\( \mathcal{D} \)-modules in birational geometry
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Introduction

These are notes written for a course I taught at Northwestern University during the Winter and Spring quarters of 2020. They cover various topics on $\mathcal{D}$-modules, Hodge modules, and especially their connections to birational geometry. At the moment they are limited to the material I was able to cover during those two quarters. My intention is however to gradually add more material, especially on various properties and applications of Hodge modules.

Among the topics to be added: vanishing and positivity package for Hodge modules; applications of Hodge modules to generic vanishing, holomorphic forms, families of varieties; Hodge ideals and applications; Hodge filtration on local cohomology.

What these notes are not meant to be is a careful introduction to $\mathcal{D}$-modules, or to mixed Hodge modules (or to birational geometry for that matter). For general $\mathcal{D}$-module theory there are several excellent references (as well as lots of online lecture notes). Among these, the main reference that covers essentially all the basic facts needed here is [HTT]. Nevertheless as these notes develop I will include more precise references, and also an introductory chapter with some background on $\mathcal{D}$-modules, as well as on constructions in birational geometry.

As for the theory of mixed Hodge modules, the fundamental references are Saito’s papers [Sa1] and [Sa2]. There are also his more recent improvement [Sa7] and survey [Sa8]. Schnell’s overview of the theory [Sch] is an excellent quick introduction to the topic, while the developing MHM project [MHM] of Sabbah and Schnell is meant to gradually become a comprehensive reference.
CHAPTER 1

Background on filtered \(\mathcal{D}\)-modules

This chapter contains a very brief review of basic definitions and facts from the theory of \(\mathcal{D}\)-modules. The main source for this material, including proofs and further details, is the book [HTT]. For more specialized facts regarding the theory of filtered \(\mathcal{D}\)-modules, the main source is [Sa1].

1.1. Generalities on \(\mathcal{D}\)-modules

We consider a smooth complex variety \(X\) of dimension \(n\). We denote by \(\mathcal{D}_X\) the sheaf of differential operators on \(X\); this is the sheaf of \(\mathcal{C}\)-subalgebras of \(\text{End}_\mathcal{C}(\mathcal{O}_X)\) generated by \(\mathcal{O}_X\) and \(T_X\), where \(\mathcal{O}_X\) acts by multiplication by functions, and we think of the tangent sheaf \(T_X\) as being the sheaf of derivations \(\text{Der}_\mathcal{C}(\mathcal{O}_X) \subset \text{End}_\mathcal{C}(\mathcal{O}_X)\).

Locally in algebraic coordinates \(x_1, \ldots, x_n\) on an affine neighborhood \(U\) around a point \(x \in X\), it can be described as follows. Consider the corresponding local basis \(\partial_1, \ldots, \partial_n\) of vector fields, dual to \(dx_1, \ldots, dx_n\). They satisfy \([\partial_i, \partial_j] = 0\) and \([\partial_i, x_j] = \delta_{i,j}\) for all \(i\) and \(j\). If we denote \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\) and \(\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}\), then we have

\[
\mathcal{D}_X(U) = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_X(U) \partial^\alpha.
\]

Therefore locally we will write sections of \(\mathcal{D}_X\) as finite sums \(P = \sum_{\alpha} a_\alpha \partial^\alpha\), with \(a_\alpha \in \mathcal{O}_X(U)\). The order of \(P\) is the maximal \(|\alpha| = \alpha_1 + \cdots + \alpha_n\) appearing in this sum.

**Remark 1.1.1 (Affine space).** When \(X = \mathbb{A}^n\), we can take \(x_1, \ldots, x_n\) to be global coordinates, and we have \(\mathcal{D}_X(X) = A_n\), the \(n\)-th Weyl algebra. In this case the theory is essentially equivalent to the theory of modules over \(A_n\); an excellent introduction, including some special features in this case, can be found in [Co].

**Definition 1.1.2.** The order filtration on \(\mathcal{D}_X\) is defined (locally) by

\[
F_k \mathcal{D}_X := \{ \sum_{\alpha} a_\alpha \partial^\alpha \mid |\alpha| \leq k \}.
\]

It can be easily checked that this induces a global filtration on \(X\), and \(F_k \mathcal{D}_X\) is called the sheaf of differential operators on \(X\) of order at most \(k\). It satisfies the following properties:

1. \(F_0 \mathcal{D}_X\) is an increasing, exhaustive filtration by locally free sheaves of finite rank.
2. \(F_0 \mathcal{D}_X = \mathcal{O}_X\) and \(F_1 \mathcal{D}_X \simeq \mathcal{O}_X \oplus T_X\).
3. \(F_k \mathcal{D}_X \cdot F_\ell \mathcal{D}_X \subseteq F_{k+\ell} \mathcal{D}_X\) for all \(k\) and \(\ell\).
4. If \(P \in F_k \mathcal{D}_X\) and \(Q \in F_\ell \mathcal{D}_X\), then \([P, Q] \in F_{k+\ell-1} \mathcal{D}_X\).
We use the notation
\[ \text{gr}_k^{\mathcal{D}} X := \mathcal{D}_k X / \mathcal{D}_{k-1} X \quad \text{and} \quad \text{gr}^\bullet \mathcal{D} X := \bigoplus_{k \in \mathbb{N}} \text{gr}_k^{\mathcal{D}} X \]
for the individual, respectively total, associated graded objects. The natural extension to differential operators of the assignment
\[ T_X(U) \longrightarrow \mathcal{O}_X(U)[y_1, \ldots, y_n], \quad \partial_i \mapsto y_i \]
induces isomorphisms
\[ \text{gr}_k^{\mathcal{D}} X \simeq \text{Sym}^k T_X \quad \text{and} \quad \text{gr}^\bullet \mathcal{D} X \simeq \text{Sym}^\bullet T_X, \]
where Sym\(^\bullet T_X\) is the symmetric algebra of the tangent sheaf of \( X \); see [HTT, §1.1]. In particular, if \( \pi: T^* X \longrightarrow X \) is the cotangent bundle of \( X \), since by the standard correspondence between vector bundles locally free sheaves (see e.g. [Ha, Ch.II, Exer. 5.17, 5.18]) we have \( T^* X \simeq \text{Spec}(\text{Sym}^\bullet T_X) \), it follows that
\[ \text{gr}^\bullet \mathcal{D} X \simeq \pi_* \mathcal{O}_{T^* X}. \]

**Definition 1.1.3.** A sheaf of \( \mathcal{O}_X \)-modules \( \mathcal{M} \) is a *left \( \mathcal{D}_X \)-module* if for every open set \( U \subseteq X \), \( \mathcal{M}(U) \) has a left \( \mathcal{D}_X(U) \)-module structure, compatible with restrictions. Informally, \( \mathcal{M} \) admits an action by differentiation. An analogous definition can be made for right \( \mathcal{D}_X \)-modules.

It is well known that the data of a \( \mathcal{D}_X \)-module structure on \( \mathcal{M} \) is equivalent to the data of a \( \mathbb{C} \)-linear map
\[ \nabla: \mathcal{M} \rightarrow \mathcal{M} \otimes \Omega^1_X \]
satisfying the properties of an integrable connection. Concretely, for every open set \( U \subseteq X \), and \( f \in \mathcal{O}_X(U) \) and \( s \in \mathcal{M}(U) \) we have
\[ \nabla(fs) = f \nabla(s) + s \otimes df, \tag{1.1.1} \]
and in addition \( \nabla \circ \nabla = 0 \). See for instance [HTT, Lemma 1.2.1]. This interpretation leads to one of the key objects associated to a \( \mathcal{D}_X \)-module:

**Definition 1.1.4.** The *de Rham complex* of \( \mathcal{M} \) is the complex
\[ \text{DR}(\mathcal{M}) = \left[ \mathcal{M} \rightarrow \Omega^1_X \otimes \mathcal{M} \rightarrow \cdots \rightarrow \Omega^n_X \otimes \mathcal{M} \right], \]
with \( \mathbb{C} \)-linear differentials induced by iterating \( \nabla \). We consider it to be placed in degrees \(-n, \ldots, 0\).

We next give a few first examples of \( \mathcal{D}_X \)-modules of a geometric nature. They will play an important role throughout.

\[^1\text{We will work with this convention, even though strictly speaking as such it is usually considered to be the de Rham complex associated to the corresponding right } \mathcal{D} \text{-module.}\]
Example 1.1.5. (1) The structure sheaf $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module, via the usual differentiation of functions. It is sometimes called the trivial $\mathcal{D}_X$-module.

(2) More generally, a vector bundle $E$ on $X$ endowed with an integrable (or flat) connection $
abla : E \rightarrow E \otimes \Omega^1_X$ is an example of a $\mathcal{D}_X$-module according to the discussion above. Example (1) is the special case corresponding to the standard differential $d : \mathcal{O}_X \rightarrow \Omega^1_X$.

It turns out that this is the only way in which a $\mathcal{D}_X$-module can be coherent as an $\mathcal{O}_X$-module; see [HTT, Theorem 1.4.10]:

**Theorem 1.1.6.** If $\mathcal{M}$ is a $\mathcal{D}_X$-module which is coherent as an $\mathcal{O}_X$-module, then $\mathcal{M}$ is locally free, hence an integrable connection.

We now turn to examples that are not coherent as $\mathcal{O}_X$-modules.

(3) Let $D$ be an effective divisor on $X$, and consider the quasi-coherent $\mathcal{O}_X$-module of rational functions which are regular away from $D$ and have with poles of arbitrary order along $D$, i.e

$$\mathcal{O}_X(*D) = \bigcup_{k \geq 0} \mathcal{O}_X(kD).$$

This is sometimes called the localization of $\mathcal{O}_X$ along $D$; indeed if $U = \text{Spec}(R)$ is an affine open set in $X$ in which $D = Z(f)$ with $f \in R$, then $\mathcal{O}_X(*D) = \text{R}_{f}$, the localization of $R$ at $f$. This has an obvious action of differential operators by the quotient rule, hence $\mathcal{O}_X(*D)$ is a left $\mathcal{D}_X$-module.

(4) Combining the two examples above, the $\mathcal{O}_X$-module $Q$ defined by

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(*D) \rightarrow Q \rightarrow 0$$

is again naturally a left $\mathcal{D}_X$-module, as the first inclusion is obviously a morphism of $\mathcal{D}_X$-modules. It is in fact well known that $Q \cong \mathcal{H}^1_D(\mathcal{O}_X)$, the first local cohomology sheaf of $\mathcal{O}_X$ along $D$.

More generally, let $Z \subset X$ be an arbitrary closed subscheme. For an integer $q \geq 0$, we denote by $\mathcal{H}^q_Z(\mathcal{O}_X)$ the $q$-th local cohomology sheaf of $\mathcal{O}_X$, with support in $Z$; see [LC, §1]. This is a quasi-coherent $\mathcal{O}_X$-module whose sections are annihilated by suitable powers of the ideal sheaf $\mathcal{I}_Z$; it depends only on the reduced structure of $Z$.

For every affine open subset $U \subseteq X$, if $R = \mathcal{O}_X(U)$ and $I = \mathcal{I}_Z(U)$, then $\mathcal{H}^q_Z(\mathcal{O}_X)(U)$ is the local cohomology module $H^q_I(R)$. One of the equivalent descriptions of this module is as follows (see for instance [LC, Theorem 2.3] and the discussion thereafter): if $I = (f_1, \ldots, f_r)$ and for a subset $J \subseteq \{1, \ldots, r\}$, we denote $f_J := \prod_{i \in J} f_i$, then there is a Čech-type complex

$$C^\bullet : 0 \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots \rightarrow C^n \rightarrow 0,$$

where

$$C^p = \bigoplus_{|J|=p} R_{f_J},$$
and we have

\[ H^q_\mathcal{D}(R) \simeq H^q(C^\bullet). \]

The differentials in \( C^\bullet \) respect the \( \mathcal{D} \)-module structure on each \( C^k \), obtained from localization as in (2) above, and consequently \( \mathcal{H}^0_Z(\mathcal{O}_X) \) is again a left \( \mathcal{D}_X \)-module.

It is often useful to consider right \( \mathcal{D}_X \)-modules. Just as with \( \mathcal{O}_X \) in the case of left \( \mathcal{D}_X \)-modules, there is also a “trivial” right \( \mathcal{D}_X \)-module, namely the canonical bundle \( \omega_X := \bigwedge^n \Omega^1_X \). Its natural right \( \mathcal{D}_X \)-module structure is given as follows: if \( x_1, \ldots, x_n \) are local algebraic coordinates on \( X \), for any \( f \in \mathcal{O}_X \) and any \( P \in \mathcal{D}_X \) the action is

\[ (f \cdot dx_1 \wedge \cdots \wedge dx_n) \cdot P = \cdot \]

Here, if \( P = \sum_a g_a \partial^a \), then \( \cdot \) is its formal adjoint.

This structure leads to an equivalence of categories between left and right \( \mathcal{D}_X \)-modules given by

\[ M \mapsto N = \omega_X \otimes \mathcal{O}_X M \quad \text{and} \quad N \mapsto M = \mathcal{H}om_{\mathcal{O}_X}(\omega_X, N). \]

See [HTT, Proposition 1.2.9 and 1.2.12] for details.

A few more words about the classes of \( \mathcal{D}_X \)-modules we are considering, and their relationship to good filtrations. First, we will essentially always work with \( \mathcal{D}_X \)-modules which are quasi-coherent as \( \mathcal{O}_X \)-modules. The sheaf \( \mathcal{D}_X \) itself is a first such example, as a union of locally free \( \mathcal{O}_X \)-modules of finite rank.

Moreover, we will usually restrict to coherent \( \mathcal{D}_X \)-modules. Recall that by definition \( M \) is coherent if, locally, it is finitely generated over \( \mathcal{D}_X \), and every submodule is locally finitely presented. However the following equivalent description is very helpful; see [HTT, Proposition 1.4.9]:

\textbf{Proposition 1.1.7.} A \( \mathcal{D}_X \)-module \( M \) is coherent if and only if it is quasi-coherent as an \( \mathcal{O}_X \)-module and locally finitely generated as a \( \mathcal{D}_X \)-module.

\section{1.2. Filtered \( \mathcal{D} \)-modules}

We say that a \( \mathcal{D}_X \)-module \( M \) is \textit{filtered} if there exists an increasing filtration \( F = F_\bullet M \) by coherent \( \mathcal{O}_X \)-modules, bounded from below and satisfying

\[ F_k \mathcal{D}_X \cdot F_{\ell} M \subseteq F_k \cdot F_{\ell} M \quad \text{for all} \quad k, \ell \in \mathbb{Z}. \]

We use the notation \((M, F)\) for this data. The filtration is called \textit{good} if the inclusions above are equalities for \( \ell \gg 0 \), which is in turn equivalent to the fact that the total associated graded object

\[ \text{gr}_F^* M = \bigoplus_k \text{gr}_k^F M = \bigoplus_k F_k M / F_{k-1} M \]

is finitely generated over \( \text{gr}_F^* \mathcal{D}_X \simeq \text{Sym}^\bullet T_X \); see [HTT, Proposition 2.1.1]. We can therefore also think of \( \text{gr}_F^* M \) as a coherent sheaf on \( T^*X \); as such we sometimes write \( \text{gr}_F^* M \) when we forget about the grading.
A basic point is that working with good filtrations places us in the category of coherent $\mathcal{D}_X$-modules; see [HTT, Theorem 2.1.3]:

**Theorem 1.2.1.** A $\mathcal{D}_X$-module is coherent if and only if it admits a (globally defined) good filtration.

Assume now that $\mathcal{M}$ has a good filtration $F_\bullet \mathcal{M}$. The compatibility of this filtration with the standard filtration on $\mathcal{D}_X$, meaning in particular that $F_1 \mathcal{D}_X \cdot F_p \mathcal{M} \subseteq F_{p+1} \mathcal{M}$ for all $p$ implies that this induces a filtration on the de Rham complex of $\mathcal{M}$ by the formula

$$F_k \text{DR}(\mathcal{M}) = \left[ F_k \mathcal{M} \to \Omega^1_X \otimes F_{k+1} \mathcal{M} \to \cdots \to \Omega^n_X \otimes F_{k+n} \mathcal{M} \right].$$

We write $\text{DR}(\mathcal{M}, F)$ when we take the filtration into account. For any integer $k$, the associated graded complex for this filtration is

$$\text{gr}^F_k \text{DR}(\mathcal{M}) = \left[ \text{gr}^F_k \mathcal{M} \to \Omega^1_X \otimes \text{gr}^F_{k+1} \mathcal{M} \to \cdots \to \Omega^n_X \otimes \text{gr}^F_{k+n} \mathcal{M} \right].$$

When descending the differentials in $F_k \text{DR}(\mathcal{M})$ to the associated graded, the second term in (1.1.1) disappears, and therefore the differentials in $\text{gr}^F_k \text{DR}(\mathcal{M})$ become $\mathcal{O}_X$-linear. Hence this is now a complex of coherent $\mathcal{O}_X$-modules in degrees $-n, \ldots, 0$, providing an object in $D^b(\text{Coh}(X))$.

