, w) = \{ i_* H \mid H \text{ is a } \mathbb{Q}-\text{Hodge structure of weight } w \}. \]

(4) For arbitrary $Z$ of dimension $d$, we say that $M$ belongs to $\text{HM}_{Z}(X, w)$ if the following is satisfied: for every function $f : U \to \mathbb{C}$ which does not vanish identically on $Z \cap U$, we require for all $k$ that
\[ \text{gr}^W_k \psi_f M \in \text{HM}_{\leq d-1}(X, w - 1 + k), \]
where $W$ is the monodromy weight filtration as in the previous section. Note that Theorem 3.2, which can be applied as discussed above, implies then that $\text{gr}^W_k \phi_f M \in \text{HM}_{\leq d-1}(X, w + k)$ as well. (Mention Schmid’s $SL_2$-orbit theorem.)

This completes inductively the definition of $\text{HM}(X, w)$ as $\bigcup_{d \geq 0} \text{HM}_{\leq d}(X, w)$, where the morphisms are morphisms of filtered regular holonomic $\mathscr{D}_X$-modules with $\mathbb{Q}$-structure. One can show that these are strictly compatible with the filtration $F$, so that in particular $\text{HM}(X, w)$ is an abelian category.

**Remark 4.1.** Strictly speaking the definition above is a definition for analytic pure Hodge modules, that can be given on any complex manifold. To have a category of algebraic Hodge modules on a smooth complex variety, one can check the properties above with respect to regular functions, but one also has to work with polarizable Hodge structures, and take into account in the definition of the notion of a polarization on a Hodge module. See for instance [Sa4, §2.2] for a discussion. See also the end of §5 below.

**Remark 4.2.** It is worth noting that the use of nearby cycles in the inductive definition of Hodge modules is rather natural, due to an important result in Hodge theory due to W. Schmid, called the $SL_2$-orbit theorem. A consequence of this result is that the nearby cycles of a polarizable VHS on the punctured disk carry a mixed Hodge structure whose Hodge filtration is the limit Hodge filtration of the variation, and whose weight filtration is the monodromy weight filtration of $N$ (up to a shift).

**Example 4.3** (Variations of Hodge structure). Going back to Example 1.3, a VHS is the most basic type of a pure Hodge module, namely what is considered to be the “smooth” case. To a $\mathbb{Q}$-VHS $V = (V, F^\bullet, V_\mathbb{Q})$ of weight $\ell$ on $X$ one associates the pure Hodge module $M = (M, F, P)$ of weight $n + \ell$, where:
\begin{itemize}
  \item $M = V$
  \item $F_p M = F^{-p} V$
  \item $P = V_\mathbb{Q}[n]$.
\end{itemize}

\[ \text{For instance, when } Z = X \text{ is a curve, then } \psi_f M \text{ is locally supported on a point, and this data corresponds to a mixed Hodge structure.} \]
Given the complicated inductive definition that depends on looking at all (locally defined) functions on $X$, it is not at all obvious that this indeed defines a pure Hodge module. This goes through several fundamental theorems in Hodge theory, or alternatively (and somewhat hiding what lies behind it) follows from functorial properties of Hodge modules established later on.

This applies even to the most basic example of this construction, namely the trivial Hodge module on $X$, i.e. the object

$$\mathcal{Q}_X^H[n] := (\mathcal{O}_X, F, \mathcal{Q}_X[n])$$

associated to the trivial variation of Hodge structure $\mathcal{Q}$ on $X$, where the filtration $F$ is the trivial filtration on $\mathcal{O}_X$ defined in Example 1.2.

We will see other examples later on, but first we need to understand some functorial properties of pure Hodge modules. On the other hand, just as with holonomic $\mathcal{D}$-modules every pure Hodge with strict support is generically “smooth”.

**Proposition 4.4.** Let $X$ be a smooth complex variety, and $Z \subseteq X$ an irreducible closed subset of dimension $m$. If $M$ is a pure Hodge module of weight $w$ on $X$, with strict support $Z$, then there exists a Zariski open subset $U \subseteq Z$ such that $M|_U$ is a $\mathbb{Q}$-VHS of weight $w - m$ on $U$, and $M$ is uniquely determined by this VHS.

**Proof.** Let $M = (\mathcal{M}, F, P) \in \text{HM}_Z(X, w)$. Then $\mathcal{M}$ is holonomic, hence by a basic result discussed last quarter there exists an open set $V \subseteq X$ intersecting $Z$ nontrivially, such that if $V = Z \cap U$, then $\mathcal{M}|_V$ is an integrable connection. Similarly, a well-known fact from the theory of perverse sheaves says that we can also take $V$ such that $P|_V = L[m]$, where $L$ is a local system on $V$ (and in fact $P$ is the intersection complex of $L$). Restricting the filtration to $V$ as well, clearly gives us the data of a $\mathbb{Q}$-VHS.

On the other hand, this VHS uniquely determines $M$ (meaning an element in $\text{HM}_Z(X, w)$ which restricts to it on $V$). By possibly shrinking $U$, we can assume that its complement in $X$ is a hypersurface. The assertion then follows from the property of $(\mathcal{M}, F)$ of being regular and quasi-unipotent along this hypersurface, as explained in the paragraph around (2.1). (Again, the perverse sheaf $P$ is determined as the intersection complex of the local system underlying the VHS.)

One of the main results of Saito’s theory is that under the extra assumption that the VHS is polarizable (which is satisfied by all those of algebro-geometric origin), the converse of this statement is true.

**Theorem 4.5 ([Sa2]).** Every polarizable $\mathbb{Q}$-VHS of weight $w - m$ on a Zariski open set $V \subseteq Z$ extends uniquely to a (polarizable) pure Hodge module in $\text{HM}_Z(X, w)$.

A nice survey of the proof of this theorem is given in [Sch, §17-18].

Now that we have at least seen one of the main definitions in the theory of Hodge modules, we need to go back and try to understand in more explicit detail some of the special properties of the filtered $\mathcal{D}$-modules that underlie these objects, and how they behave under natural functors.
5. Mixed Hodge modules. In this section we will record a few facts from the theory of mixed Hodge modules, to at least be able to place the some of the objects we’ve been looking at, like the localization $\mathcal{O}_X(*D)$ along a hypersurface, in the proper context. The relevant reference is [Sa2], while [Sch, §20-22] provides a quick overview emphasizing useful details that I will skip here.

Again, the initial definition is an analytic one, that works on an arbitrary complex manifold $X$.

**Definition 5.1.** The category $\text{MHW}(X)$ of weakly mixed Hodge modules consists of objects $(M,W^\bullet)$, where $M = (M,F,P)$ is a filtered regular holonomic $\mathcal{D}_X$-module with $\mathbb{Q}$-structure, and $W^\bullet$ is a finite decreasing filtration on $M$ by filtered regular holonomic $\mathcal{D}_X$-modules with $\mathbb{Q}$-structure, compatible with the isomorphism $\text{DR}(M) \simeq P \otimes \mathbb{Q} \mathbb{C}$, such that for all $\ell \in \mathbb{Z}$ we have

$$\text{gr}^W_\ell M \in \text{HM}(X,\ell).$$

The morphisms in this category and strict morphisms of filtered regular holonomic $\mathcal{D}_X$-module with $\mathbb{Q}$-structure that strictly respect the filtration $W^\bullet$. A weakly mixed Hodge module $M$ is graded-polarizable if each $\text{gr}^W_\ell M$ is polarizable. These form a subcategory of $\text{MHW}(X)$ denoted $\text{MHW}^p(X)$.

**Remark 5.2.** It is not hard to see that the category $\text{MHW}(X)$ is abelian; see [Sa2, Proposition 5.1.14].

Mixed Hodge modules form a subcategory of $\text{MHW}(X)$. The reason for imposing further restrictions is an issue that already appears in the theory of mixed Hodge structures, which can roughly be expressed as the fact that variations of Hodge structure can have bad singularities at the boundary, that do not allow for extensions in the category of regular $\mathcal{D}$-modules.

**Definition 5.3.** Let $(M,W^\bullet) \in \text{MHW}^p(X)$, and let $f: X \to \mathbb{C}$ be a non-constant holomorphic function on $X$. We say that $(M,W^\bullet)$ is admissible along $f$ if the following hold:

1. $(M,F)$ is regular and quasi-unipotent along $f$.
2. The three filtrations $F^\bullet M_f$, $V^\bullet M_f$, and $W^\bullet M_f$ are compatible, i.e. the order does not matter when we compute the associated graded.
3. Consider the naive limit filtrations

$$L_i(\psi_f M) = \psi_f(W_{i+1}M) \quad \text{and} \quad L_i(\varphi_{f,1}M) = \varphi_{f,1}(W_iM),$$

preserved by the nilpotent endomorphism $N = (2\pi i)^{-1} \log T_u$. Then, the relative monodromy filtrations

$$W^\bullet(\psi_f M) = W^\bullet(N,L^\bullet(\psi_f M)) \quad \text{and} \quad W^\bullet(\varphi_{f,1}M) = W^\bullet(N,L^\bullet(\varphi_{f,1}M))$$

for the action of $N$ exist; see [Sa2, 1.1.3-4].

**Remark 5.4.** It is not hard to see that condition (1) above holds in fact automatically, since it holds for each of the pure Hodge modules $\text{gr}_\ell^W M$.

With this preparation, we can define mixed Hodge modules; the definition is again inductive on dimension.
Definition 5.5. A weakly mixed Hodge module \((M, W_\bullet) \in \text{MHW}(X)\) is a mixed Hodge module if for every locally defined holomorphic function \(f : U \to \mathbb{C}\) we have:

1. The pair \((M, W_\bullet)\) is admissible along \(f\).
2. Both \((\psi f M, W_\bullet)\) and \((\varphi f M, W_\bullet)\) are mixed Hodge modules, whenever \(f^{-1}(0)\) does not contain any irreducible components of \(U \cap \text{Supp}(M)\).

We denote by \(\text{MHM}(X)\) and \(\text{MHM}^p(X)\) the full subcategories of \(\text{MHW}(X)\) and \(\text{MHW}^p(X)\) the full subcategories of mixed Hodge modules and graded-polarizable mixed Hodge modules, respectively. Morphisms are given by morphisms of filtered regular holonomic \(\mathcal{D}_X\)-modules with \(\mathbb{Q}\)-structure that are compatible with \(W_\bullet\).

The following theorem summarizes some important properties and functors associated with mixed Hodge modules.

Theorem 5.6. [Sa2] (1) The categories \(\text{MHM}(X)\) and \(\text{MHM}^p(X)\) are abelian.

(2) The categories \(\text{MHM}(X)\) and \(\text{MHM}^p(X)\) are stable under applying the nearby and vanishing cycles functors.