**Example 1.2.2 (The trivial filtered $\mathcal{D}$-module).** Consider $\mathcal{M} = \mathcal{O}_X$ with the natural left $\mathcal{D}_X$-module structure, and $F_k \mathcal{O}_X = \mathcal{O}_X$ for $k \geq 0$, while $F_k \mathcal{O}_X = 0$ for $k < 0$. The de Rham complex of $\mathcal{M}$ is

$$\text{DR}(\mathcal{O}_X) = \left[ \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^n_X \right][n].$$

For the induced filtration $F_\bullet \text{DR}(\mathcal{O}_X)$, note that

$$\text{gr}^F_{-k} \text{DR}(\mathcal{O}_X) = \Omega^k_X[n-k] \text{ for all } k.$$

Note that in the holomorphic category we would have that $\text{DR}(\mathcal{O}_X)$ is quasi-isomorphic to $\mathbb{C}[n]$ by the holomorphic Poincaré Lemma. In the algebraic category we still have that the hypercohomology $H^i(X, \text{DR}(\mathcal{O}_X))$, usually called the algebraic de Rham cohomology $H^i_{\text{dR}}(X)$, is isomorphic to the singular cohomology $H^i(X, \mathbb{C})$ by the Grothendieck comparison theorem.

**Example 1.2.3 (Variations of Hodge structure).** This example is again better phrased in the holomorphic category. Note that the previous example corresponds to the trivial variation of Hodge structure (VHS) on $X$; it can be extended to arbitrary such objects. Recall (see e.g. [Vo, Ch.10]) that a $\mathbb{Q}$-VHS of weight $\ell$ on $X$ is the data

$$V = (\mathcal{V}, F^\bullet, V_{\mathbb{Q}})$$

where:

- $V_{\mathbb{Q}}$ is a $\mathbb{Q}$-local system on $X$. 
1. BACKGROUND ON FILTERED $\mathcal{D}$-MODULES

- $\mathcal{V} = \mathcal{V}_Q \otimes Q \mathcal{O}_X$ is a vector bundle with integrable connection $\nabla$, endowed with a decreasing filtration with subbundles $F^p = F^p \mathcal{V}$ satisfying the following two properties:
  - for all $x \in X$, the data $\mathcal{V}_x = (\mathcal{V}_x, \mathcal{F}^p_x, \mathcal{V}_Q \otimes \mathcal{O}_X)$ is a Hodge structure of weight $\ell$.
  - Griffiths transversality: for each $p$, $\nabla$ induces a morphism $\nabla : F^p \to F^{p-1} \otimes \Omega^1_X$.

Recall now that we can think of $\mathcal{M} = \mathcal{V}$ as a left $\mathcal{D}_X$-module. We reindex the filtration as $F^p \mathcal{M} = F^{-p} \mathcal{V}$; this is a good filtration on $\mathcal{M}$. It is also well known that

$$\text{DR}(\mathcal{M}) \simeq \mathcal{V}_C [n],$$

where $\mathcal{V}_C = \mathcal{V}_Q \otimes Q C$. By construction the graded pieces $\text{gr}_F \mathcal{M}$ are locally free, and are sometimes known as the Hodge bundles of the VHS.

It is worth pointing out right away a well-known connection with birational geometry. The most common geometric example of a VHS is obtained as follows: let $f : Y \to X$ be a smooth projective morphism, of relative dimension $k$. Then $R^k f_* Q_Y$ supports a VHS on $X$ given by the Hodge structure on the singular cohomology $H^k$ of the fibers. Using the $\mathcal{D}$-module notation above, it is not hard to see that the lowest non-zero piece in the Hodge filtration on $\mathcal{V}$ is

$$F_{-k} \mathcal{V} = f_* \omega_X / Y.$$

A similar interpretation can be given to all $R^i f_* \omega_X / Y$. This, as well as its consequences, will be discussed more later on, and will be extended to arbitrary morphisms.

Here is also a non-example:

**Exercise 1.2.4.** If $D$ is a hypersurface in $X$, show that unless $D$ is smooth, the pole order filtration

$$F_k \mathcal{O}_X (*D) := \mathcal{O}_X \left((k + 1)D\right)$$

is not a good filtration on $\mathcal{O}_X (*D)$.

**Remark 1.2.5 (Left-right rule for filtrations).** Recall that the mapping

$$\mathcal{M} \to \mathcal{N} = \omega_X \otimes \mathcal{O}_X \mathcal{M}$$

establishes an equivalence between the categories of left and right $\mathcal{D}_X$-modules. This can be extended to filtered $\mathcal{D}_X$-modules, according to the following convention:

$$F_k \mathcal{M} = F_{-n} \mathcal{N} \otimes \mathcal{O}_X \omega_X^{-1}.$$

**Characteristic varieties and holonomic $\mathcal{D}$-modules.** Let $\mathcal{M}$ be a $\mathcal{D}_X$-module endowed with a good filtration $F_* \mathcal{M}$. Recall that the associated graded object $\text{gr}^F \mathcal{M}$ can be thought of as a coherent sheaf on $T^*X$. The characteristic variety (or singular support) of $\mathcal{M}$ is the support of this sheaf:

$$\text{Ch}(\mathcal{M}) := \text{Supp}(\text{gr}^F \mathcal{M}) \subseteq T^*X.$$

This invariant satisfies a number of basic properties; see [HTT, §2.2]:
1.3. PUSH-FORWARD OF $\mathcal{D}$-MODULES

(1) $\text{Ch}(\mathcal{M})$ is independent of the choice of good filtration $F_*\mathcal{M}$ (and therefore it is an invariant of any coherent $\mathcal{D}_X$-module).

(2) If $\pi: T^*X \to X$ is the projection map, then $\pi(\text{Ch}(\mathcal{M})) = \text{Supp}(\mathcal{M})$.

(3) $\text{Ch}(\mathcal{M})$ is conical, i.e. in each fiber over $X$ it is a cone over a subvariety in $\mathbb{P}^{n-1}$.

(4) If $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ is a short exact sequence of coherent $\mathcal{D}_X$-modules, then

$$\text{Ch}(\mathcal{N}) = \text{Ch}(\mathcal{M}) \cup \text{Ch}(\mathcal{P}).$$

(5) $\text{Ch}(\mathcal{M}) = T_X^*X$, i.e. the zero-section of $T^*X$, if and only if $\mathcal{M}$ is an integrable connection.

The fundamental result about characteristic varieties is a dimension bound; see [HTT, §2.3], and also the references therein for its history.

**Theorem 1.2.6** (Bernstein inequality). If $0 \neq \mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then for each component $Z$ of $\text{Ch}(\mathcal{M})$ we have

$$\dim Z \geq n.$$ 

A stronger statement holds true in fact: $\text{Ch}(\mathcal{M})$ is involutive with respect to the standard symplectic structure on $T^*X$. In any event, imposing equality in this bound leads to a celebrated class of $\mathcal{D}$-modules.

**Definition 1.2.7.** A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is holonomic if either $\mathcal{M} = 0$ or $\dim \text{Ch}(\mathcal{M}) = n$.

**Example 1.2.8.** An integrable connection $\mathcal{M}$ is holonomic, since $\text{Ch}(\mathcal{M}) = T_X^*X$.

It is not hard to see that holonomic $\mathcal{D}$-modules are generically as in the example above; see [HTT, Proposition 3.1.6].

**Proposition 1.2.9.** Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then there exists a dense open set $U \subseteq X$ such that $\mathcal{M}|_U$ is an integrable connection.

Further examples of holonomic $\mathcal{D}$-modules are produced once the basic properties of functors on $\mathcal{D}$-modules are recorded. One such is:

**Example 1.2.10.** The $\mathcal{D}$-modules in Example 1.1.5 (3), and therefore (4), are in fact also holonomic. In other words, the basic statement is that if $D$ is an effective divisor on $X$, then the localization $\mathcal{D}_X$-module $\mathcal{O}_X(*D)$ is holonomic. This follows from the fact that direct images of holonomic $\mathcal{D}$-modules are holonomic; in this case $\mathcal{O}_X(*D)$ is the direct image of $\mathcal{O}_U$ via the open embedding $U = X \setminus D \hookrightarrow \mathcal{O}_X$. The holonomicity of $\mathcal{O}_X(*D)$ is closely related to the existence of Bernstein-Sato polynomials.

1.3. Push-forward of $\mathcal{D}$-modules

A detailed exposition on the material in this section can be found in [HTT, §1.3 and §1.5].
Let \( f: Y \to X \) be a morphism of smooth varieties. The *transfer module* of \( f \) is defined as

\[
\mathcal{D}_{Y \to X} := \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X.
\]

This is simply \( f^*\mathcal{D}_X \) as an \( \mathcal{O}_Y \)-module, and we will use this notation when thinking of it as such; note in particular that it is filtered by \( f^*F_k\mathcal{D}_X \).

However for our purposes it is endowed with the structure of a \((\mathcal{D}_Y, f^{-1}\mathcal{D}_X)\)-bimodule. Indeed, the right \( f^{-1}\mathcal{D}_X \)-module structure comes simply from the right hand side of the tensor product. On the other hand, the left \( \mathcal{D}_Y \)-module structure is a general phenomenon:

**Remark 1.3.1 (Pullback of \( \mathcal{D} \)-modules).** If \( \mathcal{M} \) is a left \( \mathcal{D}_X \)-module, then the \( \mathcal{O}_Y \)-module \( f^*\mathcal{M} \) has a natural left \( \mathcal{D}_Y \)-module structure coming from the natural morphism \( \mathcal{D}_Y \to f^*\mathcal{D}_X \), induced in turn by the natural morphism of vector bundles \( T_Y \to f^*T_Y \).

**Example 1.3.2.** Let \( i: Y \hookrightarrow X \) be a closed embedding, with \( X \) of dimension \( n \) and \( Y \) of dimension \( r \). If we choose local algebraic coordinates \( x_1, \ldots, x_n \) on \( X \) such that \( Y = (x_{r+1} = \cdots = x_n = 0) \), then

\[
\mathcal{D}_{Y \to X} \simeq \mathcal{D}_Y \otimes \mathbb{C}[\partial_{r+1}, \ldots, \partial_n].
\]

The push-forward of \( \mathcal{D} \)-modules is a priori more naturally defined for right \( \mathcal{D} \)-modules; this is similar to the fact that we don’t have a natural push-forward of functions, but rather of distributions. As a preliminary definition, for a right \( \mathcal{D}_Y \)-module \( \mathcal{N} \), we take its push-forward to \( X \) to be

\[
f_+\mathcal{N} := f_*(\mathcal{N} \otimes_{\mathcal{D}_Y}\mathcal{D}_{Y \to X}).
\]

(We will see in a second that we should rather denote this by \( \mathcal{H}^0f_+\mathcal{N} \).) It has a right \( \mathcal{D}_X \)-module structure as follows: we use the natural right \( f^{-1}\mathcal{D}_X \)-module structure on \( \mathcal{D}_{Y \to X} \) to obtain a right \( f_*f^{-1}\mathcal{D}_X \)-module structure on \( f_+\mathcal{N} \). We then restrict scalars via the adjunction morphism \( \mathcal{D}_X \to f_*f^{-1}\mathcal{D}_X \).

Due to the use of the left exact functor \( f_* \) in combination with the right exact functor \( \otimes \), the definition above is not so well behaved, for instance with respect to the composition of morphism. This is remedied by working with derived functors. Note that in [HTT, §1.5] the functor below is denoted by \( f_f \), reminiscent of integration by fibers.

**Definition 1.3.3.** The push-forward functor on right \( \mathcal{D}_Y \) modules is defined as

\[
f_+: \mathcal{D}^b(\mathcal{D}_Y^{\text{op}}) \to \mathcal{D}^b(\mathcal{D}_X^{\text{op}}), \quad \mathcal{N}^\bullet \mapsto Rf_*(\mathcal{N}^\bullet \otimes_{\mathcal{D}_Y}\mathcal{D}_{Y \to X});
\]

where \( \mathcal{D}^b(\mathcal{D}_Y^{\text{op}}) \) is the bounded derived category of right \( \mathcal{D}_Y \)-modules, and similarly for \( \mathcal{D}^b(\mathcal{D}_X^{\text{op}}) \).\(^2\)

\(^2\)See the beginning of [HTT, §1.5] for a discussion of these derived categories.
1.3. PUSH-FORWARD OF \( \mathcal{D} \)-MODULES

Note that with this definition it is not too hard to check that for a composition of morphisms we have \((g \circ f)_+ = g_+ \circ f_+\); see [HTT, Proposition 1.5.21].

We can of course also define the push-forward functor for left \( \mathcal{D} \)-modules by applying the left-right rule at both ends of the above construction. This is packaged nicely by considering an analogue of the transfer module, namely

\[
\mathcal{D}_{X \leftarrow Y} := \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \rightarrow X} \otimes f^{-1}_* \omega_X^{-1},
\]

which is a \((f^{-1}\mathcal{D}_X, \mathcal{D}_Y)\)-bimodule. Push-forward on the bounded derived category of left \( \mathcal{D}_Y \)-modules is then defined as

\[
f_+: \mathbf{D}^b(\mathcal{D}_Y) \longrightarrow \mathbf{D}^b(\mathcal{D}_X), \quad \mathcal{M}^\bullet \mapsto Rf_* (\mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet^L).
\]

**Example 1.3.4 (Open embeddings).** Let \( j: U \hookrightarrow X \) be the embedding of an open subset of \( X \). Then \( \mathcal{D}_{X \leftarrow U} \simeq \mathcal{D}_U \), and so

\[
j_+ = Rj_*
\]

meaning \( j_+ \) is the same as the usual direct image functor on \( \mathcal{O}_U \)-modules.

In particular, if \( U = X \setminus D \), where \( D \) is an effective divisor, then

\[
j_+ \mathcal{O}_U \simeq \mathcal{O}_X(*D).
\]

(This contains the statement that \( H^i j_+ \mathcal{O}_U = 0 \) for \( i > 0 \).)

More generally, if \( Z \subseteq X \) is an arbitrary closed subset, this can be expressed in terms of local cohomology. A standard calculation (exercise!) shows that there is a short exact sequence

\[
0 \rightarrow \mathcal{O}_X \rightarrow j_* \mathcal{O}_U \rightarrow \mathcal{H}^1_Z \mathcal{O}_X \rightarrow 0,
\]

while

\[
R^k j_* \mathcal{O}_U \simeq \mathcal{H}^{k+1}_Z \mathcal{O}_X \quad \text{for} \ k \geq 1.
\]

**Example 1.3.5 (Closed embeddings).** Let \( i: Y \hookrightarrow X \) be the embedding of a closed subset of \( X \). With the notation of Example 1.3.2 (see also [HTT, Ex. 1.3.5] for more details) we have

\[
\mathcal{D}_{X \leftarrow Y} \simeq \mathbb{C}[[\partial_{r+1}, \ldots, \partial_n]] \otimes_{\mathbb{C}} \mathcal{D}_Y,
\]

with the obvious bimodule actions.

Let now \( \mathcal{M} \) be a \( \mathcal{D}_Y \)-module. Since in this case \( i_* \) is an exact functor, we have \( \mathcal{H}^k i_+ \mathcal{M} = 0 \) for \( k \neq 0 \). Moreover, using the description of \( \mathcal{D}_{X \leftarrow Y} \) above, we have

\[
\mathcal{H}^0 i_+ \mathcal{M} \simeq \mathbb{C}[[\partial_{r+1}, \ldots, \partial_n]] \otimes_{\mathbb{C}} \mathcal{M}.
\]

The left \( \mathcal{D}_X \)-module action is given by:

- \( \partial_{r+1}, \ldots, \partial_n \) act by \( \partial_j \cdot (P \otimes m) = \partial_j P \otimes m \).
- \( \partial_1, \ldots, \partial_r \) act by \( \partial_j \cdot (P \otimes m) = P \otimes \partial_j m \).
- \( f \in \mathcal{O}_X \) acts by \( f \cdot (P \otimes m) = P \otimes (f|_Y)m \).
The most important result about this functor is Kashiwara’s Theorem, which we now state. First, let’s introduce some notation:

- $\text{Mod}_{qc}(\mathcal{D}_Y)$ stands for the category of $\mathcal{D}_Y$-modules that are quasi-coherent as $\mathcal{O}_Y$-modules.
- $\text{Mod}^Y_{qc}(\mathcal{D}_X)$ stands for the category of $\mathcal{D}_X$-modules that are supported on $Y$, and are quasi-coherent as $\mathcal{O}_Y$-modules.

**Theorem 1.3.6 (Kashiwara’s Equivalence).** The functor $i_+$ induces an equivalence of categories

$$i_+: \text{Mod}_{qc}(\mathcal{D}_Y) \to \text{Mod}^Y_{qc}(\mathcal{D}_X).$$

Its inverse is given by the functor $\mathcal{H}^{n-r}i^*$, and moreover on the category $\text{Mod}^Y_{qc}(\mathcal{D}_X)$ we have $\mathcal{H}^k i^* = 0$ for $k \neq n - r$.

For a proof of this theorem see [HTT, Theorem 1.6.1]. The equivalence also restricts to the categories of coherent $\mathcal{D}$-modules on both sides, due to general results described below.

**Example 1.3.7 (Projections).** Assume that $Y = X \times Z$, with $Z$ another smooth variety, and let $f = p_1: Y \to X$ be the projection onto the first factor. In this case $f_+$ can be described as follows; see [HTT, Proposition 1.5.28].

We denote $d = \dim Z = \dim Y - \dim X$, and for each $0 \leq k \leq d$ we consider

$$\Omega^k_{Y/X} := p_2^* \Omega^k_Z.$$

The relative de Rham complex of a left $\mathcal{D}_Y$-module $\mathcal{M}$ is the complex

$$\text{DR}_{Y/X}(\mathcal{M}) : 0 \to \mathcal{M} \to \Omega^1_{Y/X} \otimes \mathcal{M} \to \cdots \to \Omega^d_{Y/X} \otimes \mathcal{M} \to 0$$

which we consider placed in degrees $-d, \ldots, 0$, and with differentials given by

$$d(\omega \otimes s) = d\omega \otimes s + \sum_{i=1}^d (dx_i \otimes \omega) \otimes \partial_i s,$$

where $x_1, \ldots, x_d$ are local coordinates on $Z$.

Then the following isomorphism holds

$$\mathcal{D}_X \overset{L}{\leftarrow} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{M} \simeq \text{DR}_{Y/X}(\mathcal{M})$$

and therefore the push-forward can be computed as

$$f_+ \mathcal{M} \simeq Rf_* \text{DR}_{Y/X}(\mathcal{M}).$$

**Remark 1.3.8.** It is worth noting that each entry $\text{DR}_{Y/X}(\mathcal{M})^k = \Omega^{n+k}_{Y/X} \otimes \mathcal{M}$ is a $f^{-1} \mathcal{D}_X$-module thanks to the rule

$$P \cdot (\omega \otimes s) = \omega \otimes ((P \otimes 1) \cdot s)$$

induced by the mapping $f^{-1} \mathcal{D}_X \to \mathcal{D}_Y$, $P \to P \otimes 1$. 
In order to establish various properties of push-forward functors, it often suffices to consider the concrete descriptions in the examples above. The reason is that every morphism $f: Y \to X$ can be written as the composition of the closed embedding $Y \hookrightarrow Y \times X$ given by the graph of $f$, followed by the second projection.

Using this approach, the following important properties of the push-forward functor can be established. The derived categories in the statement below are those corresponding to objects whose entries are quasi-coherent, coherent, or holonomic respectively; they can be shown to be equivalent to those of objects whose cohomologies are of this kind.

**Theorem 1.3.9.** Let $f: Y \to X$ be a morphism of smooth varieties. Then:

1. $f_+$ preserves quasi-coherence (over $\mathcal{O}_X$), in the sense that it induces a functor $f_+: D^{b}_{qc}(\mathcal{D}_Y) \to D^{b}_{qc}(\mathcal{D}_X)$.

2. If $f$ is projective, then $f_+$ preserves coherence, in the sense that it induces a functor $f_+: D^{b}_{c}(\mathcal{D}_Y) \to D^{b}_{c}(\mathcal{D}_X)$.

3. $f_+$ preserves holonomicity, in the sense that it induces a functor $f_+: D^{b}_{h}(\mathcal{D}_Y) \to D^{b}_{h}(\mathcal{D}_X)$.

The proof of (1) can be found in [HTT, Proposition 1.5.29], that of (2) in [HTT, Theorem 2.5.1], and that of (3) in [HTT, Theorem 3.2.3].

**Remark 1.3.10.** Note in particular that in (3) we are not imposing any conditions on the morphism $f$. It is also the case that the derived pullback functor $Lf^*$ preserves holonomicity; see [HTT, Theorem 3.2.3].
CHAPTER 2

The Bernstein-Sato polynomial

The aim of this chapter is to introduce and study the Bernstein-Sato polynomial of a regular function, and explain its connection to invariants in birational geometry. We use freely the basics of the theory of $\mathcal{D}$-modules, for instance as in [HTT].

2.1. Push-forward via a graph embedding

Let $X$ be a smooth complex variety of dimension $n$, and $f \in \mathcal{O}_X$ an arbitrary nontrivial function. Many $\mathcal{D}$-module constructions that depend on $f$ and will be studied below (like the $V$-filtration along $f$, or the associated nearby and vanishing cycles) are much easier to perform when the zero locus of $f$ is smooth. For arbitrary $f$, one usually reduces to this case using the following standard construction. Let $\iota: X \hookrightarrow X \times \mathbb{C}$, $x \mapsto (x, f(x))$ be the closed embedding given by the graph of $f$. Denote $Y = X \times \mathbb{C}$, and let $t$ be the coordinate on the second factor $\mathbb{C}$, so that $(t = 0)$ is the smooth hypersurface $X \times \{0\}$ in $Y$. For a $\mathcal{D}_X$-module $\mathcal{M}$, we consider the $\mathcal{D}_Y$-module theoretic direct image

$$\iota_+ \mathcal{M} := \mathcal{M} \otimes_{\mathbb{C}} \mathcal{O}_X[t].$$

Recall in particular that we think of the structure sheaf $\mathcal{O}_X$ as the (trivial) left $\mathcal{D}_X$-module obtained from the standard action of differential operators on functions. Furthermore, to the effective divisor $D = (f = 0)$ on $X$ we can associate the left $\mathcal{D}_X$-module $\mathcal{O}_X(\ast D)$ of functions with poles of arbitrary order along $D$; this is of course nothing but the localization $\mathcal{O}_X[\frac{1}{f}]$, with the obvious action of differential operators.

It will be especially important to have a good understanding of the $\mathcal{D}_Y$-modules $\iota_+ \mathcal{O}_X$ and $\iota_+ \mathcal{O}_X(\ast D)$, including a useful construction due to Malgrange. We focus on this next.

**Exercise 2.1.1.** (1) Show that the action of $t$ on $\iota_+ \mathcal{O}_X$ is injective, so that we have an embedding

$$\iota_+ \mathcal{O}_X \hookrightarrow (\iota_+ \mathcal{O}_X)_t.$$

(2) Show that $(\iota_+ \mathcal{O}_X)_t = \iota_+ \mathcal{O}_X(\ast D)$. (Hence $\iota_+ \mathcal{O}_X$ naturally embeds in $\iota_+ \mathcal{O}_X(\ast D)$.)

**Lemma 2.1.2.** There is an isomorphism

$$\iota_+ \mathcal{O}_X \simeq \mathcal{O}_X[t]_{f-t}/\mathcal{O}_X[t].$$
Proof. Recall that by definition we have $\mathcal{O}_X \simeq \mathcal{O}_X \otimes \mathbb{C}[\partial_t]$. If $\delta$ denotes the class of $\frac{1}{f-t}$ in $\mathcal{O}_X$, the claim is that every element in $\mathcal{O}_X[t]_{f-t}/\mathcal{O}_X[t]$ can be written uniquely as

$$\sum_{j \geq 0} h_j \partial_t^j \delta,$$

with $h_j \in \mathcal{O}_X$, only finitely many of these being nonzero. Then the asserted isomorphism is clear, as such an element can be identified with $\sum_{j \geq 0} h_j \otimes \partial_t^j \in \mathcal{O}_X[t]_{f-t}$.

Uniqueness follows from the fact that for any $k \geq 0$ the elements

$$1, \frac{1}{f-t}, \ldots, \frac{1}{(f-t)^k}$$

are linearly independent over $\mathcal{O}_X$. On the other hand, for existence note that

$$\partial_t^j \delta = j! \cdot \frac{1}{(f-t)^{j+1}},$$

while every element in $\mathcal{O}_X[t]_{f-t}/\mathcal{O}_X[t]$ is the class of

$$\frac{g_1}{f-t} + \cdots + \frac{g_k}{(f-t)^k}$$

for some $k \geq 1$, and $g_1, \ldots, g_k \in \mathcal{O}_X$. \hfill $\Box$

Remark 2.1.3. In $\mathcal{O}_X$ we have the useful formula

$$f \delta = t \delta. \quad (2.1.1)$$

Exercise 2.1.4. (2) In the interpretation given by Lemma 2.1.2, the $\mathcal{D}_X$-module structure on $\mathcal{O}_X$ is given by:

1. $g \cdot (h \partial_t^j \delta) = (gh) \partial_t^j \delta$, for $g \in \mathcal{O}_X$.
2. $t \cdot (h \partial_t^j \delta) = (fh) \partial_t^j \delta - j h \partial_t^{j-1} \delta$.
3. $\partial_t \cdot (h \partial_t^j \delta) = h \partial_t^{j+1} \delta$.
4. $D(h \partial_t^j \delta) = D(h) \partial_t^j \delta - (D(f) h) \partial_t^{j+1} \delta$.

Even when studying constructions on $\mathcal{O}_X$, it will be convenient to work in the larger $\mathcal{D}$-module $\mathcal{O}_X(\ast D)$. The advantage it provides is that multiplication by $t$ on it is bijective.

More generally let’s consider for the next few paragraphs a $\mathcal{D}_X$-module $\mathcal{M}$ on which multiplication by $f$ is bijective; in other words, $\mathcal{M}$ has a natural structure of $\mathcal{O}_X(\ast D)$-module.

Lemma 2.1.5. Under this hypothesis, multiplication by $t$ is bijective on $\mathcal{O}_X$.

Proof. We consider on $\mathcal{O}_X$ the filtration given by

$$\mathcal{G}_p = \mathcal{G}_p \mathcal{O}_X := \bigoplus_{j=0}^p \mathcal{M} \otimes \mathcal{O}_X \partial_t^j \delta.$$
By Exercise 2.1.4(2), multiplication by \( t \) preserves this filtration; moreover, for every \( p \geq 0 \), via the obvious isomorphism \( G_p/G_{p-1} \simeq \mathcal{M} \), multiplication by \( t \) gets identified with multiplication by \( f \). We thus obtain by induction on \( p \) the fact that multiplication by \( t \) on \( G_p \) is an isomorphism, which gives the conclusion as \( G_\bullet \) is exhaustive.

We now give the main construction. Let \( \mathcal{D}(t, s) \) be the subsheaf of \( \mathcal{D}_X \times C \) generated by \( \mathcal{D}_X \), \( t \), and \( s = -\partial t \). Note that \( t \) and \( s \) satisfy \( st = t(s - 1) \) and more generally

\[(2.1.2) \quad P(s)t = tP(s - 1) \quad \text{for all} \quad P \in C[s].\]

We also consider the localization \( \mathcal{D}_X(\langle t, t^{-1}, s \rangle) = \mathcal{D}_X(\langle t^{1}, t^{-1}, \partial t \rangle) \) of \( \mathcal{D}(t, s) \). (This is the push-forward of the sheaf of differential operators from \( X \times C^* \) to \( X \times C \).) Note that in this ring we have \( \partial t = -st - 1 \) and from (2.1.2) we obtain

\[(2.1.3) \quad t^{-1}P(s) = P(s - 1)t^{-1} \quad \text{for all} \quad P \in C[s].\]

A \( \mathcal{D}(t, t^{-1}, s) \)-module is simply a \( \mathcal{D}_X \)-module on which \( t \) acts bijectively.

We consider the \( \mathcal{D}_X(\langle t, t^{-1}, s \rangle) \)-module \( \mathcal{M}[s]^f \) defined as follows. As an \( \mathcal{O}_X \)-module, we have an isomorphism

\[\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X[s] \simeq \mathcal{M}[s]^f, \quad u \otimes s^j \to us^j f^s.\]

The symbol \( f^s \) is formal, and motivates the \( \mathcal{D}_X \)-action: a derivation \( D \) in \( \text{Der}_C(\mathcal{O}_X) \) acts by

\[D \cdot us^j f^s = \frac{D(f)}{f}us^{j+1} f^s + D(u)s^j f^s.\]

The action of \( s \) on \( \mathcal{M}[s]^f \) is the obvious one, while the action of \( t \) is given by the automorphism \( \ast s \to s + 1 \ast \), that is

\[us^j f^s \to fu(s + 1)^j f^s.\]

Recall that in this language \( s \) corresponds to \( -\partial t \); note also that \( f^s \) corresponds to \( \delta \).

Next comes the main technical point, observed in [Ma]. To state it, for every \( i \geq 0 \), we set

\[Q_i(x) = i! \cdot \binom{x + i - 1}{i} := \prod_{j=0}^{i-1}(x + j) \in \mathbb{Z}[x]\]

(with the convention \( Q_0 = 1 \)).

**Proposition 2.1.6.** If \( \mathcal{M} \) is a \( \mathcal{D}_X \)-module on which \( f \) acts bijectively, then we have an isomorphism of \( \mathcal{D}_X(\langle t, t^{-1}, s \rangle) \)-modules

\[\varphi: \mathcal{M}[s]^f \xrightarrow{\sim} \mathcal{M} \oplus \mathcal{M}, \quad us^j f^s \to u \otimes (\partial t)^j \delta.\]

The inverse isomorphism \( \psi \) is given by

\[u \otimes \partial^j \delta \to \frac{u}{f^j}Q_j(-s)f^s.\]
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PROOF. It is straightforward to check that $\varphi$ and $\psi$ are $D_X(t, t^{-1}, s)$-linear. In order to see that they are isomorphisms, consider on $M[s]f^s$ and $\iota_+ M$ the filtrations given by

$$G_p M[s]f^s = \bigoplus_{i=0}^{p} M s^i f^s$$
and

$$G_p \iota_+ M = \bigoplus_{j=0}^{p} M \otimes_{O_X} O_X \partial_t^j.$$

It is clear that $\varphi$ and $\psi$ preserve the filtrations. Note now that $(\partial_t)^p \delta$ is equal to $f^p \partial_t^p \delta$ plus a sum of monomials which have lower degree in $\partial_t$. Moreover, we have canonical isomorphisms

$$G_p M[s]f^s / G_{p-1} M[s]f^s \cong M \cong G_p \iota_+ M / G_{p-1} \iota_+ M,$$

and so the corresponding endomorphism of $M$ induced by $\varphi$ is given by multiplication with $(-1)^p f^p$. Since this is an isomorphism, we conclude by induction on $p$ that each induced map $G_p M[s]f^s \to G_p \iota_+ M$ is an isomorphism, and hence so is $\varphi$.