(3) If \(f : X \to Y\) is a projective morphism, then there exist direct image functors
\[
\mathcal{H}^i f_* M : \text{MHM}^p(X) \to \text{MHM}^p(Y).
\]

(4) If \(f : X \to Y\) is an arbitrary morphism, then there exist inverse image functors
\[
\mathcal{H}^i f^* M, \mathcal{H}^i f^! M : \text{MHM}^p(Y) \to \text{MHM}^p(X).
\]

There is also an important analogue of Theorem 4.5:

Theorem 5.7 ([Sa2, Theorem 3.27]). Let \(Z\) be an irreducible close subset of \(X\). Then a graded-polarizable variation of mixed Hodge structure on an open subset of \(Z\) can be extended to an object in \(\text{MHM}(X)\) supported on \(Z\) if and only if it is admissible relative to \(Z\).

Regarding the two theorems above, note that for an open embedding \(j : U \hookrightarrow X\) we cannot hope to have a general functor \(j_*\) on \(\text{MHM}^p(U)\), as the admissibility condition does not necessarily hold.

Our main interest however is to work on a smooth algebraic variety \(X\), and consider subcategory \(\text{MHM}_{\text{alg}}(X) \subseteq \text{MHM}^p(X^{an})\) of algebraic mixed Hodge modules. (Note that these are automatically polarizable.) These were originally defined in [Sa2] using a compactification of \(X\), and taking advantage of GAGA-type theorems. However in [Sa3] Saito found an alternative intrinsic definition which can be summarized as follows:

Theorem 5.8. A weakly mixed Hodge module \((M, W_\bullet)\) belongs to the category \(\text{MHM}_{\text{alg}}(X)\) if and only if \(X\) can be covered by Zariski open subsets \(U\) such that:

1. There exists \(f \in \mathcal{O}_X(U)\) such that \(U \setminus f^{-1}(0)\) is smooth and dense in \(U\).
2. The restriction of \((M, W_\bullet)\) to the open subset \(U^{an} \setminus f^{-1}(0)\) is a graded-polarizable admissible variation of mixed Hodge structure.
The pair $(M, W_*)$ is admissible along $f$.

(4) The pair $(\psi_f M, W_*(\psi_f M))$ belongs to $\text{MHM}_{\text{alg}}(f^{-1}(0))$.

Unlike the story in the analytic setting, for an open embedding $j: U \hookrightarrow X$ of smooth algebraic varieties the functors $j_*$ and $j_!$ are defined on $\text{MHM}_{\text{alg}}(U)$, and this allows one to define functors $f_*$ and $f_!$ for an arbitrary morphism of algebraic varieties.

For all practical purposes, besides the existence of these functors, what we need to know concretely at this stage is the construction of the direct image via the open embedding from the complement of an SNC divisor. The full construction of direct image via an open embedding combines this with resolution of singularities; see [Sa2, Theorem 3.27].

**Extension of a VHS across an SNC divisor.** Let $D$ be an SNC divisor in a smooth variety $X$, and denote $j: U = X \setminus D \hookrightarrow X$. We consider a polarizable VHS $V = (V, F_\bullet, V_Q)$ over $U$, with quasi-unipotent local monodromies along the components $D_i$ of $D$. In particular the eigenvalues of all residues are rational numbers. We call $M$ the associated pure Hodge module on $U$.

For $\alpha \in \mathbb{Z}$, we denote by $V^\geq \alpha$ (resp. $V^{> \alpha}$) the Deligne extension with eigenvalues of residues along the $D_i$ in $[\alpha, \alpha + 1]$ (resp. $(\alpha, \alpha + 1]$).\(^3\) Recall that $V^\geq \alpha$ is filtered by

\[ F_p V^\geq \alpha = V^\geq \alpha \cap j_* F_p V, \]

while the filtration on $V^{> \alpha}$ is defined similarly. The terms in the filtration are locally free by Schmid’s nilpotent orbit theorem (which can be extended to the quasi-unipotent case).

The mixed Hodge module $j_* M$ on $X$ is then defined as follows:

\[ j_* M = (V(*D), F_\bullet, j_* V_Q). \]

Here $V(*D)$ is Deligne’s meromorphic connection extending $V$. It has a lattice defined by $V^\geq \alpha$ for any $\alpha \in \mathbb{Q}$, i.e. $V(*D) = V^\geq \alpha \otimes \mathcal{O}_X(*D)$, and its filtration is given by

\[ F_p V(*D) = \sum_i F_i \mathcal{O}_X \cdot F_{p-i} V^{\geq-1}. \]

For the weight filtration see [Sa2, Proposition 2.8].

**Example 5.9** (Localization). When $M = \mathcal{O}_U^{[n]}$ is the trivial Hodge module, then $M = \mathcal{O}_U$, whose Deligne canonical extension is $\mathcal{O}_X$. In this case

\[ j_* \mathcal{O}_U = (\mathcal{O}_X(*D), F_\bullet, j_* \mathcal{O}_U). \]

Thus $\mathcal{O}_X(*D)$ underlies a mixed Hodge module, and the Hodge filtration can be described as follows. First, it is well known that $\mathcal{O}_U^{\geq-1} = \mathcal{O}_X(D)$, and its filtration is

\[ F_k \mathcal{O}_X(D) = \mathcal{O}_X(D) \cap j_* F_k \mathcal{O}_U = \mathcal{O}_X(D) \cap j_* \mathcal{O}_U = \mathcal{O}_X(D) \]

for all $k \geq 0$, and 0 otherwise. The formula above gives

\[ F_p \mathcal{O}_X(*D) = F_p \mathcal{O}_X \cdot \mathcal{O}_X(D) \]

\(^3\)Unfortunately we haven’t discussed yet this important construction in this course; see for instance [HTT, Theorem 5.2.17].
for \( p \geq 0 \), and 0 otherwise.

Since we have introduced this terminology, it is worth noting that the pure Hodge module extension of \( M \) with strict support \( X \) (see Theorem 4.5) is given by the minimal extension functor, which we looked at briefly earlier in this course. More precisely, as shown in [Sa2, §3.b], it is

\[
(D_X \cdot V^{>-1}, F_\bullet, j_! V_Q),
\]

where the filtration is defined as

\[
F_p(D_X \cdot V^{>-1}) = \sum_i F_i D_X \cdot F_{p-i} V^{>-1}.
\]

6. Induced \( \mathcal{D} \)-modules and filtered differential morphisms. In this section we show that complexes of filtered \( \mathcal{D}_X \)-modules are quasi-isomorphic to complexes whose entries are special types of \( \mathcal{D} \)-modules arising from plain old \( \mathcal{O}_X \)-modules. These can be further interpreted as complexes of \( \mathcal{O}_X \)-modules with a special type of \( \mathbb{C} \)-linear differentials. Since we will apply this to studying filtered push-forward, it is more convenient to work with right \( \mathcal{D}_X \)-modules.

Recall that when passing between a filtered left \( \mathcal{D}_X \)-module \((N, F)\) and its right version \((M = \omega_X \otimes_{\mathcal{O}_X} N, F)\), the filtration rule is

\[
F_{k-n} M = F_k N \otimes_{\mathcal{O}_X} \omega_X
\]

for all \( k \), where \( n = \dim X \). For instance, the trivial filtration on the right \( \mathcal{D}_X \)-module \( \omega_X \), the right analogue of Example 1.2, is given by \( F_k \omega_X = 0 \) for \( k < -n \) and \( F_k \omega_X = \omega_X \) for \( k \geq -n \).

Note once and for all that all types of filtrations appearing below satisfy the property that \( F_p = 0 \) for \( p \ll 0 \).

We begin by denoting by \( \text{FM}(\mathcal{D}_X) \) the category of filtered right \( \mathcal{D}_X \)-modules. A morphism in this category is a \( \mathcal{D}_X \)-module morphism \( f : M \to N \) such that \( f(F_k M) \subseteq F_k N \) for all \( k \). Special objects in this category are those induced by filtered \( \mathcal{O}_X \)-modules. Concretely, consider a filtered \( \mathcal{O}_X \)-module \((\mathcal{G}, F_\bullet \mathcal{G})\), with respect to the trivial filtration on \( \mathcal{O}_X \) given by \( F_k \mathcal{O}_X = 0 \) for \( k < 0 \) and \( F_k \mathcal{O}_X = \mathcal{O}_X \) for \( k \geq 0 \). We can associate to it an object in \( \text{FM}(\mathcal{D}_X) \) given by

\[
M := \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X \quad \text{and} \quad F_k M := \sum_{i=0}^k F_{k-i} \mathcal{G} \otimes F_i \mathcal{D}_X.
\]

**Definition 6.1.** An object in \( \text{FM}(\mathcal{D}_X) \) is an induced filtered \( \mathcal{D}_X \)-module if it is isomorphic to one defined as above. We use the notation \( \text{FM}_i(\mathcal{D}_X) \) for the full subcategory of \( \text{FM}(\mathcal{D}_X) \) whose objects are induced filtered \( \mathcal{D}_X \)-modules.

**Remark 6.2.** Saito [Sa1, §2.2] calls the functor

\[
\mathcal{M} := \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X \mapsto \text{DR}(\mathcal{M}) := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X \in \text{Mod}(\mathbb{C}_X).
\]

the de Rham functor on induced \( \mathcal{D} \)-modules. We have of course a canonical isomorphism \( \text{DR}(\mathcal{M}) \simeq \mathcal{G} \).
Lemma 6.3. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then there is a natural homomorphism
\[ \text{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \to \text{Hom}_{\mathcal{C}_X}(\mathcal{F}, \mathcal{G}) \]
given by $\bullet \otimes_{\mathcal{D}_X} \mathcal{O}_X$, and this homomorphism is injective.

Proof. By adjunction we have an isomorphism
\[ \text{Hom}_{\mathcal{D}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X), \]
and therefore the homomorphism in the statement corresponds to sending $\varphi$ to $\psi$ in each diagram of the form
\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{G} & \xrightarrow{\psi} & \mathcal{F}
\end{array}
\]
where the $\mathcal{O}_X$-module structure on the top sheaf is obtained by restriction of scalars from its $\mathcal{D}_X$-module structure (hence in particular $\psi$ is not $\mathcal{O}_X$-linear), and the vertical arrow is obtained by sending $s \otimes P \mapsto P(1)s$, for a section $s$ of $\mathcal{G}$ and a differential operator $P$.

Assume now that $\varphi \neq 0$. Hence there is an open set $U$ with local coordinates $x_1, \ldots, x_n$, and a section $s \in \Gamma(U, \mathcal{F})$ such that $\varphi(s) \neq 0$. We write
\[ \varphi(s) = \sum_{\alpha} t_\alpha \otimes \partial^\alpha, \]
where the sum is finite, over $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and $t_\alpha \in \Gamma(U, \mathcal{G})$. We use the standard notation $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, and we will also use $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. Let
\[ k_0 := \min \{|\alpha| \mid t_\alpha \neq 0\}, \]
and consider $\beta \in \mathbb{N}^n$ such that $|\beta| = k_0$ and $t_\beta \neq 0$. We then have
\[ \psi(s \cdot x^\beta) = \sum_{\alpha} t_\alpha \otimes \partial^\alpha(x^\beta) = \beta! \cdot t_\beta \neq 0, \]
hence $\psi \neq 0$. \qed

Definition 6.4. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. The group of \textit{differential morphisms} from $\mathcal{F}$ to $\mathcal{G}$ is defined as the image in $\text{Hom}_{\mathcal{C}_X}(\mathcal{F}, \mathcal{G})$ of the homomorphism in the Lemma above, and is denoted by $\text{Hom}_{\text{Diff}}(\mathcal{F}, \mathcal{G})$.

Note that $\text{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ admits a filtration whose $p$-th term is
\[ \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} F_p \mathcal{D}_X), \]
and therefore we can define a filtration on differential morphisms where $F_p \text{Hom}_{\text{Diff}}(\mathcal{F}, \mathcal{G})$ is (the image of) this subgroup. We call these differential morphisms \textit{of order} $\leq p$.

We now give a filtered version of this construction.
Definition 6.5. Let \((\mathcal{F}, F)\) and \((\mathcal{G}, F)\) be filtered \(\mathcal{O}_X\)-modules. The group of filtered differential morphisms
\[
\text{Hom}_{\text{Diff}}((\mathcal{F}, F), (\mathcal{G}, F))
\]
is the subgroup of \(\text{Hom}_{\text{Diff}}(\mathcal{F}, \mathcal{G})\) consisting of morphisms \(f\) satisfying, for every \(p\) and \(q\), the fact that the composition
\[
F_p \mathcal{F} \hookrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \to \mathcal{G}/F_{p-q-1}\mathcal{G}
\]
(which itself is a differential morphism) has order \(\leq q\).

Exercise .3. Restricting to filtered morphisms, the homomorphism in Lemma 6.3 induces an isomorphism
\[
\text{Hom}_{\text{FM}(D_X)}(\mathcal{F} \otimes_{\mathcal{O}_X} D_X, \mathcal{G} \otimes_{\mathcal{O}_X} D_X) \simeq \text{Hom}_{\text{Diff}}((\mathcal{F}, F), (\mathcal{G}, F)).
\]

Definition 6.6. We denote by \(\text{FM}(\mathcal{O}_X, \text{Diff})\) the additive category whose objects are filtered \(\mathcal{O}_X\)-modules, and whose morphisms are filtered differential morphisms.

Putting together all of the above, we obtain the following interpretation of the category of induced filtered \(\mathcal{D}\)-modules.

Proposition 6.7. The functor
\[
\text{DR}^{-1}: \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}_X} D_X
\]
induces an equivalence of categories
\[
\text{DR}^{-1}: \text{FM}(\mathcal{O}_X, \text{Diff}) \xrightarrow{\sim} \text{FM}(\mathcal{O}_X, \text{Diff}).
\]

It is not hard to see that this equivalence extends to an equivalence of triangulated categories
\[
(6.1) \quad \text{DR}^{-1}: D(\text{FM}(\mathcal{O}_X, \text{Diff})) \xrightarrow{\sim} D(\text{FM}(\mathcal{O}_X, \text{Diff})).
\]

However, we need a brief discussion of these and other derived categories that will be used from now on, which is done in the next remark.

Remark 6.8 (Definition of derived categories). By \(D(\text{FM}(\mathcal{O}_X))\), \(D(\text{FM}(\mathcal{O}_X))\) and all the others, we mean \(D^*\), where \(\ast\) can be either absent or any of \(\ast = -,+,b\). However the definition of these derived categories needs some explanation. I will only do this for \(D(\text{FM}(\mathcal{O}_X))\), as all the others are similar. First, note that \(\text{FM}(\mathcal{O}_X)\) is an additive category which has (co)kernels and (co)images, but it is not in general an abelian category.\(^4\) Hence we are not looking at the derived category associated to an abelian category. We form \(C^*(\text{FM}(\mathcal{O}_X))\), the category of complexes of objects in \(\text{FM}(\mathcal{O}_X)\), and then the homotopy category \(K^*(\text{FM}(\mathcal{O}_X))\), where the homotopies are required to preserve the filtrations. It is not hard to see that \(K^*(\text{FM}(\mathcal{O}_X))\) has a natural structure of triangulated category.\(^5\) Finally, the filtered derived category \(D^*(\text{FM}(\mathcal{O}_X))\) is the localization of \(K^*(\text{FM}(\mathcal{O}_X))\) at the class of filtered quasi-isomorphisms. As with the derived category of an abelian

\(^4\)Given a morphism \(\varphi: \mathcal{M} \to \mathcal{N}\) in \(\text{FM}(\mathcal{O}_X)\), it is not necessarily the case that the induced morphism \(\text{Coim}(\varphi) \to \text{Im}(\varphi)\) is an isomorphism.

\(^5\)Essentially one has to note that the cone of a morphism in \(\text{FM}(\mathcal{O}_X)\) carries a natural filtration, such that all the morphisms in the associated exact triangle are compatible with the filtrations.
category, one can show that there is a unique triangulated structure on $D^*(\text{FM}(\mathcal{D}_X))$ such that the canonical localization functor $K^*(\text{FM}(\mathcal{D}_X)) \to D^*(\text{FM}(\mathcal{D}_X))$ is exact.

On the other hand, the derived category $D(\text{FM}(\mathcal{O}_X, \text{Diff}))$ is obtained by inverting $D$-quasi-isomorphisms in $\text{FM}(\mathcal{O}_X, \text{Diff})$, meaning those morphisms that are mapped to (filtered) quasi-isomorphisms via the functor $\text{DR}^{-1}$.

We next observe that every object in $\text{FM}(\mathcal{D}_X)$ admits a finite resolution by induced filtered $\mathcal{D}_X$-modules, and use this to find a quasi-inverse for the equivalence above. To this end, recall the Spencer complex

$$0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^n T_X \to \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} T_X \to \mathcal{D}_X \to 0,$$

placed in degrees $-n, \cdots, 0$. The differentials are such that this complex is isomorphic in local coordinates $x_1, \ldots, x_n$ to the Koszul complex $K(\mathcal{D}_X; \partial_1, \ldots, \partial_n)[n]$ associated to the (right) action of $\partial_1, \ldots, \partial_n$ on $\mathcal{D}_X$. We consider this to be a complex of filtered left $\mathcal{D}_X$-modules, where the filtration on $\mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^i T_X$ is given by

$$F_k(\mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^i T_X) := F_{k+i} \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^i T_X.$$

This complex is filtered quasi-isomorphic to the left $\mathcal{D}_X$-module $\mathcal{O}_X$ with the trivial filtration; see [HTT, Lemma 1.5.27].

Consider now an arbitrary $(\mathcal{M}, F) \in \text{FM}(\mathcal{D}_X)$. Recall that $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ has a natural right $\mathcal{D}_X$-module structure (see e.g. [HTT, Proposition 1.2.9(ii)]). Applying $\mathcal{M} \otimes_{\mathcal{O}_X} \bullet$ to the complex above corresponds to the Spencer complex of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X$:

$$0 \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^n T_X \to \cdots \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} T_X \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to 0.$$

**Proposition 6.9.** The complex in (6.2) is a complex of filtered induced right $\mathcal{D}_X$-modules, quasi-isomorphic to $(\mathcal{M}, F)$.

**Proof.** By the same [HTT, Proposition 1.2.9(ii)], all the terms $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^i T_X$ have a natural (filtered) right $\mathcal{D}_X$-module structure, and it is not hard to check that there is a filtered isomorphism

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^i T_X \simeq \mathcal{M} \otimes_{\mathcal{O}_X} \wedge^i T_X \otimes_{\mathcal{O}_X} \mathcal{D}_X,$$

with the obvious right $\mathcal{D}_X$-module structure on the right hand side (exercise!). This realizes our complex as a complex of induced $\mathcal{D}_X$-modules, using the natural $\mathcal{O}_X$-module structure on $\mathcal{M} \otimes_{\mathcal{O}_X} \wedge^i T_X$.

On the other hand, there is a natural map

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \mathcal{M}, \quad m \otimes P \mapsto P(1)m,$$

and this is a surjective filtered right $\mathcal{D}_X$-module morphism. Placing this at the right end of the complex in (6.2), the claim is that it induces a filtered quasi-isomorphism between this complex and $(\mathcal{M}, F)$ (which then finishes the proof). This is of course equivalent to saying that the complex in (6.2) has no cohomology except at the right-most term.

---

\(^6\)Note that this map is induced by tensoring with $\mathcal{M}$ over $\mathcal{O}_X$ the natural map $\mathcal{D}_X \to \mathcal{O}_X$, taking an operator $P$ to $P(1)$ (and realizing the quasi-isomorphism between the Spencer complex and $\mathcal{O}_X$).
Let’s reinterpret this for convenience in terms of the corresponding complex in \( \text{FM}(\mathcal{O}_X, \text{Diff}) \). Concretely, our complex is obtained by applying \( \text{DR}^{-1} \) to the complex in \( \text{FM}(\mathcal{O}_X, \text{Diff}) \)

\[
\widetilde{\text{DR}}(\mathcal{M}, F) : 0 \to \mathcal{M} \otimes_{\mathcal{O}_X} \wedge^n T_X \to \cdots \to \mathcal{M} \otimes_{\mathcal{O}_X} T_X \to \mathcal{M} \to 0.
\]

(we are using the notation introduced in [Sa1, §2]), where the filtration on this complex obtained by setting

\[
F_p(\mathcal{M} \otimes_{\mathcal{O}_X} \wedge^i T_X) = F_{p+i} \mathcal{M} \otimes_{\mathcal{O}_X} \wedge^i T_X.
\]

Note that \( \widetilde{\text{DR}}(\mathcal{M}, F) \) is obtained by applying \( (\mathcal{M}, F) \otimes_{\mathcal{O}_X} \bullet \) to the Spencer complex resolving \( \mathcal{O}_X \), and therefore in local coordinates \( x_1, \ldots, x_n \) it is isomorphic to the Koszul complex \( K(\mathcal{M}; \partial_1, \ldots, \partial_n)[n] \) associated to the elements \( \partial_1, \ldots, \partial_n \) acting on \( \mathcal{M} \), hence the assertion. □

Restating the last part of the proof of the Proposition above, we have the following:

**Corollary 6.10.** For every \( (\mathcal{M}, F) \in \text{FM}(\mathcal{D}_X) \) there is a natural quasi-isomorphism of filtered complexes of right \( \mathcal{D}_X \)-modules

\[
\text{DR}^{-1} \widetilde{\text{DR}}(\mathcal{M}, F) \to (\mathcal{M}, F).
\]

Via standard homological algebra, this discussion leads to the following equivalence of filtered derived categories:

**Proposition 6.11.** The natural functor \( \mathcal{D}(\text{FM}(\mathcal{D}_X)) \to \mathcal{D}(\text{FM}(\mathcal{D}_X)) \) is an equivalence of categories.