The formula for the inverse isomorphism $\psi$ follows if we show that in $\iota_+ O_X(*D)$ we have

$$Q_j(\partial_t)\delta = f^j \partial_t^j \delta \quad \text{for all} \quad j \geq 0.$$

We argue by induction on $j$, the case $j = 0$ being obvious. Assuming the formula for some $j$, we apply (2.1.3) and the fact that $\partial_t = -st^{-1}$ to write

$$f^{j+1} \partial_t^{j+1} \delta = f \partial_t Q_j(-s) \delta = -f st^{-1} Q_j(-s) \delta = f(-s) Q_j(-s - 1) t^{-1} \delta = Q_{j+1}(-s) \delta.$$

This completes the proof of the proposition. \hfill $\square$

2.2. Definition, existence, and examples

In this section $X$ is a smooth variety over $C$ (or more generally over a field of characteristic 0), and $f$ is a non-invertible regular function on $X$. The following theorem was proved by Bernstein when $f$ is a polynomial, and by Björk and Kashiwara in general.\footnote{They also proved it in the case of germs of holomorphic functions on complex manifolds.}

THEOREM 2.2.1. There exists a polynomial $b(s) \in C[s]$, and a polynomial $P(s) \in D_X[s]$ whose coefficients are differential operators on $X$, such that the relation

$$P(s)f^{s+1} = b(s) \cdot f^s$$

holds formally in the $D_X$-module $O_X[\frac{1}{s}, s] \cdot f^s$. (Here $f^{s+1}$ stands for $f \cdot f^s$.)

REMARK 2.2.2. Recall that we have discussed the $D$-module in the statement of the Theorem in §2.1 above; the action of derivations on $X$ on its elements is given by

$$D(w f^s) = (D(w) + sw \frac{D(f)}{f}) f^s.$$

It carries an obvious action of $s$, as part of its $D_X(t, t^{-1}, s)$-module structure.

In the case of polynomials $f \in C[X_1, \ldots, X_n]$, the proof of this theorem is a simple application of the fact that $O_C[\frac{1}{s}](s) \cdot f^s$ is holonomic as a module over the Weyl algebra $A_n(C(s)) \cong A_n(C)(s)$ associated to the field $C(s)$. This is proved using the Bernstein filtration on $A_n$: see [Co, Ch.10, §3]. In the general case this filtration is not available, but we follow a similar approach replacing it by general properties of holonomic $D$-modules.
Proof of Theorem 2.2.1. We denote by $C(s)$ the field of rational functions in the variable $s$. We also denote by $j: U \hookrightarrow X$ the natural inclusion of $U = X \setminus Z(f)$. Via base field extension we can consider $X$ and $U$ as being defined over $C(s)$; we use the notation

$$X_s := X \times_{\text{Spec } C} \text{Spec } C(s) \quad \text{and} \quad U_s := U \times_{\text{Spec } C} \text{Spec } C(s),$$

with the corresponding inclusion $j_s: U_s \hookrightarrow X_s$. We then have

$$\mathcal{D}_{U_s} \cong \mathcal{D}_U \times_C C(s) \quad \text{and} \quad \mathcal{D}_{X_s} \cong \mathcal{D}_X \times_C C(s).$$

Thinking of $f^s$ as a formal symbol as before, we now consider the $\mathcal{D}_{U_s}$-module $M := \mathcal{O}_{U_s} \cdot f^s$, where the action of a derivation on $X$ is given by

$$D(gf^s) = (D(g) + sgD(f))f^s.$$

(Note that $f$ is invertible on $U_s$.) Note that $M$ is a holonomic $\mathcal{D}_{U_s}$-module. Indeed, by analogy with the trivial filtration on $\mathcal{O}_{U_s}$, on $M$ we can consider the filtration given by

$$F_k M = \mathcal{O}_{U_s} \cdot f^s \quad \text{for} \quad k \geq 0 \quad \text{and} \quad F_k M = 0 \quad \text{for} \quad k < 0.$$

This is a good filtration such that

$$\text{Ch}(M) = \text{Ch}(\mathcal{O}_U)_s = (T^*_U s),$$

the scalar extension of the zero section of $T^* U$, hence holonomicity is clear. Now the main claim is:

Claim. The $\mathcal{D}_{X_s}$-submodule

$$\mathcal{N} := \mathcal{D}_{X_s} f^s \subseteq j_{s+} M$$

is holonomic.

In order to show this, we note that the construction of a maximal holonomic submodule of a finitely generated $\mathcal{D}$-module works over arbitrary base fields, and is functorial (GIVE REFERENCE). Hence there exists a maximal holonomic submodule $\mathcal{N}' \subseteq \mathcal{N}$, compatible with restriction to open sets. Note now that

$$\mathcal{N}|_{U_s} \subseteq \mathcal{M} = \mathcal{O}_{U_s} \cdot f^s.$$

By the observation above, we conclude that $\mathcal{N}|_{U_s}$ is holonomic, hence

$$\mathcal{N}'|_{U_s} = \mathcal{N}|_{U_s}.$$

In other words, in the short exact sequence of $\mathcal{D}_{X_s}$-modules

$$0 \longrightarrow \mathcal{N}' \longrightarrow \mathcal{N} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

the quotient $\mathcal{Q}$ is supported on the zero locus of $f$ in $X_s$. Consequently, if we look at the section $f^s$ of $\mathcal{N}$, there exists an integer $k_0 \geq 0$ such that $f^{k_0} f^s \in \mathcal{N}'$. Hence

$$\mathcal{D}_{X_s} f^{k_0} f^s \subseteq \mathcal{N}'$$

and so $\mathcal{D}_{X_s} f^{k_0} f^s$ is holonomic. Finally note that we have an isomorphism of $\mathcal{D}_{X_s}$-modules

$$\mathcal{D}_{X_s} f^s \xrightarrow{\sim} \mathcal{D}_{X_s} f^{k_0} f^s, \quad P(s)f^s \mapsto P(s + k_0)f^{k_0} f^s$$

induced by the automorphism $s \mapsto s + k$ of $\mathcal{D}_{X_s}$. 
Having established the claim, let us conclude the proof of the main statement. Consider the chain of submodules
\[
\cdots \subseteq \mathcal{D}_X f^2 f^s \subseteq \mathcal{D}_X f f^s \subseteq \mathcal{D}_X f^s.
\]
Since \( \mathcal{D}_X f^s \) is holonomic, hence of finite length, this chain stabilizes. There exists therefore an integer \( m \geq 0 \) such that
\[
f^m f^s \in \mathcal{D}_X f^{m+1} f^s.
\]
Applying now the similar automorphism \( s \mapsto s - m \), we conclude that \( f^s \in \mathcal{D}_X f f^s \), and so there exists \( Q(s) \in \mathcal{D}_X = \mathcal{D}_X \otimes \mathbb{C} \mathbb{C}(s) \) such that
\[
f^s = Q(s) f f^s.
\]
Clearing denominators, this operator can be rewritten as \( Q(s) = P(s)/b(s) \), where \( P(s) \in \mathcal{D}_X[s] \) and \( b(s) \in \mathbb{C}(s) \), so the identity becomes
\[
b(s) \cdot f^s = P(s) f f^s,
\]
which is what we were after. \( \square \)

**Definition 2.2.3.** The set of all polynomials \( b(s) \) satisfying an identity as in Theorem 2.2.1 clearly forms an ideal in the polynomial ring \( \mathbb{C}[s] \). The monic generator of this ideal is called the **Bernstein-Sato polynomial** of \( f \), and is denoted \( b_f(s) \).

There is also a local version of the Bernstein-Sato polynomial. We discuss this next, together with its relationship with the global version above.

**Lemma 2.2.4.** If \( x \) is a point in \( X \), then there exists an open neighborhood \( U \) of \( x \) such that for any other open neighborhood \( V \) we have
\[
b_{f|U}(s) \mid b_{f|V}(s).
\]

**Proof.** For simplicity, let’s denote \( b_U(s) = b_{f|U}(s) \). Start with any open neighborhood \( x \in U_0 \) and assume that it does not satisfy the property we want. There exists then another neighborhood \( x \in U_1 \) such that \( b_{U_0} \) does not divide \( b_{U_1} \). As \( b_{U_0 \cap U_1} \mid b_{U_1} \) since we can restrict the Bernstein-Sato identity on \( U_1 \) to \( U_0 \cap U_1 \), and similarly for \( U_0 \), it follows that \( b_{U_0 \cap U_1} \) is a proper factor of \( b_{U_0} \).

If the neighborhood \( U_0 \cap U_1 \) again does not satisfy the property in the statement, then by a similar argument there exists a neighborhood \( x \in U_2 \) such that \( b_{U_0 \cap U_1} \) is a proper factor of \( b_{U_0 \cap U_1 \cap U_2} \). Continuing this way, since \( b_{U_0} \) has finitely many factors at some point the polynomial has to stabilize, and we obtain an open set \( U_0 \cap U_1 \cap \ldots \cap U_r \) satisfying the assertion. \( \square \)

**Definition 2.2.5.** The **local Bernstein-Sato polynomial** of \( f \) at \( x \) is
\[
b_{f,x}(s) := b_{f|U}(s),
\]
where \( U \) is an open neighborhood of \( x \) as in Lemma 2.2.4.
**Proposition 2.2.6.** If $X$ is affine, then the global and local Bernstein-Sato polynomials are related by the formula\[ b_f(s) = \operatorname{lcm}_{x \in X} b_{f,x}(s). \]

In fact, let $\{U_i\}_{i \in I}$ be any open cover of $X$. Then\[ b_f(s) = \operatorname{lcm}_{i \in I} b_{f|U_i}(s). \]

**Proof.** Denoting as in the previous proof $b_{U_i} = b_{f|U_i}$, since clearly $b_{U_i} | b_f$ for all $i$ we have\[ b'(s) := \operatorname{lcm}_{i \in I} b_{U_i}(s) | b_f(s). \]

Consider now the Bernstein-Sato identity on each $U_i$, namely\[ b_{U_i}(s) f_{U_i}^s = P_i(s) f_{U_i}^{s+1} \text{ with } P_i(s) \in \mathcal{D}_{U_i}(s). \]

Since $U_i$ is an open set in the affine variety $X$, there exist for each $i$ and operator $Q_i(s) \in \mathcal{D}X[s]$, and $g_i \in \mathcal{O}_X(X)$, such that $g_i P_i(s) = Q_i(s)$. We then have the identity\[ g_i b_{U_i}(s) f^s = Q_i(s) f^{s+1} \]
on $X$, which implies that $g_i b'(s) f^s \in \mathcal{D}_X[s] f^{s+1}$. Define now\[ I = \{ g \in \mathcal{O}_X(X) \mid g b'(s) f^s \in \mathcal{D}_X[s] f^{s+1} \}. \]

This is clearly an ideal in $\mathcal{O}_X(X)$, and for each $i \in I$ we have that $g_i \in I \setminus \mathfrak{m}_x$ for every $x \in U_i$. It follows that $I = (1)$, hence $b_f(s) | b'(s)$. \(\Box\)

**Remark 2.2.7.** On an arbitrary $X$, the result of Proposition 2.2.6 continues to hold if we think of the Bernstein-Sato polynomial as being the monic polynomial of minimal degree such that\[ b(s) f^s \in \mathcal{D}_X[s] f^{s+1} \]
in a sheaf-theoretic sense.

**Exercise 2.2.8.** If $g$ is an invertible function on $X$, then $b_{gf}(s) = b_f(s)$.

**Remark 2.2.9 (Bernstein-Sato polynomials of divisors).** Let $D$ be an arbitrary effective divisor on $X$. For any two functions $f_1, f_2$ defining $D$ on an open set $U \subset X$, there exists $g \in \mathcal{O}_X^*(U)$ such that $f_1 = f_2 g$. The results and exercise above imply then that it makes sense to define a Bernstein-Sato polynomial $b_D(s)$ associated to $D$, and\[ b_D(s) = \operatorname{lcm}_{x \in D} b_{f,x}(s), \]
where $f$ is any locally defining equation for $D$ in a neighborhood of $X$.

We next list a few basic facts regarding Bernstein-Sato polynomials.

**Remark 2.2.10.** (1) For $f$ invertible we could simply take $b_f(s) = 1$. This is why we restrict to $f$ non-invertible.

(2) If $f$ is arbitrary, then we have\[ (s + 1) | b_f(s). \]
Indeed, take \( s = -1 \) in the identity in Theorem 2.2.1, to obtain
\[
bf(-1) = P(-1) \frac{1}{f}.
\]
Since \( f \) is not constant, this is only possible when \( bf(-1) = 0 \).

(3) If \( f \) is smooth, by a simple reduction we can assume that \( f = x_1 \) in local algebraic coordinates \( x_1, \ldots, x_n \). It follows that \( bf(s) = s + 1 \), due to (2) and the formula
\[
\partial_{x_1} x_1^{s+1} = (s + 1)x_1^s.
\]
The converse is also true, meaning that if \( bf(s) = (s + 1) \), then \( f \) is smooth; see ??.

In the rest of the section we discuss a few standard examples.

**Example 2.2.11.** (1) Let \( f = x_1^2 + \cdots + x_n^2 \), and consider the Laplace operator
\[
\Delta = \partial^2_1 + \cdots + \partial^2_n.
\]
A simple calculation gives
\[
\Delta f^{s+1} = 4(s + 1)(s + \frac{n}{2}) f^s.
\]
According to the last comment in the Remark above, we obtain that
\[
bf(s) = (s + 1)(s + \frac{n}{2}).
\]
We will also obtain this as a special case of the general calculation for all weighted homogeneous singularities, Theorem 2.3.4 below.

(2) Let \( f = x^2 + y^3 \) be a cusp in the plane. A well-known, though tedious and not easily motivated, calculation is that
\[
= (s + 1)(s + \frac{5}{6})(s + \frac{7}{6}) f^s,
\]
and in fact
\[
bf(s) = (s + 1)(s + \frac{5}{6})(s + \frac{7}{6}).
\]
This again will be implied by the general result for weighted homogeneous singularities.

(3) Let \( f = \det(x_{ij}) \) be the determinant of a generic matrix in \( n \times n \) variables. A formula attributed to Cayley is
\[
\det(\partial/\partial x_{ij}) f^{s+1} = (s + 1)(s + 2) \cdots (s + n) f^s.
\]
We actually have \( bf(s) = (s + 1)(s + 2) \cdots (s + n) \), and this will also be a special case of the general result below.

(4) The case of a divisor with SNC support can also be computed explicitly. Let
\[
f = x_1^{a_1} \cdots x_n^{a_n}, \quad a_i \in \mathbb{N}.
\]
Note that in one variable \( x \), and \( a \geq 1 \), we have the formula
\[
\partial^a x^{as+a} = a^a (s + 1)(s + 1 - \frac{1}{a}) \cdots (s + 1 - \frac{a-1}{a}) x^{as}.
\]
Therefore using the operator
\[ P = \prod_{i=1}^{n} \frac{1}{a_i} \cdot \partial_1^{a_1} \cdots \partial_n^{a_n}, \]
a straightforward calculation gives
\[ Pf^{s+1} = b_{a_1, \ldots, a_n}(s)f^s, \quad b_{a_1, \ldots, a_n}(s) := \prod_{i=1}^{n} \left( \prod_{k=0}^{a_i-1} \left( s + 1 - \frac{k}{a_i} \right) \right) \]
and so \( b_{a_1, \ldots, a_n} \mid b_f \). We can in fact see that \( b_f = b_{a_1, \ldots, a_n} \) as follows. Starting with an expression
\[ \left( \sum_{\alpha, \beta, j} a_j^{\alpha} s^j x_1^\alpha \partial_1^\beta \right) x_1^{a(s+1)} = c(s) \cdot x_1^a, \]
where we use the notation \( x_1^a = x_1^{a_1} \cdots x_n^{a_n} \), etc., by comparing the terms containing \( x_1^a \) on both sides we see that the only contribution coming from the left hand side is from terms of the form \( a_j^{\beta} s^j x_1^\beta \partial_1^\beta \) with \( a_j^{\beta} \neq 0 \) and \( \beta_i = \alpha_i + a_i \). We then have that \( \beta_i \geq a_i \) for all \( i \), hence employing the one variable formula above repeatedly we see that each \( \partial_1^\beta x_1^{a(s+1)} \) contributes a polynomial (in \( s \)) term divisible by \( b_{a_1, \ldots, a_n} \). It follows that \( b_{a_1, \ldots, a_n} \mid c(s) \).

(5) If \( f(x_1, \ldots, x_n) = g(x_1, \ldots, x_m) \cdot h(x_{m+1}, \ldots, x_n) \), it is immediate to see that
\[ b_f \mid b_g \cdot b_h. \]

Whether equality holds seems to be an open problem. Note that for arbitrary \( f = g \cdot h \) it is very easy to produce examples where this divisibility does not hold (and there is no reason for it to do so). Consider for instance the triple point \( f = xy(x + y) \) in \( \mathbb{C}^2 \), and take \( g = xy \) and \( h = x + y \). Then
\[ b_g(s) \cdot b_h(s) = (s + 1)^3 \quad \text{while} \quad (s + \frac{2}{3}) \mid b_f(s). \]

For the last statement one can for instance use the general formula for weighted homogeneous singularities in Theorem 2.3.4, or the fact that the log canonical threshold of \( f \) is 2/3 combined with Theorem 2.7.2 below.