Moreover, restricted to filtered induced \( \mathcal{D} \)-modules, the functor \( \widetilde{\text{DR}} \) provides a quasi-inverse for the functor \( \text{DR}^{-1} \) in (6.1).

Together with the equivalence in (6.1), the Proposition above shows that in order to study operations on \( \mathcal{D}(\text{FM}(\mathcal{D}_X)) \) we may restrict to complexes of induced \( \mathcal{D} \)-modules, or to filtered differential complexes. We will take advantage of this below, when defining the push-forward functor for filtered \( \mathcal{D} \)-modules.

Proposition 6.9 gives us a canonical approach to finding resolutions by induced \( \mathcal{D} \)-modules. Other explicit resolutions may however be more meaningful and easier to work with. The following extended example is very important for applications.

**Example 6.12 (Localization along an SNC divisor).** Let \( E \) be a reduced simple normal crossing (SNC) divisor on a smooth \( n \)-dimensional variety \( Y \).\(^7\) Recall that \( \omega_Y(*E) \simeq \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(*E) \) stands for the right \( \mathcal{D} \)-module version of the localization along the divisor \( E \), in other words the sheaf of \( n \)-forms with arbitrary poles along \( E \).

We endow \( \omega_Y(*E) \) with what we will call the Hodge filtration, namely

\[
F_k \omega_Y(*E) := \omega_Y(E) \cdot F_{k+n} \mathcal{D}_Y \quad \text{for} \quad k \geq -n.
\]

\(^7\) We use this notation since in practice we will consider this setting on a log resolution \( f: Y \to X \) of a pair \( (X, D) \), with \( E = f^{-1}(D)_{\text{red}} \).
For instance, the first two nonzero terms are
\[ F_{-n}\omega_Y(\ast E) = \omega_Y(E) \quad \text{and} \quad F_{-n+1}\omega_Y(\ast E) = \omega_Y(2E) \cdot \text{Jac}(E), \]
where \( \text{Jac}(E) \) is the Jacobian ideal of \( E \), i.e. \( F_1\mathcal{D}_Y \cdot \mathcal{O}_Y(-E) \), whose zero locus is the singular locus of \( E \).

To gain intuition for what comes next, recall that the right \( \mathcal{D} \)-module \( \omega_Y \) has a standard filtered resolution
\[ 0 \to \mathcal{D}_Y \to \Omega^1_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \cdots \to \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \omega_Y \to 0 \]
by induced \( \mathcal{D}_Y \)-modules. This is simply the resolution of \( \omega_Y \) (with trivial filtration) described by the procedure in Proposition 6.9; see also [HTT, Lemma 1.2.57]. It is a simple check that \( \overline{\text{DR}}(\omega_X) \), i.e. the associated complex in \( \text{FM}(\mathcal{O}_X, \text{Diff}) \), is the standard de Rham complex
\[ 0 \to \mathcal{O}_Y \xrightarrow{d} \Omega^1_Y \to \cdots \to \omega_Y \to 0. \]

A similar type of resolution by right induced \( \mathcal{D}_Y \)-modules can be found for \( \omega_Y(\ast E) \), only this time it will correspond to the de Rham complex with log poles along \( E \).

**Proposition 6.13.** The right \( \mathcal{D}_Y \)-module \( \omega_Y(\ast E) \) has a filtered resolution by induced \( \mathcal{D}_Y \)-modules, given by
\[ 0 \to \mathcal{D}_Y \to \Omega^1_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \cdots \to \omega_Y(\ast E) \to 0. \]

Here the morphism
\[ \omega_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \omega_Y(\ast E) \]
is given by \( \omega \otimes P \to \omega \cdot P \) (the \( \mathcal{D} \)-module operation), and for each \( p \) the morphism
\[ \Omega^p_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \Omega^{p+1}_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \]
is given by \( \omega \otimes P \mapsto d\omega \otimes P + \sum_{i=1}^n (dx_i \wedge \omega) \otimes \partial_i P \), in local coordinates \( x_1, \ldots, x_n \).

**Proof.** It is not hard to check that the expression in the statement is indeed a complex, which we call \( A^\bullet \). We consider on \( \Omega^p_Y(\log E) \) the filtration
\[ F_i\Omega^p_Y(\log E) = \begin{cases} \Omega^p_Y(\log E) & \text{if } i \geq -p \\ 0 & \text{if } i < -p, \end{cases} \]
and on \( \Omega^p_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \) the tensor product filtration. This filters \( A^\bullet \) by subcomplexes \( F_{k-n}A^\bullet \) given by
\[ \cdots \to \Omega^{n-1}_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \omega_Y(E) \otimes_{\mathcal{O}_Y} F_k\mathcal{D}_Y \to F_k\omega_Y(\ast E) \to 0 \]
for each \( k \geq 0 \). Note that they can be rewritten as
\[ \cdots \to \omega_Y(E) \otimes T_Y(- \log E) \otimes_{\mathcal{O}_Y} F_k\mathcal{D}_Y \xrightarrow{\delta_k} \omega_Y(E) \otimes_{\mathcal{O}_Y} F_k\mathcal{D}_Y \to F_k\omega_Y(\ast E) \to 0, \]
where \( T_Y(- \log E) \) is the dual of \( \Omega^1_Y(\log E) \), and we use the isomorphisms \( \omega_Y(E) \otimes \wedge^i T_Y(- \log E) \cong \Omega^{n-i}_Y(\log E) \).

It is clear directly from the definition that every such complex is exact at the term \( F_k\omega_Y(\ast E) \). We now check that they are exact at the term \( \omega_Y(E) \otimes_{\mathcal{O}_Y} F_k\mathcal{D}_Y \). Let us assume
that, in the local coordinates $x_1, \ldots, x_n$, the divisor $E$ is given by $x_1 \cdots x_r = 0$. Using the notation $\omega = dx_1 \wedge \cdots \wedge dx_n$, we consider an element

$$u = \frac{\omega}{x_1 \cdots x_r} \otimes \sum_{|\alpha| \leq k} g_\alpha \partial^\alpha$$

mapping to 0 in $F_k \omega_Y(*E) = \omega_Y(E) \cdot F_k \mathcal{D}_Y$. This means that

$$\sum_{|\alpha| \leq k} \alpha_1! \cdots \alpha_r! \cdot g_\alpha \cdot x_1^{-\alpha_1} \cdots x_r^{-\alpha_r} = 0.$$

We show that $u$ is in the image of the morphism $\beta_k$ by using a descending induction on $|\alpha|$. What we need to prove is the following claim: for each $\alpha$ in the sum above, with $|\alpha| = k$, there exists some $i$ with $\alpha_i > 0$ such that $x_i$ divides $g_\alpha$. If so, an easy calculation shows that the term $u_\alpha = \omega x_1 \cdots x_r \otimes g_\alpha \partial^\alpha$ is in the image of $\beta_k$, and hence it is enough to prove the statement for $u - u_\alpha$. Repeating this a finite number of times, we can reduce to the case when all $|\alpha| \leq k - 1$. But the claim is clear: if $x_i$ did not divide $g_\alpha$ for all $i$ with $\alpha_i > 0$, then the Laurent monomial $x_1^{-\alpha_1} \cdots x_r^{-\alpha_r}$ would appear in the term $g_\alpha \cdot x_1^{-\alpha_1} \cdots x_r^{-\alpha_r}$ of the sum above, but in none of the other terms.

To check the rest of the statement, note that after discarding the term on the right, the associated graded complexes

$$\cdots \rightarrow \omega_Y(E) \otimes \bigwedge^2 T_Y(-\log E) \otimes_{\mathcal{O}_Y} S^{k-2} T_Y \longrightarrow$$

$$\longrightarrow \omega_Y(E) \otimes T_Y(-\log E) \otimes_{\mathcal{O}_Y} S^{k-1} T_Y \longrightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} S^k T_Y \longrightarrow 0$$

are acyclic. Indeed, each such complex is, up to a twist, an Eagon-Northcott complex associated to the inclusion of vector bundles of the same rank

$$\varphi: T_Y(-\log E) \rightarrow T_Y.$$

Concretely, in the notation on [La, p.323], the complex above is $(EN_k)$ tensored by $\omega_Y(E)$. According to [La, Theorem B.2.2(iii)], $(EN_k)$ is acyclic provided that

$$\text{codim } D_{n-\ell}(\varphi) \geq \ell \quad \text{for all } 1 \leq \ell \leq \min\{k, n\},$$

where

$$D_s(\varphi) = \{y \in Y \mid \text{rk}(\varphi_y) \leq s\}$$

are the deneracy loci of $\varphi$. But locally $\varphi$ is given by the diagonal matrix

$$\text{Diag}(x_1, \ldots, x_r, 1, \ldots, 1)$$

so this condition is verified by a simple calculation. \qed

**Remark 6.14.** It is again an immediate check that the associated filtered differential complex $\mathcal{D}(\omega_Y(*E))$ is precisely the well-known de Rham complex of holomorphic forms with log poles along $E$, namely

$$0 \rightarrow \mathcal{O}_Y \xrightarrow{d} \Omega^1_Y(\log E) \xrightarrow{d} \cdots \xrightarrow{d} \omega_Y(E) \longrightarrow 0.$$
7. Push-forward of filtered \( \mathcal{D} \)-modules. We want to enhance the definition of push-forward of \( \mathcal{D} \)-modules discussed in the first part of the course to the filtered setting, following a construction due to Saito [Sa, §2.1-2.3]. This is more natural in the setting of right \( \mathcal{D} \)-modules. The usual left-right transformation allows us to recover the corresponding construction for left \( \mathcal{D} \)-modules.

Let \( f : X \to Y \) be a morphism of smooth complex varieties. Recall that the associated transfer module

\[
\mathcal{D}_{X \to Y} := \mathcal{O}_X \otimes f^{-1} \mathcal{O}_Y f^{-1} \mathcal{D}_Y
\]

has the structure of a \( (\mathcal{D}_X, f^{-1} \mathcal{D}_Y) \)-bimodule, and is used to define the push-forward functor at the level of derived categories by the formula

\[
f_+ : D(\mathcal{D}_X) \to D(\mathcal{D}_Y), \quad M^* \mapsto Rf_*(M^* \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}).
\]

(For details, a good refresher is [HTT, §1.5], where this functor is denoted by \( \int f \).) Here we loosely use the symbol \( D(\mathcal{D}_X) \) to stand for \( D^qcoh(\mathcal{D}_X) \), where \( * \) can be any either absent, or any of \(-, + \) or \( b \) for instance; recall that all the \( \mathcal{D} \)-modules we work with are assumed to be quasi-coherent. However if \( f \) is proper, which is our main focus, this induces a functor

\[
f_+ : D^bcoh(\mathcal{D}_X) \to D^bcoh(\mathcal{D}_Y),
\]

between the bounded derived categories of coherent \( \mathcal{D} \)-modules.