(6) Hyperplane arrangements. TO ADD.

2.3. Quasi-homogeneous singularities

In this section we study an extended example, proving a general formula for the Bernstein-Sato polynomial of a quasi-homogeneous isolated singularity.\(^3\)

We start by recalling a few basic notions from singularity theory. For a polynomial \( f \in \mathbb{C}[X] = \mathbb{C}[X_1, \ldots, X_n] \), inside the ring of convergent power series \( \mathbb{C}\{X\} \) we consider

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\(^2\)A priori there may be terms of other type in the differential operator, but after differentiation their contributions must cancel each other.

\(^3\)I thank Mingyi Zhang for giving lectures at Northwestern on this topic, which I am following here.
the associated Jacobian ideal
\[ J(f) := \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \subseteq \mathbb{C}\{X\}. \]
Similarly, the Tjurina ideal is
\[ (f, J(f)) = \left( f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \subseteq \mathbb{C}\{X\}. \]
The Milnor and Tjurina algebras associated to \( f \) are
\[ M_f := \mathbb{C}\{X\}/J(f) \quad \text{and} \quad T_f := \mathbb{C}\{X\}/(f, J(f)), \]
while the corresponding Milnor and Tjurina numbers are
\[ \mu_f := \dim_{\mathbb{C}} M_f \quad \text{and} \quad \tau_f := \dim_{\mathbb{C}} T_f. \]

If \( w = (w_1, \ldots, w_n) \in \mathbb{Q}^n_{>0} \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), we denote
\[ |w| = w_1 + \cdots + w_n \quad \text{and} \quad \langle w, \alpha \rangle = w_1\alpha_1 + \cdots + w_n\alpha_n \in \mathbb{Q}_{>0}. \]

**Definition 2.3.1.** We say that a polynomial \( f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha X^\alpha \in \mathbb{C}[X] \) is quasi-homogeneous (or weighted homogeneous) of type \((w; \rho)\) if for every \( \alpha \in \mathbb{N}^n \) such that \( a_\alpha \neq 0 \) we have
\[ \rho(X^\alpha) := \langle w, \alpha \rangle = \rho. \]
We simply say that \( f \) is quasi-homogeneous or weighted-homogeneous (with respect to the weights \( w_1, \ldots, w_n \)) if \( \rho = 1 \).

**Remark 2.3.2.** Note that the weights are not invariant under linear change of coordinates.

In what follows we assume that \( f(0) = 0 \), and \( 0 \) is an isolated singular point of \( Z(f) \).
It is a well-known fact that this condition is equivalent to \( \mu_f < \infty \), and also to \( \tau_f < \infty \); see e.g. \([GLS, \text{Lemma } 2.3]\). Under this assumption, a theorem of K. Saito \([KSa]\) states that a polynomial \( f \) is quasi-homogeneous (of any weight) after some biholomorphic change of coordinates if and only if the following equivalent conditions hold:
\[ f \in J(f) \iff \mu_f = \tau_f. \]

**Example 2.3.3.** (1) The most common examples of quasi-homogeneous polynomials are the diagonal hypersurfaces
\[ f = X_1^{a_1} + \cdots + X_n^{a_n} \in \mathbb{C}[X], \]
with weights \( w_i = \frac{1}{a_i} \) for \( i = 1, \ldots, n \).

(2) Let \( f = x^3 + xy^3 \in \mathbb{C}[X, Y] \). This is quasi-homogeneous, with weights \( w_1 = \frac{1}{3} \) and \( w_2 = \frac{2}{9} \). We have \( \tau_f = \mu_f = 7 \), and a monomial basis of the Milnor algebra seen as a \( \mathbb{C} \)-vector space is \( 1, x, x^2, y, xy, x^2y, y^2 \).

(3) Let \( f = x^5 + y^5 + x^2y^2 \in \mathbb{C}[X, Y] \). Then \( f \) is not a quasi-homogeneous polynomial. Indeed, it is not hard to compute that
\[ 10 = \tau_f < \mu_f = 11. \]
Let’s now fix a quasi-homogeneous polynomial \( f \in \mathbb{C}[X] \) with an isolated singularity, of weights \( w_1, \ldots, w_n \), and denote \( \mu = \mu_f \). We write
\[
M_f = \bigoplus_{i=1}^{\mu} \mathbb{C} \cdot e_i,
\]
where \( e_i \) are a monomial basis for the Milnor algebra as a \( \mathbb{C} \)-vector space. We also consider the set of rational numbers
\[
\Sigma = \{ \rho(e_1), \ldots, \rho(e_n) \},
\]
where each number appears without repetitions. With this notation, the main result is the following theorem obtained in [BGM]; see also [BGMM] for an extension to the case of polynomials which are non-degenerate with respect to their Newton polygon.

**Theorem 2.3.4.** The Bernstein-Sato polynomial of \( f \) is
\[
b_f(s) = (s + 1) \cdot \prod_{\rho \in \Sigma} (s + |w| + \rho).
\]

**Remark 2.3.5.** The statement of the theorem implies that all of the roots of \( b_f(s) \) different from \(-1\) are simple, while \(-1\) appears with multiplicity 1 or 2.

Before proving the theorem, we discuss some preliminaries. First, with respect to this set of weights, we define the *Euler vector field* as
\[
\chi := \sum_{i=1}^{n} w_i x_i \partial_{x_i}.
\]
Quasi-homogeneity immediately implies that we have the identity
\[
\chi(f) = f.
\]

**Lemma 2.3.6.** For every \( u \in \mathbb{C}\{X\} \) and every \( \rho \in \mathbb{Q} \) we have
\[
(s + |w| + \rho)uf^s = \left( \sum_{i=1}^{n} w_i \partial_{x_i}(x_i u) + pu - \chi(u) \right)f^s.
\]

**Proof.** This is a simple exercise, using the identities:
\[
\sum_{i=1}^{n} w_i \partial_{x_i} x_i = \chi + |w| \quad \text{(as operators)}
\]
and
\[
\chi(uf^s) = su f^s + \chi(u) f^s.
\]

**Proof of Theorem 2.3.4.** Step 1. In this first step we show that
\[
b_f(s) \mid (s + 1) \cdot \prod_{\rho \in \Sigma} (s + |w| + \rho).
\]
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By definition, this follows if we show
\[(2.3.1) \quad (s + 1) \cdot \prod_{\rho \in \Sigma} (s + |w| + \rho) \in \mathcal{D}_{\mathbb{C}^n}[s]f^s.\]

This in turn follows by setting \(u = 1\) in the following more general:

**Claim:** For a quasi-homogeneous representative \(u\) of an element in \(M_f\), we have
\[(s + 1) \prod_{\rho \in \Sigma, \rho \geq \rho(u)} (s + |w| + \rho)u^s f^s = \sum_{i=1}^n A_i \partial_{x_i} \cdot f^{s+1}, \quad A_i \in A_n.\]

This can be proven by descending induction on the weight \(\rho(u)\). First, if \(\rho(u)\) is the maximal value in \(\Sigma\), then \(x_i u \in J(f)\) for all \(i\). Since \(\chi(u) = \rho(u)u\), by Lemma 2.3.6 we have
\[(s + |w| + \rho)uf^s = \left(\sum_{i=1}^n w_i \partial_{x_i}(x_i u)\right)f^s.\]

Modulo \(A_n \cdot J(f) \cdot f^s\), this last term is equal to
\[\sum_{i=1}^n \partial_{x_i} \cdot v_i,\]
with \(v_i\) a quasi-homogeneous element in \(M_f\) of weight \(\rho(u) + w_i\).

We now assume that the Claim is true for any \(u\) as above with \(\rho(u) > \nu \in \Sigma\), and take a quasi-homogeneous representative \(u'\) of an element in \(M_f\), such that \(\rho(u') = \nu\). By the inductive assumption, for each \(1 \leq i \leq n\) we have
\[\prod_{\rho \geq \rho(x_i u')} (s + |w| + \rho)(x_i u') f^s \in A_n \cdot J(f) \cdot f^s.\]

Notice that the action of any polynomial in \(s\) on \(A_n \cdot J(f) \cdot f^s\) stays in \(A_n \cdot J(f) \cdot f^s\), because
\[s \cdot \frac{\partial f}{\partial x_i} \cdot f^s = -(1 - w_i) \partial_i f f^s + \chi \cdot \frac{\partial f}{\partial x_i} \cdot f^s.\]

We can then multiply the products above by suitable factors of the form \((s + |w| + \rho')\), to get
\[\prod_{\rho \in \Sigma, \rho > \nu} (s + |w| + \rho)(x_i u') f^s \in A_n \cdot J(f) \cdot f^s\]
for all \(i\). Using now Lemma 2.3.6, a straightforward calculation (acting \(w_i \partial_i\) on these products and taking the sum) leads to
\[\prod_{\rho \in \Sigma, \rho \geq \nu} (s + |w| + \rho)u' f^s \in A_n \cdot J(f) \cdot f^s,\]
with \(A_i \in A_n\). Finally, multiplying both sides by \((s + 1)\) and applying the formula
\[(s + 1)\partial_{x_i} f^s = \partial_{x_i} f^{s+1},\]
we obtain the Claim.
Step 2. According to Step 1, we are left with proving that for every \( \rho \in \Sigma \) we have
\[
\tilde{b}_f(-|w| - \rho) = 0,
\]
where \( \tilde{b}_f = b_f(s)/(s + 1) \) is the reduced Bernstein-Sato polynomial of \( f \).

Note first that for the operator \( P \) in the functional equation
\[
P(s)f^{s+1} = b_f(s)f^s
\]
we have \( P(-1) \cdot 1 = 0 \), hence it is easy to check that we can write
\[
P(s) = (s + 1)Q(s) + \sum_{i=1}^{n} A_i \partial_{x_i}, \quad \text{with } Q(s) \in \mathbb{C}[s] \text{ and } A_i \in A_n[s].
\]
Consequently, we obtain
\[
\tilde{b}_f(s)f^s = (Q(s)f + \sum_{i=1}^{n} A_i \partial_{x_i}) \cdot f^s.
\]

Let now \( u \) be a monomial representing an element in \( M_f \), of weight \( \rho = \rho(u) \in \Sigma \). Multiplying the identity above on the left by \( u \), we obtain
\[
(2.3.2) \quad \tilde{b}_f(s) \cdot uf^s = (Q'(s)f + \sum_{i=1}^{n} A_i' \partial_{x_i}) \cdot f^s,
\]
where \( Q' = uQ \), and similarly for \( A_i \). On the other hand, we use Lemma 2.3.6 to get
\[
(2.3.3) \quad (s + |w| + \rho) \cdot uf^s = (\sum_{i=1}^{n} w_i \partial_{x_i}(x_i u)) f^s.
\]
Let’s assume now that the conclusion is false, so that the polynomials \( (s + |w| + \rho) \) and \( \tilde{b}_f(s) \) are coprime. In this case there exist polynomials \( p(s), q(s) \in \mathbb{C}[s] \) such that
\[
p(s)(s + |w| + \rho) + q(s)\tilde{b}_f(s) = 1.
\]
Using (2.3.2) and (2.3.3), we then obtain
\[
u f^s = \left( p(s)\left(\sum_{i=1}^{n} w_i \partial_{x_i}(x_i u)\right) + q(s)\left(Q'(s)f + \sum_{i=1}^{n} A_i' \partial_{x_i}\right)\right) f^s.
\]
Recalling that \( \chi(f^s) = sf^s \) (see the proof of Lemma 2.3.6) and \( f \in J(f) \), a straightforward calculation shows that an operator of the form
\[
R = u - \sum_{i=1}^{n} B_i \partial_{x_i} - \sum_{i=1}^{n} \partial_{x_i} C_i, \quad B_i, C_i \in A_n
\]
is in the annihilator of \( f^s \) in \( A_n \). However the Lemma below says that \( R \) belongs to \( \sum_{a \in \mathbb{N}} \partial^a J(f) \), and so it follows that \( u \in J(f) \), which is a contradiction.

\[\square\]

Lemma 2.3.7. For any polynomial \( f \) with an isolated singularity at the origin, the annihilator of \( f^s \) in the Weil algebra \( A_n \) is generated by the operators
\[
\frac{\partial f}{\partial x_i} \cdot \partial_{x_j} - \frac{\partial f}{\partial x_j} \cdot \partial_{x_i}, \quad 1 \leq i < j \leq n.
\]
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Proof. This is [Ya, Theorem 2.19]; see also [Gr, Appendix B]. It is a consequence of the fact that \( \{ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_1} \} \) form a regular sequence in the case of an isolated singularity. □

2.4. Analytic continuation of the archimedean zeta function

We present here one of the original motivations for the introduction of Bernstein-Sato polynomials, namely Bernstein’s approach to Gel’fand’s problem on the analytic continuation of complex powers of polynomials. The original problem was stated in the context of real polynomials; we will discuss its complex version.

For an introduction to what comes next, we recall the following statement from complex analysis:

Exercise 2.4.1. Let \( f: \mathbb{R}^n \to \mathbb{R} \) be a continuous function such that \( f(x) > 0 \) for all \( x \in \mathbb{R}^n \), and let \( \varphi \in C_0^0(\mathbb{R}^n) \) be a continuous complex \( \mathbb{C} \)-valued function with compact support. Then the function

\[
Z_\varphi: \mathbb{C} \to \mathbb{C}, \quad s \mapsto \int_{\mathbb{R}^n} f(x)^s \varphi(x) dx
\]

is an analytic function, where the meaning of \( f(x)^s \) is \( e^{s \log f} \).

Let now \( f \in \mathbb{C}[x_1, \ldots, x_n] \) be a nonconstant polynomial, and \( s \in \mathbb{C} \). We will consider \( |f|^{2s} \) as a distribution depending on the complex parameter \( s \), in the sense described below. Note first that this time \( f \) has zeros, and therefore the function \( |f|^2 \) does not satisfy the strict positivity condition in Exercise 2.4.1 any more.

On the other hand, it is easy to see that the function \( |f(x)|^{2s} \) is continuous in \( x \) if we restrict to the case when \( s \in \mathbb{H}_0 := \{ s \in \mathbb{C} \mid \text{Re}(s) > 0 \} \).

For any continuous \( \mathbb{C} \)-valued function with compact support \( \varphi \in C_0^0(\mathbb{C}^n) \), we then still have a well-defined function

\[
Z_\varphi: \mathbb{H}_0 \to \mathbb{C}, \quad s \mapsto \int_{\mathbb{C}^n} |f(x)|^{2s} \varphi(x) dx.
\]

Proposition 2.4.2. \( Z_\varphi \) is an analytic function on \( \mathbb{H} \).

Proof. First proof. The statement is an immediate application of the complex version of the theorem on differentiation under the integral sign; see e.g. [?]. Indeed, the function \( |f(x)|^{2s} \varphi(x) \) is integrable for each \( s \in \mathbb{H}_0 \), and is analytic as a function of \( s \), hence with bounded partial derivatives on the support of \( \varphi \). Therefore applying \( \partial/\partial \bar{z} \) commutes with the integral sign, and the statement follows.

Second proof. Let me also include a slightly more tedious second proof, which has the potentially useful advantage of providing the coefficients of a power series expansion in a neighborhood of a point in \( \mathbb{H}_0 \).
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Fix a point \( s_0 \in \mathbb{H}_0 \). We show that there exists \( \varepsilon > 0 \) such that for all \( s \in \mathbb{H} \) with \( |s - s_0| < \varepsilon \) we have

\[
Z_\varphi(s) = \sum_{k=0}^{\infty} a_k (s - s_0)^k,
\]
a convergent power series with \( a_k \in \mathbb{C} \).

For \( s \in \mathbb{H}_0 \) close enough to \( s_0 \), using the usual expansion of the exponential function we have

\[
|f(x)|^{2s} \varphi(x) = \sum_{k=0}^{\infty} \frac{(2\log|f(x)|)^k |f(x)|^{2s_0} \varphi(x)}{k!} \cdot (s - s_0)^k.
\]

(2.4.1)

Since \( \varphi \) has compact support, we can find \( C > 0 \) such that

\[
\text{Supp}(\varphi) \subseteq K := [-C, C]^n.
\]

Claim. There exist real numbers \( \varepsilon > 0 \) and \( M > 0 \) such that

\[
a_k := \sup_{x \in K} \left| \frac{(2\log|f(x)|)^k |f(x)|^{2s_0}}{k!} \right| \leq \frac{M}{\varepsilon^k}, \quad \forall k \in \mathbb{N},
\]

with the convention that the quantity we are measuring is 0 at \( x \in K \) such that \( f(x) = 0 \). Assuming the Claim, we obtain that the series in (2.4.1) converges absolutely and uniformly on \( K \times \{|s - s_0| < \delta\} \) for any \( 0 < \delta < \varepsilon \), hence

\[
Z_\varphi(s) = \sum_{k=0}^{\infty} \int_{\mathbb{C}^n} \frac{(2\log|f(x)|)^k |f(x)|^{2s_0}}{k!} \varphi \cdot (s - s_0)^k \quad \text{for} \quad |s - s_0| < \varepsilon,
\]

which concludes the proof.

We are left with proving the Claim. Denoting \( r_0 = \text{Re}(s_0) \in \mathbb{R}_{>0} \), we note that

\[
k! \cdot a_k = \sup_{x \in K} |(2\log|f(x)|)^k |f(x)|^{2s_0}| = |(2\log|f(x)|)^k |f(x)|^{r_0}| = \frac{2^k}{r_0^k} |(\log|f(x)|)^k |f(x)|^{r_0}|.
\]

We consider \( M \in \mathbb{R}_{>0} \) such that \( |f(x)|^{r_0} \leq M \) for all \( x \in K \), so that

\[
a_k \leq \frac{\sup_{x \in (0, M]} 2^k |(\log x)^k x|}{k! r_0^k} \leq \frac{2^k \cdot \max \{M (\log M)^k, k^k e^{-k}\}}{k! r_0^k},
\]

where the second inequality uses the easily checked fact that the \( x(\log x)^k \) takes its minimum on \((0, 1] \) at \( x = e^{-k} \). Finally, Stirling’s formula says that

\[
k! \sim \sqrt{2\pi k} \cdot k^k e^{-k},
\]

which implies that, choosing a suitable \( M > 0 \), any \( 0 < \varepsilon < r_0 \) will do for the Claim.

The problem proposed by Gel’fand was whether one can analytically continue \( Z_\varphi \) to a meromorphic function on \( \mathbb{C} \); he also asked whether its poles lie in a finite number of arithmetic progressions. The existence of the Bernstein-Sato polynomial leads to a positive answer.
THEOREM 2.4.3. With the notation above, for every smooth complex-valued function with compact support \( \varphi \in C^\infty_c(\mathbb{C}^n) \), \( Z_\varphi \) admits an analytic continuation to \( \mathbb{C} \) as a meromorphic function whose poles are of the form \( \alpha - m \), where \( \alpha \) is a root of the Bernstein-Sato polynomial \( b_f(s) \) and \( m \in \mathbb{N} \). In particular all the poles are negative rational numbers.

Before giving the proof of the theorem, we record a technical point that will be useful later on as well. We start with the defining Bernstein-Sato formula

\[
b(s)f^s = P(s)f^{s+1}, \quad \text{with} \quad P \in A_n[s],
\]

where \( A_n \) is the Weyl algebra, and for simplicity we denote \( b(s) = b_f(s) \). We also consider the conjugate \( \bar{P}(s) \), where we replace \( x_i \) by \( \bar{x}_i \) and \( \partial_{x_i} \) by \( \partial_{\bar{x}_i} \).

**Lemma 2.4.4.** For every \( s \in \mathbb{C} \) we have

\[
b(s)^2|f|^{2s} = P(s)P(s)|f|^{2(s+1)}.
\]

**Proof.** Conjugating the Bernstein-Sato formula above gives

\[
b(s)f^s = \bar{P}(s)f^{s+1}.
\]

(Note that by Kashiwara’s theorem, \( b(s) \) has rational coefficients.) Hence multiplying the two formulas leads to

\[
b(s)^2|f|^{2s} = (P(s)f^{s+1}) \cdot (\bar{P}(s)f^{s+1}).
\]

The fact that the coefficients of \( P(s) \) are antiholomorphic differential operators implies that

\[
\bar{P}(s)(f^{s+1} \cdot \bar{f}^{s+1}) = f^{s+1} \cdot \bar{P}(s)f^{s+1}, \tag{4}
\]

and therefore

\[
P(s)\bar{P}(s)|f|^{2(s+1)} = P(s)\bar{P}(s)(f^{s+1} \cdot \bar{f}^{s+1}) = (P(s)f^{s+1}) \cdot (\bar{P}(s)f^{s+1})
\]

as similarly since \( P \) is holomorphic we have \( P(s)(\bar{P}(s)f^{s+1}) = 0 \). \qed

**Proof. (of Theorem 2.4.3).** For every \( m \in \mathbb{N} \), denote

\[
\mathbb{H}_m := \{ s \in \mathbb{C} \mid \text{Re}(s) > -m \}.
\]

We have seen that all such \( Z_\varphi \) are holomorphic on \( \mathbb{H}_0 \), and we show by induction on \( m \) that they can be extended to meromorphic functions on \( \mathbb{H}_m \) with poles as stated.

Going back to the identity

\[
b(s) \cdot f^s = P(s) \cdot f^{s+1},
\]

we write

\[
P(s) = \sum_{i=0}^{p} Q_i \cdot s^i, \quad Q_i \in A_n.
\]

We then have

\[
P(s)\bar{P}(s) = \sum_{i=0}^{2p} R_i \cdot s^i,
\]

\(^4\text{Apply one } \partial/\partial \bar{z}_i \text{ at a time, and use the product rule.}\)
for operators $R_i = \overline{R}_i$. For $s \in \mathbb{H}_0$ we can then write
\[
b(s)^2 \int_{\mathbb{C}^n} |f(x)|^{2s} \varphi(x) = \sum_{i=0}^{2p} s^i \int_{\mathbb{C}^n} (R_i|f(x)|^{2(s+1)}) \cdot \varphi(x).
\]
Now a simple application of integration by parts (see Exercise 2.4.5 below) shows that
\[
\int_{\mathbb{C}^n} (R_i|f(x)|^{2(s+1)}) \cdot \varphi(x) = \int_{\mathbb{C}^n} |f(x)|^{2(s+1)} \cdot (R_i\varphi)(x).
\]
As a consequence, for $s \in \mathbb{H}_0$ we obtain
\[
Z_\varphi(s) = \sum_{i=0}^{2p} s^i Z_{R_i\varphi}(s+1) \frac{b(s)^2}{b(s)^2}.
\]
But we are assuming by induction that $Z_{R_i\varphi}(s)$ can be extended to a meromorphic function with poles as prescribed on $\mathbb{H}_m$. Therefore the identity shows that $Z_\varphi(s)$ can be extended to a meromorphic function $\mathbb{H}_{m+1}$, whose poles are among the roots of $b(s)$ and the poles of the extension of $Z_{R_i\varphi}(s+1)$ to $\mathbb{H}_{m+1}$, again of the prescribed form. The last statement follows from Kashiwara’s result, Theorem 2.5.1. \(\square\)

**Exercise 2.4.5.** Let $Q \in A_n$, and define $Q'$ to be its image under the (anti-)automorphism of $A_n$ sending $x_i$ to $x_i$ and $\partial_i$ to $-\partial_i$. For every complex-valued $\varphi \in C_c^\infty(\mathbb{C}^n)$ we have
\[
\int_{\mathbb{C}^n} (Q|f(x)|^{2(s+1)}) \cdot \varphi(x) = \int_{\mathbb{C}^n} |f(x)|^{2(s+1)} \cdot (Q'\varphi)(x),
\]
and similarly for $\overline{Q}$. (Hint: apply the change of variables formula when $Q = \partial_i$, for each $i$, and then use induction on the degree of $Q$.)

**Remark 2.4.6 (The distribution $|f|^{2s}$).** Usually the quantity $|f|^{2s}$ is thought of as a distribution, which on any $\varphi \in C_c^\infty(\mathbb{C}^n)$ is defined by
\[
(|f|^{2s}, \varphi) := \int_{\mathbb{C}^n} |f(x)|^{2s} \varphi(x).
\]
This depends holomorphically on $s$ for Re$(s) > 0$, and the theorem above says that it can be continued meromorphically to $\mathbb{C}$, with poles of the form $\alpha - m$, where $\alpha$ is a root of the Bernstein-Sato polynomial $b_f(s)$ and $m \in \mathbb{N}$. Here if $P(\cdot)$ denotes the set of poles, by definition we have
\[
P(|f|^{2s}) := \bigcup_\varphi P(Z_\varphi(s)).
\]

**Note.** Bernstein-Gel’fand and Atyiah originally answered Gel’fand’s question making use of resolution of singularities. The proof presented here, making use instead of the existence of the Bernstein-Sato polynomial, was given later by Bernstein in [Be]. The exposition draws also on the lecture notes [Ay], [Gr].
2.5. Kashiwara’s rationality theorem

In this section we prove the main theorem in [Ka1]. It is worth noting that Kashiwara’s proof also works when \( f \) is the germ of an analytic function on a complex manifold.

**Theorem 2.5.1.** For every non-invertible regular function \( f \) on a smooth complex variety \( X \), the roots of \( b_f(s) \) are negative rational numbers.

The module \( \mathcal{N}_f \). We start by enhancing the constructions described in §2.1. We denote as always by \( D \) the effective divisor on \( X \) corresponding to \( f \).

We consider the \( \mathcal{D}_X \)-submodule

\[
\mathcal{N}_f := \mathcal{D}_X[s]f^s \subseteq \mathcal{O}_X \left[ \frac{1}{f}, s \right] f^s \simeq \mathcal{O}_X(*D).
\]

Since \( \partial t \delta \in \mathcal{I}_+ \mathcal{O}_X \), we have in fact \( \mathcal{N}_f \subseteq \mathcal{I}_+ \mathcal{O}_X \).

**Lemma 2.5.2.** Multiplication by \( t \) leaves \( \mathcal{N}_f \) invariant, so we have an endomorphism \( t : \mathcal{N}_f \to \mathcal{N}_f \).

**Proof.** Recall that the action of \( t \) on \( \mathcal{O}_X \left[ \frac{1}{f}, s \right] f^s \) is given by the automorphism “\( s \to s + 1 \)”. Hence if \( Q(s)f^s \in \mathcal{N}_f \), using (2.1.2) we have

\[
t \cdot Q(s)f^s = (Q(s + 1) \cdot f) \cdot f^s \in \mathcal{N}_f.
\]

Observe that the image \( t\mathcal{N}_f \) of the endomorphism in the Lemma is \( \mathcal{D}_X[s]f \cdot f^s = \mathcal{D}_X[\partial t]t \delta \). Another remark is that the action of \( s \) clearly leaves \( \mathcal{N}_f \) invariant as well, so using the notation above \( \mathcal{N}_f \) is a \( \mathcal{D}_X(t,s) \)-module.

**Proposition 2.5.3.** The action of \( s \) induces an endomorphism

\[
s : \mathcal{N}_f/t\mathcal{N}_f \longrightarrow \mathcal{N}_f/t\mathcal{N}_f,
\]

which has a minimal polynomial equal to the Bernstein-Sato polynomial \( b_f(s) \).

**Proof.** For the first assertion we only need to check that \( s(t\mathcal{N}_f) \subseteq t\mathcal{N}_f \). But this is clear since by (2.1.2) we have

\[
s \cdot t \cdot \mathcal{N}_f = t \cdot (s - 1) \cdot \mathcal{N}_f.
\]

For the second assertion, we first show that \( b_f(s) \) is identically zero on \( \mathcal{N}_f/t\mathcal{N}_f \). Let \( Q(s)f^s \in \mathcal{N}_f \). Then, since \( b_f(s) \) has constant coefficients, we have

\[
b_f(s)Q(s)f^s = Q(s)b_f(s)f^s = Q(s)P(s)f \cdot f^s \in t\mathcal{N}_f.
\]

On the other hand, say that \( h(s) \in \mathbb{C}[s] \) is identically zero on \( \mathcal{N}_f/t\mathcal{N}_f \). This means in particular that

\[
h(s)f^s \in t\mathcal{N}_f,
\]

which means that there exists \( P(s) \in \mathcal{D}_X[s] \) such that \( h(s)f^s = P(s)f \cdot f^s \). By the definition of the Bernstein-Sato polynomial we then have that \( b_f(s)h(s) \).

\( \square \)
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Remark 2.5.4. The previous result implies in particular that $\mathcal{N}_f/t\mathcal{N}_f$ is finitely generated over $\mathscr{D}_X$. More precisely, if $\deg b_f(s) = d$, then it is generated by the classes of $\delta, s\delta, \ldots, s^{d-1}\delta$. Kashiwara showed in fact the stronger statement that $\mathcal{N}_f/t\mathcal{N}_f$ is holonomic; more below.

We record one more basic fact about $\mathcal{N}_f$:

Lemma 2.5.5. We have

$$(\mathcal{N}_f)_t = (\iota_+ \mathcal{O}_X)_t = \iota_+ \mathcal{O}_X (\ast D).$$

Proof. We need to show that for every $w \in \iota_+ \mathcal{O}_X$, there exists $m \in \mathbb{N}$ such that $t^m w \in \mathcal{N}_f$. Write

$$w = \sum_{j=0}^{p} h_j \partial_t^j \delta, \quad h_j \in \mathcal{O}_X,$$

and note that $t(h_j \partial_t^j \delta) = h_j(t \partial_t^j \delta)$. Hence it suffices to show that

$$t^j \partial_t^j \delta \in \mathcal{N}_f = \mathcal{D}_X[\partial_t] \delta.$$

But this follows from the next Exercise, applied with $i = j$. □

Exercise 2.5.6. For all integers $i \geq j \geq 0$, we have the operator identity

$$t^i \partial_t^j = \prod_{k=1}^{j} (\partial_t - (i - k + 1)) \cdot t^{i-j}.$$

Main construction. We now describe the main technical construction of Kashiwara. We consider a log resolution $\mu: Y \rightarrow X$ of $D$, which is an isomorphism away from $D$, see §2.6, and we denote $f' = f \circ \mu$.

As in the discussion above, we denote

$$\mathcal{N}_f := \mathcal{D}_X[s] \cdot f^* \quad \text{and} \quad \mathcal{N}_{f'} := \mathcal{D}_{X'}[s] \cdot f'^*.$$

We also define

$$\mathcal{N} := \mathcal{H}^0 \mu_* \mathcal{N}_{f'}.$$

We will see below that we can attach to $\mathcal{N}$ a Bernstein-Sato polynomial $b_{\mathcal{N}}(s)$, and that $b_{\mathcal{N}}(s) \mid b_{f'}(s)$. Hence one of the key points in the proof of Kashiwara’s theorem will be to compare $b_f(s) = b_{\mathcal{N}}(s)$ with $b_{\mathcal{N}}(s)$. This is however not immediate; it turns out that $\mathcal{N}_f$ is a special type of sub-quotient of $\mathcal{N}$, but this requires a bit of work.

To this end, we construct a distinguished section $u \in \Gamma(X, \mathcal{N})$. First, since $\mu$ is birational, there is a natural nontrivial (hence injective) map $p^*\omega_X \rightarrow \omega_Y$, hence a natural section $s: \mathcal{O}_Y \rightarrow \omega_{Y/X}$. On the other hand, recall that by definition

$$\mathcal{N} = R^0 \mu_* \left( \mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{O}_Y} \mathcal{N}_{f'} \right),$$

where

$$\mathcal{D}_{X \leftarrow Y} = \mu^{-1} \mathcal{D}_X \otimes_{\mu^{-1} \mathcal{O}_X} \omega_{Y/X}.$$
is the transfer \((\mu^{-1} \mathcal{D}_X, \mathcal{D}_Y)\)-bimodule. Note that \(1 \in \mathcal{D}_X\) and \(s\) give rise to a homomorphism

\[ \mathcal{O}_Y \to \mathcal{D}_{X \leftarrow Y} \]

of \((\mu^{-1} \mathcal{O}_X, \mathcal{O}_Y)\)-bimodules. On the other hand the section \(f^{t^s}\) of \(\mathcal{N}_f\) gives rise to a \(\mathcal{O}_Y\)-module homomorphism \(\mathcal{O}_Y \to \mathcal{N}_f\). Taking the tensor product of these two we obtain a morphism

\[ \mathcal{O}_Y \to \mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{O}_Y} \mathcal{N}_f \]

in the derived category of \(\mu^{-1} \mathcal{O}_X\)-modules, and applying \(R^0 \mu_*\) to this we obtain our desired section \(u \in \Gamma(X, \mathcal{N})\). We define

\[ \mathcal{N}' := \mathcal{D}[s] \cdot u \subseteq \mathcal{N}, \]

a \(\mathcal{D}[s]\)-submodule of \(\mathcal{N}\).