Note furthermore that \( \mathcal{D}_{X \to Y} \) has a natural filtration given by \( f^* F_k \mathcal{D}_Y \). More precisely, the sheaf \( f^{-1} \mathcal{D}_Y \) carries a filtration induced by the standard filtration on \( \mathcal{D}_Y \).

By analogy with the previous section, we can also considered the categories \( FM_i(f^{-1} \mathcal{D}_Y) \) and \( FM_i(f^{-1} \mathcal{D}_Y) \) of filtered \( f^{-1} \mathcal{D}_Y \)-modules and filtered induced \( f^{-1} \mathcal{D}_Y \)-modules respectively, where the latter are isomorphic to filtered \( f^{-1} \mathcal{D}_Y \)-modules of the form \( \mathcal{G} \otimes f^{-1} \mathcal{O}_Y \), with \( \mathcal{G} \) a filtered \( f^{-1} \mathcal{O}_Y \)-module.

We define the functor

\[
DR_{X/Y} : FM_i(\mathcal{D}_X) \to FM_i(f^{-1} \mathcal{D}_Y), \quad (\mathcal{M}, F) \mapsto (\mathcal{M}, F) \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}.
\]

This is indeed well defined, since if \( \mathcal{M} = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X \), with \( \mathcal{G} \) a filtered \( \mathcal{O}_X \)-module, then we have an isomorphism of filtered \( f^{-1} \mathcal{D}_Y \)-modules

\[
DR_{X/Y}(\mathcal{M}) \simeq \mathcal{G} \otimes_{f^{-1} \mathcal{O}_Y} f^{-1} \mathcal{D}_Y.
\]

**Exercise 4.** Show that \( DR_{X/Y} \) takes filtered quasi-isomorphisms of complexes in \( FM_i(\mathcal{D}_X) \) to filtered quasi-isomorphisms of complexes in \( FM_i(f^{-1} \mathcal{D}_Y) \).

We next use the definitions and notation on derived categories discussed in Remark 6.8. Given the exercise above, we have an induced functor

\[
DR_{X/Y} : D(FM_i(\mathcal{D}_X)) \to D(FM_i(f^{-1} \mathcal{D}_Y)).
\]

In combination with Proposition 6.11 and its obvious analogue, we can in turn see this as a functor

\[
DR_{X/Y} : D(FM(\mathcal{D}_X)) \to D(FM(f^{-1} \mathcal{D}_Y)).
\]
We next define a direct image functor
\[ D(\text{FM}(f^{-1}\mathcal{D}_Y)) \rightarrow D(\text{FM}(\mathcal{D}_Y)), \]
which composed with \( \text{DR}_{X/Y} \) will give rise to our desired filtered direct image functor
\[ f_*: D(\text{FM}(\mathcal{D}_X)) \rightarrow D(\text{FM}(\mathcal{D}_Y)). \]

**Definition 7.1.** Let \((\mathcal{M}, F)\) be a filtered \( f^{-1}\mathcal{D}_Y\)-module. Its *topological direct image* is defined as
\[ f_*^{\mathcal{N}}(\mathcal{M}, F) = (\mathcal{N}, F), \]
where
\[ \mathcal{N} := \bigcup_{k \in \mathbb{Z}} f_* F_k \mathcal{M} \subseteq f_* \mathcal{M} \quad \text{and} \quad F_k \mathcal{N} := f_* F_k \mathcal{M}. \]

Here on the right hand side we use the standard sheaf-theoretic direct image.

**Remark 7.2.** This definition can be made in great generality, and usually it is not necessarily the case that \( \mathcal{N} = f_* \mathcal{M} \). However this is always true in the case we are interested in, namely the case of algebraic varieties (since every open set is quasi-compact), and also in the case of complex analytic varieties if \( f \) is proper.

We therefore obtain a functor
\[ f_*: \text{FM}(f^{-1}\mathcal{D}_Y) \rightarrow \text{FM}(\mathcal{D}_Y). \]

We would like to extend this functor to the derived category \( D(\text{FM}(f^{-1}\mathcal{D}_Y)) \), in order to finish our construction.

First recall that to every module \( \mathcal{M} \) over a sheaf of rings on \( X \), in particular over \( f^{-1}\mathcal{D}_Y \), we can associate the flasque sheaf of discontinuous sections \( \mathcal{I}^0(\mathcal{M}) \) defined on every open set \( U \subseteq X \) by
\[ \Gamma(U, \mathcal{I}^0(\mathcal{M})) = \prod_{x \in U} \mathcal{M}_x, \]
and we have a functorial inclusion \( \mathcal{M} \hookrightarrow \mathcal{I}^0(\mathcal{M}) \).

Let’s now consider a filtered version. To \((\mathcal{M}, F)\in\text{FM}(f^{-1}\mathcal{D}_Y)\) we associate \( \mathcal{I}_0(\mathcal{M}, F) \), namely the filtered \( f^{-1}\mathcal{D}_Y\)-module \((\mathcal{N}, F)\) given by
\[ \mathcal{N} := \bigcup_{k \in \mathbb{Z}} \mathcal{I}^0(F_k \mathcal{M}) \subseteq \mathcal{I}^0(\mathcal{M}) \quad \text{and} \quad F_k \mathcal{N} := \mathcal{I}^0(F_k \mathcal{M}). \]

We have a filtered inclusion \((\mathcal{M}, F) \hookrightarrow \mathcal{I}^0(\mathcal{M}, F), \) and we define
\[ \mathcal{I}^1(\mathcal{M}, F) : = \text{Coker}(i). \]

Continuing in this fashion, we obtain a complex
\[ 0 \rightarrow \mathcal{I}^0(\mathcal{M}, F) \rightarrow \mathcal{I}^1(\mathcal{M}, F) \rightarrow \cdots \]
in \( C^+(\text{FM}(f^{-1}\mathcal{D}_Y)) \) which is filtered quasi-isomorphic to \((\mathcal{M}, F), \) and consequently a functor
\[ \mathcal{I}^\bullet: \text{FM}(f^{-1}\mathcal{D}_Y) \rightarrow C^+(\text{FM}(f^{-1}\mathcal{D}_Y)). \]
We thus obtain a finite \( \text{in C} \) of \( \tilde{\mathcal{I}} \) over, all the entries and \( X \).

This follows by construction and the assumption on \( N \). Note now that in the context we are considering, basic properties of higher direct images tell us that there exists an integer \( N > 0 \) such that for every sheaf \( \mathcal{F} \) of abelian groups on \( X \) we have \( R^i f_* \mathcal{F} = 0 \) for \( i > N \). (In the setting of algebraic varieties we can in fact take \( N = \dim X \).) We modify our resolution by taking

\[
\tilde{\mathcal{I}}^j(M, F) = \mathcal{I}^j(M, F) \quad \text{for} \quad j \leq N, \quad \tilde{\mathcal{I}}^j(M, F) = 0 \quad \text{for} \quad j > N + 1,
\]

and

\[
\tilde{\mathcal{I}}^{N+1}(M, F) = \text{Coker}(\mathcal{I}^{N-1}(M, F) \to \mathcal{I}^N(M, F)).
\]

We thus obtain a finite resolution

\[
\tilde{\mathcal{I}}^\bullet(M, F) : 0 \to \tilde{\mathcal{I}}^0(M, F) \to \tilde{\mathcal{I}}^1(M, F) \to \cdots \to \tilde{\mathcal{I}}^N+1(M, F) \to 0
\]

of \( (M, F) \) with filtered sheaves having the same properties as those in \( \mathcal{I}^\bullet(M, F) \). Moreover, all the entries \( \tilde{\mathcal{I}}^j(M, F) \) are filtered \( f \)-acyclic in the sense that

\[
R^i f_*(F_k \tilde{\mathcal{I}}^j(M, F)) = 0 \quad \text{for all} \quad k \in \mathbb{Z}, \quad i > 0.
\]

This follows by construction and the assumption on \( N \) (exercise!).

We can extend this construction to complexes. If \( C^\bullet = (\mathcal{M}^\bullet, F_\bullet \mathcal{M}^\bullet) \) is an object in \( C(\text{FM}(f^{-1} \mathcal{D}_Y)) \), we can form the double complex \( \tilde{\mathcal{I}}^p(\mathcal{M}^q, F_\bullet \mathcal{M}^q) \) and define \( \tilde{\mathcal{I}}^\bullet(C^\bullet) \) to be the total complex of this double complex. We thus have a functor

\[
\tilde{\mathcal{I}}^\bullet : \text{C}(\text{FM}(f^{-1} \mathcal{D}_Y)) \to \text{C}(\text{FM}(f^{-1} \mathcal{D}_Y))
\]

such that \( C^\bullet \) is filtered quasi-isomorphic to \( \tilde{\mathcal{I}}^\bullet(C^\bullet) \), with \(-, + \) and bounded versions.

Finally, this allows us to define the exact functor of triangulated categories we are interested in, as

\[
Rf_* : \text{D}(\text{FM}(f^{-1} \mathcal{D}_Y)) \to \text{D}(\text{FM}(\mathcal{D}_Y)), \quad Rf_* C^\bullet := f_*(\tilde{\mathcal{I}}^\bullet(C^\bullet)).
\]

This functor is well defined thanks to the following

**Exercise 5.** Show that if \( A^\bullet \to B^\bullet \) is a filtered quasi-isomorphism in \( C(\text{FM}(f^{-1} \mathcal{D}_Y)) \), then the induced \( f_*(\tilde{\mathcal{I}}^\bullet(A^\bullet)) \to f_*(\tilde{\mathcal{I}}^\bullet(B^\bullet)) \) is a filtered quasi-isomorphism as well.

The following property is a direct consequence of the definition and of filtered \( f \)-acyclicity:

**Corollary 7.3.** If \( C^\bullet \) represents an object in \( \text{D}(\text{FM}(f^{-1} \mathcal{D}_Y)) \), then

\[
\mathcal{H}^i F_k(Rf_* C^\bullet) \simeq R^i f_*(F_k C^\bullet).
\]

Note that this allows us to obtain the filtration on each \( R^i f_* C^\bullet \) as follows:

\[
F_k R^i f_* C^\bullet = \text{Im}[\mathcal{H}^i F_k(Rf_* C^\bullet) \to \mathcal{H}^i(Rf_* C^\bullet)] = \text{Im}[R^i f_*(F_k C^\bullet) \to R^i f_* C^\bullet].
\]

It is however not necessarily the case that this last map is injective, and therefore the filtration is in general not simply given by \( R^i f_*(F_k C^\bullet) \). That this is actually true for those filtered \( \mathcal{D} \)-modules that underlie Hodge modules is a deep property of Hodge-theoretic flavor that we will analyze in the next section.
Finally, as mentioned above, composing $Rf_*$ with $\text{DR}_{X/Y}$, we obtain the desired filtered direct image functor
\[ f_+: D(\text{FM}(\mathcal{D}_X)) \rightarrow D(\text{FM}(\mathcal{D}_Y)). \]
If $f$ is proper, this induces a functor
\[ f_+: D^b(\text{cof}(\text{FM}(\mathcal{D}_X))) \rightarrow D^b(\text{cof}(\text{FM}(\mathcal{D}_Y))). \]

Remark 7.4. It is immediate from the definitions that if we forget the filtration, this functor coincides with the usual direct image functor $f_+$ on the derived category of $\mathcal{D}_X$-modules recalled at the beginning of this section.