**Lemma 2.5.7.** The module \(\mathcal{N}'\) satisfies the following properties:

1. It sits in a diagram of \(\mathcal{D}(s, t)\)-modules

\[
\begin{array}{ccc}
\mathcal{N}' & \xrightarrow{i} & \mathcal{N} \\
\downarrow & & \downarrow \\
\mathcal{N}_f & &
\end{array}
\]

where \(i\) is the inclusion map, and \(g\) is surjective.

2. It is a coherent \(\mathcal{D}_X\)-module.

**Proof.** To see that \(\mathcal{N}'\) is a \(\mathcal{D}_X(s, t)\)-module, we only need to check that it is preserved by the action of \(t\). Denoting by \(1_{X \leftarrow Y}\) the section of \(\mathcal{D}_{X \leftarrow Y}\) considered above, we have

\[ t \cdot (1_{X \leftarrow Y} \otimes f^{t^s}) = 1_{X \leftarrow Y} \otimes t f^{t^s} = 1_{X \leftarrow Y} \otimes f f^{t^s} = f (1_{X \leftarrow Y} \otimes f^{t^s}), \]

which indeed shows that \(t \cdot \mathcal{N}' \subseteq \mathcal{N}'\).

To complete the proof of (1), it suffices to check that the mapping

\[ \mathcal{D}_X[s] \cdot u \to \mathcal{D}_X[s] \cdot f^s, \quad P(s)u \mapsto P(s)f^s \]

is well defined. But assuming that \(P(s)u = 0\), and recalling that \(u\) and \(f^s\) coincide on the open set \(U\), it follows that \(P(s)f^s\) is zero on \(U\) and therefore annihilated by a power of \(f\). This means that in fact \(P(s)f^s = 0\), since \(\mathcal{D}_X[s] \cdot f^s\) is torsion-free as an \(\mathcal{O}_X\)-module; indeed, it is a submodule of \(\mathcal{O}_X[1/t, s]f^s\), which is isomorphic to \(\mathcal{O}_X[1/t, s]f^s\) as an \(\mathcal{O}_X\)-module, and therefore torsion-free.

The assertion in (2) follows immediately from (1), since \(\mathcal{N}\) is a coherent \(\mathcal{D}_X\)-module by the behavior of direct images under projective morphisms.

**Bernstein-sato polynomials of special \(\mathcal{D}\)-modules.** We will make use of the following general theorem; see [Ka2, Theorem 4.45].

**Theorem 2.5.8.** Let \(\mathcal{M}\) and \(\mathcal{N}\) be holonomic \(\mathcal{D}_X\)-modules. Then \(\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})_x\) is a finite dimensional \(\mathbb{C}\)-vector space for every \(x \in X\).
2.5. KASHIWARA’S RATIONALITY THEOREM

This allows us to deduce:

**Proposition 2.5.9.** Let $\mathcal{M}$ be a $\mathcal{D}_X[s]$-module, which is holonomic as $\mathcal{D}_X$-module. Then there exists a polynomial $b(s) \in C[s]$ such that $b(s) \cdot \mathcal{M} = 0$. (Therefore the action of $s$ on $\mathcal{M}$ has a minimal polynomial.)

**Proof.** By Theorem 2.5.8, we know that $\text{End}_{\mathcal{D}_X}(\mathcal{M})_x$ is a finite dimensional $C$-vector space for all $x \in X$. It follows that there exists a polynomial $b_x(s)$ such that $b_x(s) \cdot \mathcal{M}_x = 0$, and this continues to hold on a neighborhood of $x$. We can now cover $X$ with finitely many such neighborhoods, and take $b(s)$ to be the least common multiple of the respective $b_x(s)$. □

**Lemma 2.5.10.** Let $\mathcal{N}$ be a $\mathcal{D}_X(s,t)$-module which is holonomic as a $\mathcal{D}_X$-module. Then there exists an integer $N \geq 0$ such that $t^N \cdot \mathcal{N} = 0$.

**Proof.** Let $b(s)$ be a polynomial as in Proposition 2.5.9, so that $b(s) \cdot \mathcal{N} = 0$. If

$$b(s) = \prod_{i=1}^r (s + \alpha_i)^{m_i},$$

then every $\mathcal{N}_x$ has a decomposition into subspaces on which some $s + \alpha_i$ acts nilpotently. Therefore for $N \gg 0$, the action of $b(s + N)$ on $\mathcal{N}$ is bijective. Recalling that $P(s)t = tP(s - 1)$ for every $P \in C[s]$, we have

$$b(s + N) \cdot t^N \cdot \mathcal{N} = t^N \cdot b(s) \cdot \mathcal{N} = 0,$$

hence $t^N \cdot \mathcal{N} = 0$. □

Let now $\mathcal{N}$ be a module $\mathcal{D}_X(s,t)$-module, assumed moreover to be coherent over $\mathcal{D}_X$, and such that $\mathcal{N}/t\mathcal{N}$ is holonomic. By Proposition 2.5.9, the action of $s$ on $\mathcal{N}/t\mathcal{N}$ has a minimal polynomial, which we denote $b_N(s)$. In other words, this is the monic polynomial of minimal degree such that

$$b_N(s) \cdot \mathcal{N} \subseteq t\mathcal{N}.$$

**Example 2.5.11.** We have then seen in Proposition 2.5.3 that $b_f(s) = b_N(s)$. In fact Kashiwara shows in [Ka1], in the general setting of germs of holomorphic functions, that $\mathcal{N}_f/t\mathcal{N}_f$ is holonomic, which by the discussion above leads to another proof of the existence of the Bernstein-Sato polynomial.

**Lemma 2.5.12.** Let $\mathcal{N}$ be as above, and $\mathcal{N}'$ a $\mathcal{D}_X(s,t)$-submodule of $\mathcal{N}$. Assume that $\mathcal{N}/\mathcal{N}'$ is a holonomic $\mathcal{D}_X$-module. Then there exists an integer $N \geq 0$ such that

$$b_N(s) \mid b_N(s)b_N(s+1) \cdots b_N(s+N).$$

**Proof.** Note that $b_N(s)$ makes sense, as $\mathcal{N}/t\mathcal{N}$ is holonomic as well; see the Exercise below.

Now since $\mathcal{N}/\mathcal{N}'$ is holonomic, Lemma 2.5.10 implies that there exists $N \geq 0$ such that $t^N \cdot (\mathcal{N}/\mathcal{N}') = 0$, or equivalently

$$t^N \cdot \mathcal{N} \subseteq \mathcal{N}'.$$
As above, for every \( j \geq 0 \) we have
\[
b_N(s + j)t^j \cdot \mathcal{N} = t^j b_N(s) \cdot \mathcal{N} \subseteq t^{j+1} \cdot \mathcal{N}.
\]
Applying this repeatedly, we obtain
\[
b_N(s + N) \cdots b_N(s) \cdot \mathcal{N}' \subseteq b_N(s + N) \cdots b_N(s) \cdot \mathcal{N} \subseteq t^{N+1} \cdot \mathcal{N} \subseteq t \cdot \mathcal{N}',
\]
which implies the conclusion by the definition of \( b_N'(s) \).

**Exercise 2.5.13.** Show that in the situation above, i.e. \( \mathcal{N}' \) is a \( \mathcal{D}_X(s,t) \)-submodule of a module \( \mathcal{N} \) having the property that \( \mathcal{N}/t\mathcal{N} \) is holonomic, we have that \( \mathcal{N}'/t\mathcal{N}' \) is holonomic as well.

**Proof of the Theorem.** We go back to the main construction at the beginning of the section. A crucial technical result of Kashiwara, see [Ka1, Lemma 5.7] or [Ka2, p.113], is the following. Strictly speaking, in the theorem below one needs to assume that the zero locus of the Jacobian ideal \( \text{Jac}(f) \) (generated locally by \( \partial f/\partial x_1, \ldots, \partial f/\partial x_n \)) is contained in the zero locus of \( f \). This however can be accomplished after passing to an open neighborhood of the zero locus of \( f \), which is of course all we care about if we want to study \( b_f(s) \). Recall that \( \mathcal{N} = \mathcal{H}^0 \mu_* \mathcal{N}_f'. \)

**Theorem 2.5.14.** The \( \mathcal{D}_X \)-module \( \mathcal{N} \) is subholonomic. More precisely we have
\[
\text{Ch}(\mathcal{N}) = W_f \cup \Lambda,
\]
where \( \Lambda \) is a Lagrangian subvariety of \( T^*X \), and \( W_f \) is the closure of the subset
\[
\{(x, s \text{dlog} f(x) \mid f(x) \neq 0, \ s \in \mathbb{C} \} \subset T^*X,
\]
which is involutive and \((n + 1)\)-dimensional.

**Corollary 2.5.15.** The \( \mathcal{D}_X \)-module \( \mathcal{N}/t\mathcal{N} \) is holonomic.

**Proof.** This is true for an arbitrary subholonomic \( \mathcal{D} \)-module preserved by the action of \( t \). Indeed, consider the short exact sequence
\[
0 \rightarrow \mathcal{N} \xrightarrow{t} \mathcal{N} \rightarrow \mathcal{N}/t\mathcal{N} \rightarrow 0.
\]
It implies that the characteristic variety of \( \mathcal{N}/t\mathcal{N} \) satisfies
\[
\text{Char}(\mathcal{N}/t\mathcal{N}) \subseteq \text{Char}(\mathcal{N}),
\]
but also that its multiplicity along each irreducible component of \( \text{Char}(\mathcal{N}) \) is equal to zero. Hence its dimension is strictly smaller than that of \( \text{Char}(\mathcal{N}) \), so it can only be equal to \( n = \dim X \).

By the discussion above, the Corollary implies that there exists a Bernstein-Sato polynomial \( b_N(s) \); it satisfies the following:

**Lemma 2.5.16.** We have the divisibility
\[
b_N(s) \mid b_f'(s).
\]
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**Proof.** By definition we have \( b_{f'}(s) \cdot N' \subseteq tN' \), so there exists a \( \mathcal{D}_X \)-module endomorphism \( \varphi: \mathcal{N}' \to \mathcal{N}' \) such that \( b_{f'} = t\varphi \).

We can apply the functor \( H^0_{\mu_+} \) to this identity; since the push-forward is in the sense of \( \mathcal{D}_Y \)-modules, it commutes with the \( s \) and \( t \) action, hence we obtain an induced endomorphism

\[
b_{f'}: \mathcal{N} \to \mathcal{N}, \quad b_{f'} = t\psi,
\]

where \( \psi = H^0_{\mu_+}(\varphi) \). This immediately gives the conclusion. \( \square \)

We can now address the main result of the section.

**Proof of Theorem 2.5.1.** With the notation above, by Example 2.2.11(4) the statement holds for \( b_{f'}(s) \). The result follows from the following stronger statement: there exists an integer \( N \geq 0 \) such that

\[
(2.5.1) \quad b_f(s) \mid b_{f'}(s)b_{f'}(s+1) \cdots b_{f'}(s+N).
\]

To show (2.5.1), we start by considering the \( \mathcal{D}_X(s,t) \)-module

\[
\mathcal{M} := \mathcal{N}/\mathcal{N}'.
\]

This is a coherent \( \mathcal{D}_X \)-module, and we claim that it is holonomic. Indeed, on one hand we have

\[
\text{Ch}(\mathcal{M}) \subseteq \text{Ch}(\mathcal{N}) = W_f \cup \Lambda,
\]

using Theorem 2.5.14. On the other hand, by construction \( \text{Supp}(\mathcal{M}) \subseteq D = (f = 0) \), since by construction \( \mathcal{N} \) and \( \mathcal{N}' \) coincide away from \( D \). Hence in fact

\[
\text{Ch}(\mathcal{M}) \subseteq (W_f \cap \pi_X^{-1}(D)) \cup \Lambda.
\]

It remains to note that by definition \( W_f \) dominates \( X \), and therefore the right hand side has dimension \( n \).

We can therefore apply Lemma 2.5.12, since by Corollary 2.5.15 we know that \( \mathcal{N}/t\mathcal{N}' \) is holonomic. We conclude that there exists an integer \( N \geq 0 \) such that

\[
(2.5.1) \quad b_{\mathcal{N}'}(s) \mid b_{\mathcal{N}'}(s)b_{\mathcal{N}'}(s+1) \cdots b_{\mathcal{N}'}(s+N).
\]

By Lemma 2.5.16 we also know that \( b_{\mathcal{N}'}(s) \mid b_{f'}(s) \). It suffices then to show that \( b_f(s) \mid b_{\mathcal{N}'}(s) \).

But this is clear, since \( \mathcal{N}_f \) is a quotient of \( \mathcal{N}' \), so \( b_{\mathcal{N}'}(s) \cdot \mathcal{N}_f \subseteq t\mathcal{N}_f \). \( \square \)

2.6. Log resolutions, log canonical thresholds, multiplier ideals

In this section we review a few basic concepts from birational geometry. Most of the material below is covered in great detail in [La, Ch.9]. For general singularities of pairs, a great introduction is [KM].

We start by recalling a few notions and facts related to resolution of singularities. We always consider \( X \) to be a smooth complex variety of dimension \( n \).
2. THE BERNSTEIN-SATO POLYNOMIAL

Definition 2.6.1. Let $D$ be an effective $\mathbb{Q}$-divisor on $X$. A log resolution of the pair $(X, D)$ is a proper birational morphism $\mu: Y \to X$, where $Y$ is a smooth variety and the divisor $\mu^* D + \text{Exc}(\mu)$ has simple normal crossing (SNC) support.

Log resolutions are known to exist in characteristic 0 by Hironaka’s theorem. Moreover, one can always obtain such a resolution as a sequence of blow-ups along smooth centers contained in the locus where $D$ does not have SNC support. Hence we will often assume that $\mu$ is an isomorphism away from this locus, so in particular away from $U = X \setminus \text{Supp}(D)$.

Remark 2.6.2. For concrete calculations, the existence of such a resolution can be rephrased locally as saying that for any regular function $f$ on $X$, there exists a proper birational map as above, and around each point $y \in Y$ a system of algebraic coordinates $y_1, \ldots, y_n$ such that in this neighborhood
\[
\mu^* f = h \cdot y_1^{a_1} \cdot \cdots \cdot y_n^{a_n},
\]
with $k_i \geq 0$ and $h$ an invertible function.

We will write
\begin{equation}
(2.6.1) \quad f^* D = \sum_{i=1}^{m} a_i \cdot E_i,
\end{equation}
where $a_i \geq 0$ are rational numbers, and $E_i$ are either components of the proper transform $\tilde{D}$ or exceptional divisors. Another standard point to note is that while we cannot talk about canonically defined divisors $K_X$ and $K_Y$, there is a canonically defined relative canonical divisor $K_{Y/X}$, namely the zero locus of the Jacobian
\[
\text{Jac}(\mu) = \det(\frac{\partial \mu_i}{\partial y_j})_{i,j}
\]
of the map $\mu$. This supported on the exceptional locus of $\mu$. We will also write
\begin{equation}
(2.6.2) \quad K_{Y/X} := \text{div}(\text{Jac}(\mu)) = \sum_{i=1}^{m} b_i \cdot E_i.
\end{equation}
with $b_i \geq 0$. When $\mu$ is an isomorphism away from $U = X \setminus \text{Supp}(D)$, all exceptional divisors appear nontrivially in both sums.

Example 2.6.3. Let’s review a few well-known examples; all the statements are left as exercises.

1. If $\mu: Y = \text{Bl}_W X \to X$ is the blow-up of $X$ along a smooth subvariety of codimension $c$, and $F$ is the exceptional divisor over $W$, then
\[
K_{Y/X} = (c - 1)F.
\]
In particular, if $W = \{x\}$ is a point, then $K_{Y/X} = (n - 1)F$.