8. Strictness. A special property that is crucial in the theory of filtered $\mathcal{D}$-modules underlying Hodge modules is the strictness of the filtration.

Definition 8.1. Let $f: (\mathcal{M}, F) \rightarrow (\mathcal{N}, F)$ be a morphism of filtered $\mathcal{D}_X$-modules. Then $f$ is called strict if
\[ f(F_k\mathcal{M}) = F_k\mathcal{N} \cap f(\mathcal{M}) \quad \text{for all } k. \]
Similarly, a complex of filtered $\mathcal{D}_X$-modules $(\mathcal{M}^\bullet, F^\bullet\mathcal{M}^\bullet)$ is called strict if all of its differentials are strict. Via a standard argument, the notion of strictness makes sense more generally for objects in the derived category $D(\text{FM}(\mathcal{D}_X))$ of filtered $\mathcal{D}_X$-modules.

Note that it is only in the case of a strict complex that the cohomologies of $\mathcal{M}^\bullet$ can also be seen as filtered $\mathcal{D}_X$-modules. An equivalent interpretation is given by the following:

Exercise .6. The complex $(\mathcal{M}^\bullet, F^\bullet\mathcal{M}^\bullet)$ is strict if and only if, for every $i, k \in \mathbb{Z}$, the induced morphism
\[ H^i F_k\mathcal{M}^\bullet \rightarrow H^i \mathcal{M}^\bullet \]
is injective.

As a preview, a crucial property of the filtered $\mathcal{D}$-modules underlying Hodge modules will be the following. If $f: X \rightarrow Y$ is a proper morphism of smooth varieties, and $(\mathcal{M}, F)$ is one such filtered $\mathcal{D}_X$-module, then $f_+(\mathcal{M}, F)$ is strict as an object in $D(\text{FM}(\mathcal{D}_Y))$; here $f_+$ is the filtered direct image functor discussed in the previous section. By the Exercise above, this means that
\[ H^i F_kf_+(\mathcal{M}, F) \rightarrow H^i f_+(\mathcal{M}, F) \]
is injective for all integers $i$ and $k$. By Corollary 7.3 and the discussion right after, this is equivalent to the injectivity of the natural morphism

\[ R^if_*\left( F_k(M \otimes_{\mathcal{D}_X} \mathcal{D}_{X \rightarrow Y}) \right) \rightarrow R^if_*\left( M \otimes_{\mathcal{D}_X} \mathcal{D}_{X \rightarrow Y} \right). \]

Moreover, the image of this morphism, isomorphic to the term on the left hand side, is the term $F_kH^if_+(\mathcal{M}, F)$.

As a conclusion, in the strict case the cohomologies of direct images of filtered $\mathcal{D}$-modules are themselves filtered $\mathcal{D}$-modules, and sometimes it’s possible to have a reasonably good grasp of the filtration on such direct images.
Example 8.2 (Absolute case). The absolute case gives a good idea of the meaning of strictness, and how it is natural in Hodge theory. In this context it can be seen as a generalization of the degeneration at $E_1$ of the classical Hodge-to-de Rham spectral sequence. Concretely, let $X$ be a smooth variety, and $(\mathcal{M}, F)$ a filtered $\mathcal{D}$-module on $X$. The natural inclusion of complexes $F_k \text{DR}(\mathcal{M}) \hookrightarrow \text{DR}(\mathcal{M})$ induces, after passing to cohomology, a morphism

$$\varphi_{k,i} : H^i(X, F_k \text{DR}(\mathcal{M})) \longrightarrow H^i(X, \text{DR}(\mathcal{M})).$$

Now for the constant map $f : X \rightarrow \text{pt}$, the definition of pushforward gives

$$f_+ \mathcal{M} \simeq \mathbb{R} \Gamma(X, \text{DR}(\mathcal{M})), $$

and by the discussion above, the image of $\varphi_{k,i}$ is $F_k H^i(X, \text{DR}(\mathcal{M}))$. The strictness of $f_+(\mathcal{M}, F)$ is captured by the injectivity of $\varphi_{k,i}$ for all $k$ and $i$, which is in turn equivalent to

$$\text{gr}_k^F H^i(X, \text{DR}(\mathcal{M})) \simeq H^i(X, \text{gr}_k^F \text{DR}(\mathcal{M})).$$

It is not hard to see that this is the same as the $E_1$-degeneration of the Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = H^{p+q}(X, \text{gr}_q^F \text{DR}(\mathcal{M})) \implies H^{p+q}(X, \text{DR}(\mathcal{M})).$$

where on the left hand side we have the hypercohomology of a complex of coherent $\mathcal{O}_X$-modules, and on the right hand side the cohomology of the perverse sheaf $\text{DR}(\mathcal{M})$.

For instance, when $X$ is projective and $\mathcal{M}$ is given by the trivial VHS $\mathbb{C}_X$, then this is the degeneration of the classical Hodge-to-de Rham spectral sequence, and same for any other VHS; see Examples 1.2, 1.3 and 4.3. This property extends to the filtered $\mathcal{D}$-modules associated to Hodge modules on $X$.

9. Localization and Hodge ideals. We now focus on one extended example, where many of the definitions and techniques in the previous sections can be seen in action.

Throughout this section $X$ is a smooth complex variety of dimension $n$ and $D$ is a reduced effective divisor on $X$. Recall that to this data we can associate the left $\mathcal{D}_X$-module of functions with poles along $D$,

$$\mathcal{O}_X(*D) = \bigcup_{k \geq 0} \mathcal{O}_X(kD),$$

i.e. the localization of $\mathcal{O}_X$ along $D$. If $f$ is a local defining equation for $D$, then this is $\mathcal{O}_X[\frac{1}{f}]$, with the obvious action of differential operators. The associated right $\mathcal{D}_X$-module is denoted $\omega_X(*D)$.

This $\mathcal{D}$-module underlies the mixed Hodge module $j_* \mathbb{Q}_U^H[n]$, where $U = X \setminus D$ and $j : U \hookrightarrow X$ is the inclusion map. It therefore comes with an attached Hodge filtration $F_k \omega_X(*D)$, where $k \geq 0$. When $D$ is an SNC divisor we have already seen this in Example 5.9, while in general one uses resolution of singularities and the push-forward functor; see Lemma 9.6 below, and the discussion thereafter.
Note that there is also a more obvious filtration on \( \mathcal{O}_X(\ast D) \), namely the *pole order filtration*, whose nonzero terms are taken by convention to be

\[
P_k \mathcal{O}_X(\ast D) = \mathcal{O}_X((k+1)D) \quad \text{for} \quad k \geq 0.
\]

This filtration is in general too coarse to be of much use; it is usually not even a good filtration. We will see however that it is very useful to measure the Hodge filtration against it, and indeed we have:

**Lemma 9.1.** For every \( k \geq 0 \), we have an inclusion

\[
F_k \mathcal{O}_X(\ast D) \subseteq P_k \mathcal{O}_X(\ast D).
\]

It is possible to give a proof of this Lemma using the interaction between the \( F \)-filtration and the \( V \)-filtration. We will however give a proof using resolution of singularities below.

**Definition 9.2 (Hodge ideals).** Given the inclusion in Lemma 9.1, for each \( k \geq 0 \) we define the ideal sheaf \( I_k(D) \) on \( X \) by the formula

\[
F_k \mathcal{O}_X(\ast D) = \mathcal{O}_X((k+1)D) \otimes I_k(D).
\]

We call \( I_k(D) \) the \( k \)-th Hodge ideal of \( D \).

In what follows we will analyze these ideals (and in the process prove Lemma 9.1 on which their definition relies) by first looking at the SNC case and then making use of log resolutions and filtered push-forward for Hodge modules.

**Simple normal crossings case.** When \( D \) is a simple normal crossing divisor, we saw in Example 5.9 that

\[
F_k \mathcal{O}_X(\ast D) = F_k \mathcal{D}_X \cdot \mathcal{O}_X(D).
\]

**Exercise .7.** Verify directly in this setting that indeed \( F_k \mathcal{D}_X \cdot \mathcal{O}_X(D) \subseteq \mathcal{O}_X((k+1)D) \) for all \( k \geq 0 \).

We conclude that in this case the ideals \( I_k(D) \) are given by the following expression:

\[
(9.1) \quad I_k(D) = (F_k \mathcal{D}_X \cdot \mathcal{O}_X(D)) \otimes \mathcal{O}_X(-(k+1)D) \quad \text{for all} \quad k \geq 0, \quad ^{8}
\]

Note that in particular \( I_0(D) = \mathcal{O}_X \). We can in fact completely describe a set of generators for all of these ideals in local coordinates.

**Proposition 9.3.** Suppose that around a point \( p \in X \) we have coordinates \( x_1, \ldots, x_n \) such that \( D \) is defined by \( (x_1 \cdots x_r = 0) \). Then, for every \( k \geq 0 \), the ideal \( I_k(D) \) is generated around \( p \) by

\[
\{ x_1^{a_1} \cdots x_r^{a_r} \mid 0 \leq a_i \leq k, \sum_i a_i = k(r-1) \}.
\]

In particular, if \( r = 1 \) (that is, when \( D \) is smooth), we have \( I_k(D) = \mathcal{O}_X \) and if \( r = 2 \), then \( I_k(D) = (x_1, x_2)^k \).

\(^8\text{We will see below that one can also define Hodge ideals directly using log resolutions, without appealing to the theory of Hodge modules, in which case we would need to take this as the definition.} \)
Proof. It is clear that $F_k \mathcal{D}_X \cdot \mathcal{O}_X(D)$ is generated as an $\mathcal{O}_X$-module by 
\[ \{ x_1^{-b_1} \cdots x_r^{-b_r} \mid b_i \geq 1, \sum b_i = r + k \}. \]
According to (9.1), the expression for $I_k(D)$ now follows by multiplying these generators by $(x_1 \cdots x_r)^{k+1}$. The assertions in the special cases $r = 1$ and $r = 2$ are clear. □

For convenience, let’s record separately the case $r = 1$ in the Proposition above.

Corollary 9.4. If $D$ is a smooth divisor, then $I_k(D) = \mathcal{O}_X$ for all $k \geq 0$.

Remark 9.5. It turns out that the converse of this statement holds as well, but this requires some serious work; see [MP1, Theorem A].

The general case. When $D$ is arbitrary, we consider a log resolution $f : Y \to X$ of the pair $(X, D)$ which is an isomorphism over $U = X \setminus D$, and let $E = (f^*D)_{\text{red}}$, an SNC divisor.

Lemma 9.6. There is a natural isomorphism
\[ f_+ \omega_Y(*E) \simeq H^0 f_+ \omega_Y(*E) \simeq \omega_X(*D). \]

Proof. Let $V = Y \setminus E = f^{-1}(U)$, and denote by $j_U$ and $j_V$ the inclusions of $U$ and $V$ into $X$ and $Y$ respectively. Note that $j_U^* \omega_U \simeq \omega_X(*D)$ and $j_V^* \omega_V \simeq \omega_Y(*E)$. The result then follows from the fact that $j_U = f \circ j_V$. □

We therefore need to understand the isomorphism of filtered $\mathcal{D}$-modules
\[ f_+ (\omega_Y(*E), F) \simeq (\omega_X(*D), F), \]
where in both cases we consider the Hodge filtration underlying the mixed Hodge module structure, and the filtered push-forward is the construction described in Section 7. Since
\[ f_+ \omega_Y(*E) \simeq Rf_*(\omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \to X}), \]
the first step is to consider a convenient (filtered) representative for $\omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \to X}$. It turns out that this object is supported only in degree zero; even though not entirely necessarily, let’s go one step further and provide a precise description.

First, recall that since $\mathcal{D}_{Y \to X}$ is a left $\mathcal{D}_Y$-module, we have a canonical morphism of $(\mathcal{D}_Y, f^{-1}\mathcal{O}_X)$ bimodules
\[ \varphi : \mathcal{D}_Y \longrightarrow f^* \mathcal{D}_X \]
that maps 1 to 1, and is clearly an isomorphism over $V = Y \setminus E$. Since $\mathcal{D}_Y$ is torsion-free, we conclude that $\varphi$ is injective, with cokernel supported on $E$. For each $k$, we have induced inclusions $F_k \mathcal{D}_Y \hookrightarrow f^* F_k \mathcal{D}_X$ of $(\mathcal{O}_Y, f^{-1}\mathcal{O}_X)$ bimodules.

Proposition 9.7. The canonical morphism induced by $\varphi$ in the derived category of right $f^{-1}\mathcal{O}_X$-modules
\[ \omega_Y(*E) \longrightarrow \omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \to X} \]
is an isomorphism.
The proof we give below is inspired in part by arguments in [HTT, §5.2].

**Lemma 9.8.** The induced morphism
\[ \text{Id} \otimes \varphi : \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \]
is an isomorphism.

**Proof.** It suffices to show that the induced mappings
\[ \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} F_k \mathcal{D}_Y \rightarrow \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} f^* F_k \mathcal{D}_X \]
are all isomorphisms for \( k \geq 0 \). But this follows immediately from Lemma 9.9 below (note that since \( \mathcal{O}_Y(*E) \) is flat over \( \mathcal{O}_Y \), these maps are injective). \( \square \)

In the proof above we used the following well-known observation; see [HTT, Lemma 5.2.7].

**Lemma 9.9.** If \( \mathcal{F} \) is a coherent \( \mathcal{O}_X(*D) \)-module supported on \( D \), then \( \mathcal{F} = 0 \).

Let us now use the notation
\[ \mathcal{D}_Y(*E) := \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y. \]
This is a sheaf of rings, and one can identify it with the subalgebra of \( \text{End}_{\mathcal{O}_Y}(\mathcal{O}_Y(*E)) \) generated by \( \mathcal{D}_Y \) and \( \mathcal{O}_Y(*E) \). Note that since \( \mathcal{O}_Y(*E) \) is a flat \( \mathcal{O}_Y \)-module, we have \( \mathcal{D}_Y(*E) \simeq \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \). A basic fact is the following:

**Lemma 9.10.** The canonical morphism
\[ \mathcal{D}_Y(*E) \rightarrow \mathcal{D}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow \mathcal{D}_Y(*E) \]
induced by \( \varphi \) is an isomorphism.

**Proof.** Via the isomorphism \( \mathcal{D}_Y(*E) \simeq \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \), the morphism in the statement gets identified to the morphism
\[ \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \]
induced by \( \varphi \). Moreover, since \( \mathcal{O}_Y(*E) \) is a flat \( \mathcal{O}_Y \)-module, the morphism (9.2) gets identified with the isomorphism in Lemma 9.8. \( \square \)

**Proof of Proposition 9.7.** Via the right \( \mathcal{D} \)-module structure on \( \omega_Y \), we have that \( \omega_Y(*E) \) has a natural right \( \mathcal{D}_Y(*E) \)-module structure. The morphism in the proposition gets identified with the morphism
\[ \omega_Y(*E) \otimes_{\mathcal{O}_Y(*E)} \mathcal{D}_Y(*E) \rightarrow \omega_Y(*E) \otimes_{\mathcal{O}_Y(*E)} \mathcal{D}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow \mathcal{D}_Y(*E) \]
induced by \( \varphi \). In turn, this is obtained by applying \( \omega_Y(*E) \otimes_{\mathcal{O}_Y(*E)} \) to the isomorphism in Lemma 9.10, hence it is an isomorphism. \( \square \)
Next we address the filtration. On the tensor product $\omega_Y(*E) \otimes \mathcal{D}_Y \to X$ we consider the tensor product filtration, that is,

$$F_k(\omega_Y(*E) \otimes \mathcal{D}_Y \to X) := \text{Im} \left[ \bigoplus_{i \geq -n} F_i \omega_Y(*E) \otimes \mathcal{D}_Y \to \omega_Y(*E) \otimes \mathcal{D}_Y \to X \right],$$

where the map in the parenthesis is the natural map between the tensor product over $\mathcal{O}_Y$ and that over $\mathcal{D}_Y$.

**Lemma 9.11.** The definition above simplifies to

$$F_k(\omega_Y(*E) \otimes \mathcal{D}_Y \to X) = \text{Im} [\omega_Y(E) \otimes \mathcal{O}_Y, f^* F_{k+n} \mathcal{D}_X \to \omega_Y(*E) \otimes \mathcal{D}_Y \to X].$$

**Proof.** Fix $i \geq -n$ and recall that $F_i \omega_Y(*E) = \omega_Y(E) \cdot F_i \mathcal{D}_Y$. The factor $F_i \mathcal{D}_Y$ can be moved over the tensor product once we pass to the image in the tensor product over $\mathcal{D}_Y$, and moreover we have an inclusion

$$F_{i+n} \mathcal{D}_Y \cdot f^* F_{k-i} \mathcal{D}_X \subseteq f^* F_{k+n} \mathcal{D}_X.$$

Therefore inside $\omega_Y(*E) \otimes \mathcal{D}_Y \to X$, the image of $F_i \omega_Y(*E) \otimes \mathcal{O}_Y, f^* F_{k-i} \mathcal{D}_X$ is contained in the image of $\omega_Y(E) \otimes \mathcal{O}_Y, f^* F_{k+n} \mathcal{D}_X$. \hfill $\square$

Propositions 6.13 and 9.7 have the following immediate consequence:

**Corollary 9.12.** On $Y$ there is a filtered complex of right $f^{-1} \mathcal{D}_X$-modules

$$0 \to f^* \mathcal{D}_X \to \Omega^1_Y(\log E) \otimes \mathcal{O}_Y, f^* \mathcal{D}_X \to \cdots$$

$$\cdots \to \Omega^{n-1}_Y(\log E) \otimes \mathcal{O}_Y, f^* \mathcal{D}_X \to \omega_Y(E) \otimes \mathcal{O}_Y, f^* \mathcal{D}_X \to \omega_Y(*E) \otimes \mathcal{O}_Y, \mathcal{D}_Y \to X \to 0$$

which is exact (though not necessarily filtered exact).

**Proof.** It follows from Proposition 6.13 that the complex

$$0 \to f^* \mathcal{D}_X \to \Omega^1_Y(\log E) \otimes \mathcal{O}_Y, f^* \mathcal{D}_X \to \cdots \to \omega_Y(E) \otimes \mathcal{O}_Y, f^* \mathcal{D}_X \to 0$$

represents the object $\omega_Y(*E) \otimes \mathcal{O}_Y, \mathcal{D}_Y \to X$ in the derived category, hence Proposition 9.7 implies the exactness of the entire complex in the statement. \hfill $\square$

With these preparations in place, let’s go back to understanding Hodge ideals. We denote by $A^\bullet$ the complex

$$0 \to f^* \mathcal{D}_X \to \Omega^1_Y(\log E) \otimes \mathcal{O}_Y, f^* \mathcal{D}_X \to \cdots \to \omega_Y(E) \otimes \mathcal{O}_Y, f^* \mathcal{D}_X \to 0$$

placed in degrees $-n, \ldots, 0$. It is filtered by the subcomplexes $C^\bullet_{k-n} = F_{k-n} A^\bullet$ of $A^\bullet$, for $k \geq 0$, given by

$$0 \to f^* F_{k-n} \mathcal{D}_X \to \Omega^1_Y(\log E) \otimes \mathcal{O}_Y, f^* F_{k-n+1} \mathcal{D}_X \to \cdots$$

$$\cdots \to \Omega^{n-1}_Y(\log E) \otimes \mathcal{O}_Y, f^* F_{k-1} \mathcal{D}_X \to \omega_Y(E) \otimes \mathcal{O}_Y, f^* F_k \mathcal{D}_X \to 0.$$

With this filtration, it represents the object

$$\omega_Y(*E) \otimes \mathcal{O}_Y, \mathcal{D}_Y \to X \simeq \omega_Y(*E) \otimes \mathcal{O}_Y, \mathcal{D}_Y \to X$$
Filtered \( \mathcal{D} \)-modules

in the derived category of filtered right \( f^{-1}\mathcal{D}_X \)-modules. In other words, in the language of Section 7, we have

\[
\text{DR}_{Y/X}((\omega_Y(*E), F)) \simeq (A^*, F).
\]

**Remark 9.13.** Moreover, we have seen in Corollary 9.12 that the natural mapping

\[
A^* \rightarrow H^0 A^* \simeq \omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_Y \rightarrow X
\]

is a quasi-isomorphism.; again though, this is not necessarily a filtered quasi-isomorphism, meaning \( C_{k-n}^* \) are not necessarily exact away from the right-most term.

Lemma 9.6 combined with Corollary 7.3 and the subsequent discussion then gives

\[(9.3) \quad F_{k-n}\omega_X(*D) = F_{k-n} \mathcal{H}^0 f_* \omega_Y(*E) \simeq \text{Im} \left[ R^0 f_* C_{k-n}^* \rightarrow R^0 f_* A^* \right],\]

where the map of \( \mathcal{O}_X \)-modules in the parenthesis is induced by the inclusion \( C_{k-n}^* \hookrightarrow A^* \).

With this description, we are now finally able to prove the crucial Lemma 9.1.

**Proof of Lemma 9.1.** Passing to right \( \mathcal{D} \)-modules, equivalently we need to show

\[
F_{k-n}\omega_X(*D) \subseteq \omega_X((k+1)D)
\]

for all \( k \). Since \( D \) is reduced, we can find an open subset \( U \subseteq X \) with the property that \( \text{codim}(X \smallsetminus U, U) \geq 2 \), the induced morphism \( f^{-1}(U) \rightarrow U \) is an isomorphism, and \( D|_U \) is a smooth (possibly disconnected) divisor. Let \( j: U \hookrightarrow X \) be the inclusion. By assumption, on \( f^{-1}(U) \) we have \( \mathcal{D}_{Y \rightarrow X} = \mathcal{D}_Y \). From Proposition 6.13, on \( U \) we obtain

\[
F_{k-n}\omega_X(*D) = \omega_X((k+1)D).
\]

Now \( F_{k-n}\omega_X(*D) \) is torsion-free, being a subsheaf of \( \omega_X(*D) \), and so the following canonical map is injective:

\[
F_{k-n}\omega_X(*D) \rightarrow j_*(F_{k-n}\omega_X(*D)|_U) = j_*(\omega_X((k+1)D)|_U) = \omega_X((k+1)D).
\]

This completes the proof. \( \square \)

Recall that this means that we can define a sequence of coherent sheaves of ideals \( I_k(D) \) associated to \( D \), by the formula

\[
F_{k-n}\omega_X(*D) = \omega_X((k+1)D) \otimes I_k(D).
\]

**Remark 9.14.** The formula above, together with (9.3), can be alternatively taken as a definition of the Hodge ideals \( I_k(D) \) using log resolutions, without appealing to the theory of mixed Hodge modules. In this approach however, one has to prove that the definition is independent of the choice of log resolution, which can be done using a Nakano-type local vanishing theorem; see [MP1, Theorem 11.1].

**Computations for** \( k = 0, 1 \). The first in the sequence of Hodge ideals can be identified with a multiplier ideal; for the general theory of multiplier ideals see [La, Ch.9] or Section 6 in the notes on the Bernstein-Sato polynomial in this course.

**Proposition 9.15.** We have

\[
I_0(D) = \mathcal{J}(X, (1 - \epsilon)D),
\]

the multiplier ideal associated to the \( \mathbb{Q} \)-divisor \((1 - \epsilon)D\) on \( X \), for any \( 0 < \epsilon \ll 1 \).
Proof. Recall that $C_{-n} = \omega_Y(E)$, hence
\[ F_0 \omega_X(*D) = \text{Im}(f_\ast \omega_Y(E) \to \omega_X(D)) = f_\ast \mathcal{O}_Y(K_{Y/X} + E - f^\ast D) \otimes \omega_X(D). \]
Therefore the statement to be proved is that
\[ f_\ast \mathcal{O}_Y(K_{Y/X} + E - f^\ast D) = \mathcal{J}(X, (1 - \epsilon)D). \]
On the other hand, the right-hand side is by definition
\[ f_\ast \mathcal{O}_Y(K_{Y/X} - \lfloor (1 - \epsilon)f^\ast D \rfloor) = f_\ast \mathcal{O}_Y(K_{Y/X} + (f^\ast D)_{\text{red}} - f^\ast D), \]
which implies the desired equality. □

The following is a direct consequence of the definition; see Exercise 10 in the notes on the Bernstein-Sato polynomial.

**Corollary 9.16.** We have $\mathcal{I}_0(D) = \mathcal{O}_X$ if and only if the pair $(X, D)$ is log-canonical.

We now move to a description of the case $k = 1$. We begin by noting the following fact about the complex $C_{1-n}$.

**Lemma 9.17.** The morphism
\[ \Omega_{Y}^{n-1}(\log E) \longrightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} f^\ast F_1 \mathcal{D}_X \]
is injective.

**Proof.** Note that we have a commutative diagram
\[
\begin{array}{ccc}
\Omega_{Y}^{n-1}(\log E) & \xrightarrow{\beta} & \omega_Y(E) \otimes_{\mathcal{O}_Y} F_1 \mathcal{D}_Y \\
\downarrow{\text{Id}} & & \downarrow{\gamma} \\
\Omega_{Y}^{n-1}(\log E) & \xrightarrow{\alpha} & \omega_Y(E) \otimes_{\mathcal{O}_Y} f^\ast F_1 \mathcal{D}_X,
\end{array}
\]
in which $\gamma$ is the canonical inclusion. Since $\beta$ is injective by Proposition 6.13, it follows that $\alpha$ is injective, too. □

Denoting by $\mathcal{F}_1$ the cokernel of the map in the Lemma above, we therefore have a short exact sequence
\[ 0 \to \Omega_{Y}^{n-1}(\log E) \to \omega_Y(E) \otimes f^\ast F_1 \mathcal{D}_X \to \mathcal{F}_1 \to 0, \]
hence in particular $\mathcal{F}_1$ is quasi-isomorphic to the complex $C_{1-n}$. We have:

**Corollary 9.18.** We have an isomorphism
\[ \omega_X(2D) \otimes I_1(D) \simeq f_\ast \mathcal{F}_1. \]

**Proof.** It is clear by definition and the exact sequence above that
\[ R^0 f_\ast C_{1-n} \simeq f_\ast \mathcal{F}_1. \]
By definition we therefore have that $\omega_X(2D) \otimes I_1(D)$ is isomorphic to the image of the canonical morphism $f_\ast \mathcal{F}_1 \to \omega_X(*D)$. However it is not hard to see that this morphism
is injective; in this case this can be seen directly (exercise!), but note that it is in fact a
special case of the general Corollary 9.21 (1) below.

\[ \square \]

**Corollary 9.19.** There is a four-term exact sequence

\[ 0 \to f_* \Omega_Y^1(\log E) \to \omega_X(D) \otimes I_0(D) \otimes F_1 \mathcal{D}_X \xrightarrow{\psi} \omega_X(2D) \otimes I_1(D) \to R^1 f_* \Omega_Y^1(\log E) \to 0, \]

where the image of the map \( \psi \) corresponds to the image of the natural filtered \( \mathcal{D}_X \)-module map

\[ F_{-n} \omega_X(*D) \cdot F_1 \mathcal{D}_X \subseteq F_{1-n} \omega_X(*D). \]

**Proof.** The exact sequence is obtained by pushing forward the short exact sequence (9.4). The zero on the right comes from applying the projection formula, and using the fact that

\[ R^i f_* \omega_Y(E) = 0 \quad \text{for all} \quad i > 0. \]

This is the well-known Local Vanishing theorem for multiplier ideals, but see also Corollary 9.21 (2) below and the Remark thereafter. \( \square \)

**Remark 9.20.** The Corollary above shows the main obstruction to a having an easy calculation of \( I_1(D) \) once we have good understanding of the multiplier ideal \( I_0(D) \), namely the appearance of the higher direct image \( R^1 f_* \Omega_X^{n-1}(\log E) \). In general this sheaf does not vanish, and is not straightforward to compute. Similar phenomena occur for the higher Hodge ideals \( I_k(D) \).

**Applications of strictness.** Recall the definition of the Hodge filtration on \( \omega_X(*D) \) as in (9.3). This is a filtration on the push-forward of a mixed Hodge module, and is therefore strict. This has useful consequences:

**Corollary 9.21.** The following hold for every \( k \geq 0 \):

1. The map

\[ R^0 f_* C_{k-n}^\bullet \longrightarrow R^0 f_* A^\bullet = \omega_X(*D) \]

is injective, and therefore

\[ \omega_X((k+1)) \otimes I_k(D) \simeq R^0 f_* C_{k-n}^\bullet. \]

2. (Local vanishing for \( I_k(D) \).) We have

\[ R^i f_* C_{k-n}^\bullet = 0 \quad \text{for} \quad i \neq 0. \]

**Proof.** We have \( R^i f_* A^\bullet = 0 \) for all \( i \neq 0 \) by Lemma 9.6 and Proposition 9.7. On the other hand, the strictness property (8.1) of the Hodge filtration implies that \( R^i f_* C_{k-n}^\bullet \) injects in \( R^i f_* A^\bullet \). \( \square \)

**Remark 9.22.** The case \( k = 0 \) in part (2) of the Corollary is the well-known Local Vanishing for multiplier ideals (see [La, Theorem 9.4.1]), a consequence of the Kawamata-Viehweg vanishing theorem; in this case \( C_{-n}^\bullet = \omega_Y(E) \).
Chain of inclusions. From the $\mathcal{D}$-module filtration property
\[ F_1 \mathcal{D}_X \cdot F_{k-1} \mathcal{O}_X(*D) \subseteq F_k \mathcal{O}_X(*D) \]
it follows immediately that
\[ I_{k-1}(D) \cdot \mathcal{O}_X(-D) \subseteq I_k(D) \]
for each $k \geq 1$. However, the Hodge ideals also satisfy another natural but more subtle sequence of inclusions.

**Proposition 9.23.** For every reduced effective divisor $D$ on the smooth variety $X$, and for every $k \geq 1$, we have
\[ I_k(D) \subseteq I_{k-1}(D). \]

**Proof.** Consider the canonical inclusion
\[ \iota: \mathcal{O}_X \hookrightarrow \mathcal{O}_X(*D) \]
of filtered left $\mathcal{D}_X$-modules that underlie mixed Hodge modules. Since the category MHM($X$) of mixed Hodge modules on $X$ is abelian, the cokernel $\mathcal{M}$ of $\iota$ underlies a mixed Hodge module on $X$ too, and it is clear that $\mathcal{M}$ has support $D$. Since morphisms between Hodge $\mathcal{D}$-modules preserve the filtrations and are strict, for each $k \geq 0$ we have a short exact sequence
\[ 0 \rightarrow F_k \mathcal{O}_X \rightarrow F_k \mathcal{O}_X(*D) \rightarrow F_k \mathcal{M} \rightarrow 0. \]
Recall now that $F_k \mathcal{O}_X = \mathcal{O}_X$ for all $k \geq 0$. On the other hand, if $f$ is a local equation of $D$, then by Lemma 2.7 we have
\[ f \cdot F_k \mathcal{M} \subseteq F_{k-1} \mathcal{M}. \]
It then follows that $f \cdot F_k \mathcal{O}_X(*D) \subseteq F_{k-1} \mathcal{O}_X(*D)$ as well, which implies the assertion in the theorem by definition of Hodge ideals. \qed

**References**