2. If $D = (x^2 + y^3 = 0) \subset \mathbb{C}^2$ is a cusp, then a log resolution $\mu: Y \to X = \mathbb{C}^2$ of $(X, D)$ can be obtained as the composition of three successive blow-ups at points; see the picture
in [La, Example 9.1.13]. If we denote by $F_1$, $F_2$ and $F_3$ the exceptional divisors arising from the three blow-ups (in this order), then an easy calculation gives

\[(2.6.3) \quad \mu^*D = \tilde{D} + 2F_1 + 3F_2 + 6F_3 \quad \text{and} \quad K_{Y/X} = F_1 + 2F_2 + 4F_3.\]

(3) Let $D = (f = 0)$ and assume for simplicity that $f(0) = 0$. Write $f = f_m + f_{m+1} + \cdots$, with $f_i$ homogeneous of degree $i$ for all $i$, and $f_m \neq 0$, so that $\text{mult}_0D = m$. Recall that the tangent cone of $D$ at 0 is

\[TC_xD = (f_m = 0) \subset \mathbb{A}^n,\]
while the projectivized tangent cone is $\mathbb{P}(TC_0D) \subset \mathbb{P}^{n-1}$. We say that $D$ has an ordinary singularity at 0 if $\mathbb{P}(TC_0D)$ is smooth.

The main example is when $D$ is the cone in $\mathbb{A}^n$ over a smooth hypersurface in $\mathbb{P}^{n-1}$. For instance one can take $f = x_1^{m_1} + \cdots + x_n^{m_n}$, the Fermat hypersurface of degree $m$.

If $x \in D$ is an ordinary singularity of multiplicity $m$, then it is an isolated singularity, and a log resolution is given by $\mu : Y = \text{Bl}_xX \to X$. We then have

\[\mu^*D = \tilde{D} + mF \quad \text{and} \quad K_{Y/X} = (n-1)F.\]

**Definition 2.6.4.** (1) Let $\mu : Y \to X$ be a log resolution of $(X, D)$, and fix a prime divisor $E$ on $Y$. The discrepancy of $E$ (with respect to $D$) is

\[a(E) = a(E; X, D) := \text{ord}_E(K_{Y/X} - \mu^*D) \in \mathbb{Q}.\]

Consequently we have

\[K_Y - \mu^*(K_X + D) = \sum_E a(E) \cdot E,\]

where the sum is taken over all prime divisors $E$ in $X$, or equivalently

\[K_Y + \tilde{D} - \mu^*(K_X + D) = \sum_{E \text{ exceptional}} a(E) \cdot E.\]

With the notation introduced above, we of course have $a(E_i) = b_i - a_i$.

(2) The pair $(X, D)$ is called log-canonical if $a(E) \geq -1$ for all exceptional divisors $E$ on any log resolution. It is called klt (Kawamata log terminal) if $a(E) > -1$ for all prime divisors $E$ on any log resolution.

(3) The log canonical threshold of $D$ is

\[\lct(D) := \inf\{c \in \mathbb{Q} \mid (X, cD) \text{ is not log canonical}\}.\]

In particular, if $D$ is an integral divisor, then $(X, D)$ is log canonical if and only if $\lct(D) = 1$. There is also a more refined local invariant, defined for any $x \in X$ to be

\[\lct_x(D) := \inf\{c \in \mathbb{Q} \mid (X, cD) \text{ is not log canonical around } x\}.\]

**Exercise 2.6.5.** We have $\lct(D) = \min_{x \in X} \lct_x(D)$.

**Lemma 2.6.6.** We have $\lct_x(D) \in \mathbb{Q}$, and the infimum in the definition is in fact a minimum that can be computed on any resolution of singularities.
Proof. We fix a log resolution $\mu : Y \rightarrow X,$ and use the notation introduced in (2.6.1) and (2.6.2). We then have, for $c \in \mathbb{Q},$
\[
K_Y - \mu^*(K_X + cD) = \sum_{i=1}^{m} (b_i - ca_i) E_i,
\]
and so with respect to the pair $(X, cD)$ we get, for every exceptional divisor $E_i,$ that
\[
a(E_i) \geq -1 \iff b_i - ca_i \geq -1 \iff \frac{b_i + 1}{a_i} \geq c.
\]
Hence we obtain that
\[
\text{lct}_x(D) = \min_{1 \leq i \leq m} \frac{b_i + 1}{a_i},
\]
where the minimum is taken over all $E_i$ such that $x \in \mu(E_i),$ at least after showing that this quantity is independent of the choice of log resolution. This is left as an exercise, but note that it will also follow from the analytic interpretation given below. \hfill \square

Exercise 2.6.7. Show that the function $x \mapsto \text{lct}_x(D)$ is lower semicontinuous on $X.$

Remark 2.6.8. When $D$ is a reduced integral divisor, the proof above shows that
\[
\text{lct}(D) = \min \{1, \min_{E_i \text{ exceptional}} \frac{b_i + 1}{a_i}\}.
\]

Example 2.6.9. Going back to Example 2.6.3, we see that:
(1) The log canonical threshold of the cusp $(x^2 + y^3 = 0)$ is $5/6.$
(2) If $D$ has an ordinary singularity of multiplicity $m$ at $x,$ then $\text{lct}_x(D) = \min \{1, \frac{n}{m}\}.$

We now come to a concept of great importance in modern birational geometry.

Definition 2.6.10. The multiplier ideal of the $\mathbb{Q}$-divisor $D$ on $X$ is defined as
\[
\mathcal{J}(D) := \mu_* \mathcal{O}_Y(K_{Y/X} - [\mu^*D]) \subseteq \mathcal{O}_X,
\]
where $\mu : Y \rightarrow X$ is any log resolution of $(X,D).$ The fact that it is an ideal sheaf follows since clearly $\mathcal{J}(D) \subseteq \mu_* \omega_{Y/X} \simeq \mathcal{O}_X.$

It is well known, and originally due to Esnault-Viehweg, that this definition is independent of the choice of resolution of singularities. By dominating any two log resolutions by a third, this reduces to the following statement, which can be found in [La, Lemma 9.2.19].

Lemma 2.6.11. Let $D$ be an effective $\mathbb{Q}$-divisor on $X$ with SNC support, and $\mu : Y \rightarrow X$ a log resolution of $(X,D).$ Then
\[
\mu_* \mathcal{O}_Y(K_{Y/X} - [\mu^*D]) \simeq \mathcal{O}_X(-[D]).
\]

The behavior of multiplier ideals under birational maps is not hard to describe:
Exercise 2.6.12 (Birational transformation rule). Show that if \( \mu : Y \to X \) is a proper birational map of smooth varieties, and \( D \) an effective \( \mathbb{Q} \)-divisor on \( X \), then
\[
\mathcal{J}(D) \otimes_{\mathcal{O}_X} \omega_X \cong \mu_* \mathcal{O}_Y(\mathcal{J}(\mu^* D) \otimes_{\mathcal{O}_Y} \omega_Y).
\]
(Hint: take a log resolution of the pair \((Y, \mu^* D + \text{Exc}(\mu))\), and use the projection formula.)

Exercise 2.6.13. The notions of singularities of pairs defined above can be interpreted in terms of multiplier ideals as follows:
\((X, D)\) is klt \iff \( \mathcal{J}(D) = \mathcal{O}_X \)
and
\((X, D)\) is log canonical \iff \( \mathcal{J}((1-\varepsilon)D) = \mathcal{O}_X \) for all \( 0 < \varepsilon < 1 \).

In fact we have
\[
\text{lct}(D) = \inf \{ c \in \mathbb{Q} \mid \mathcal{J}(cD) \neq \mathcal{O}_X \}
\]
and more precisely
\[
\text{lct}_x(D) = \inf \{ c \in \mathbb{Q} \mid \mathcal{J}(cD)_x \subseteq \mathfrak{m}_x \}.
\]

Example 2.6.14. (1) Using the definition and the projection formula, if \( D \) is an integral effective divisor we have \( \mathcal{J}(D) = \mathcal{O}_X(-D) \). For the same reason, if \( D \) is an arbitrary effective \( \mathbb{Q} \)-divisor and \( E \) is an integral divisor, then
\[
\mathcal{J}(D + E) = \mathcal{J}(D) \otimes \mathcal{O}_X(-E).
\]
(2) If \( D \) has SNC support, then \( \mathcal{J}(D) = \mathcal{O}_X(-[D]) \).
(3) If \( D = (x^2 + y^3 = 0) \subset \mathbb{C}^2 \), using the resolution and calculations in Example 2.6.3(2) we have
\[
\mathcal{J}(cD) = \mu_* \mathcal{O}_Y(K_{Y/X} - [c \mu^* D]) =
\mu_* \mathcal{O}_Y((1 - [2c])E_1 + (2 - [3c])E_2 + (4 - [6c])E_3 - [c \tilde{D}]).
\]
We can focus on the case \( 0 < c < 1 \) (see (1)), and so we obtain \( \mathcal{J}(cD) = \mathcal{O}_X \) as long as all the coefficients in the parenthesis are nonnegative, i.e.
\[
\mathcal{J}(cD) = \mathcal{O}_X \iff 0 < c < 5/6
\]
and
\[
\mathcal{J}(cD) = \mu_* \mathcal{O}_Y(-E_3) = \mathfrak{m}_0 \iff 5/6 \leq c < 1.
\]
(4) A more general example that will be discussed later in the analytic setting is that of a general \( D = (x^a + y^b = 0) \subset \mathbb{C}^2 \), with \( a, b \geq 2 \). We will see in Example 2.6.25 that
\[
\mathcal{J}(cD) = \mathcal{O}_X \iff c < \frac{1}{a} + \frac{1}{b},
\]
so in particular lct(D) = 1/a + 1/b.

The log canonical threshold is the first in a sequence of rational numbers describing the “jumps” of the multiplier ideals associated to multiples of a fixed divisor.
Proposition 2.6.15. If $D$ is an effective $\mathbb{Q}$-divisor and $x \in X$ is a point, then there exists an increasing sequence of rational numbers

$$0 = c_0 < c_1 < c_2 < \cdots$$

such that:

• $\mathcal{J}(cD)_x = \mathcal{J}(c_iD)_x$ for $c \in [c_i, c_{i+1})$.
• $\mathcal{J}(c_{i+1}D)_x \neq \mathcal{J}(c_iD)_x$ for all $i$.

(Here by convention $\mathcal{J}(0 \cdot D) = \mathcal{O}_X$.) In particular $c_1 = \operatorname{lct}_x(D)$.

Proof. Just as in the proof of Lemma 2.6.6, we can write

$$\mathcal{J}(cD) = \mu_* \mathcal{O}_Y \left( \sum_{i=1}^m (b_i - [ca_i])E_i \right),$$

and clearly the coefficients $b_i - [ca_i]$ are constant on intervals as indicated. Moreover, the endpoints of these intervals belong to the set

$$\left\{ \frac{b_i + m}{a_i} \mid \text{some i and some } m \geq 1 \right\} \subset \mathbb{Q}. \qed$$

Definition 2.6.16. The numbers $c_i$ in Proposition 2.6.15 are called the jumping numbers (or jumping coefficients) of $D$ at $x$.

Example 2.6.17. (1) If $D$ is an integral divisor, we have noted earlier that $\mathcal{J}((c+1)D) = \mathcal{J}(cD) \otimes \mathcal{O}_X(-D)$, and therefore $c$ is a jumping number for $D$ if and only if $c+1$ is one. So all the jumping numbers are determined by those in the interval $[0, 1]$, which form a finite set according to the formula at the end of the proof of Proposition 2.6.15.

(2) We will see later that if $f = x_1^{d_1} + \cdots + x_n^{d_n}$, then the jumping numbers of $f$ are all the rational numbers of the form

$$\frac{e_1 + 1}{d_1} + \cdots + \frac{e_n + 1}{d_n}, \quad \text{for all } e_1, \ldots, e_n \in \mathbb{N}.$$

We record the following for later use:

Lemma 2.6.18. The jumping numbers of $D$ at $x$ satisfy the inequalities

$$c_{i+1} \leq c_1 + c_i.$$

Proof. This is a consequence of the Subadditivity Theorem for multiplier ideals, for which I refer to [La, Theorem 9.5.20]. It says that for any $\mathbb{Q}$-divisors $D_1$ and $D_2$ on $X$ we have

$$\mathcal{J}(D_1 + D_2) \subseteq \mathcal{J}(D_1) \cdot \mathcal{J}(D_2).$$

In our case, we then have

$$\mathcal{J}((c_i + 1)cD) \subseteq \mathcal{J}(cD) \cdot \mathcal{J}(cD) \not\subseteq \mathcal{J}(cD),$$

and therefore by definition $c_{i+1} \leq c_1 + c_i$. \qed
Analytic interpretation. An excellent source for this material is [De, §5.B].

Let \( D = \sum_{i=1}^{k} a_i D_i \) be an effective divisor on \( X \), with \( D_i \) prime divisors and \( a_i \in \mathbb{Q} \). We fix an open set \( U \) on which we have \( D_i = (f_i = 0) \) for a regular function \( f_i \). We next view \( X \) as a complex manifold.

**Definition 2.6.19.** The **analytic multiplier ideal** of \( D \) (on \( U \)) is

\[
J_{\text{an}}(D) := \{ g \in \mathcal{O}_X(U) \mid \prod_{i=1}^{k} \frac{|g|^2}{|f_i|^{2a_i}} \text{ is locally integrable} \}.
\]

In particular, if \( D \) is an integral divisor given by \( f \in \mathcal{O}_X(U) \), then for every \( c \in \mathbb{Q}_{>0} \) we have

\[
J_{\text{an}}(cD) := \{ g \in \mathcal{O}_X(U) \mid \frac{|g|^2}{|f|^{2c}} \text{ is locally integrable} \}.
\]

It is not hard to see that the local definitions glue to give a global sheaf of ideals on \( X^{\text{an}} \).

**Remark 2.6.20 (Plurisubharmonic functions).** The multiplier ideal of \( D \) defined above is a special example of a more general analytic notion. To this end, note that

\[
\varphi_D := \sum_{i=1}^{k} a_i \log |f_i|
\]

is a **plurisubharmonic** function\(^6\) (since \( \log |z| \) is so). Now for any plurisubharmonic function \( \varphi \) on \( X \), one defines its multiplier ideal \( J(\varphi) \) via

\[
J(\varphi)(U) := \{ g \in \mathcal{O}_X(U) \mid |g|^2 \cdot e^{-2\varphi} \text{ is locally integrable} \}.
\]

It can be shown that this is a coherent sheaf of ideals; see [De, Proposition 5.7].

It turns out that analytic multiplier ideals satisfy the same birational transformation formula as the algebraic ones:

**Lemma 2.6.21.** Let \( \mu: Y \to X \) be a proper bimeromorphic holomorphic map, and \( \varphi \) any plurisubharmonic function on \( X \). Then

\[
J(\varphi) \otimes_{\mathcal{O}_X} \omega_X \simeq \mu_* \mathcal{O}_Y(J(\varphi \circ \mu) \otimes_{\mathcal{O}_Y} \omega_Y).
\]

**Proof.** Using the definition of \( J(\varphi) \), we can interprete the sheaf \( J(\varphi) \otimes_{\mathcal{O}_X} \omega_X \) as the subsheaf of \( \omega_X \) consisting, for each open set \( U \), of \( n \)-forms \( \omega \) such that

\[
\omega \wedge \overline{\omega} \cdot e^{-2\varphi} \text{ is locally integrable on } U.
\]

\(^5\)Recall that this means that its integral with respect to the Lebesgue measure is finite on every compact subset of \( U \).

\(^6\)This means a function \( \varphi: X \to [-\infty, \infty) \) that is upper semicontinuous, locally integrable, and satisfies the mean-value inequality

\[
(\varphi \circ \gamma)(0) \leq \frac{1}{\pi} \int_{\Delta} (\varphi \circ \gamma) \, d\mu
\]

for any holomorphic map \( \gamma: \Delta \to X \) from the open unit disk \( \Delta \subset \mathbb{C} \). See [De, §1B] for basic properties.
It is in fact enough to restrict to forms defined on the open set $V \subseteq U$ over which $\mu$ is a biholomorphism, as they automatically extend over the complement of $V$ by virtue of being locally $L^2$. Thus the change of variables formula gives
\[
\int_U \omega \wedge \overline{\omega} \cdot e^{-2\varphi} = \int_{\mu^{-1}(U)} \mu^* \omega \wedge \mu^* \overline{\omega} \cdot e^{-2(\varphi \circ \mu)},
\]
and therefore
\[
\omega \in \Gamma(U, \mathcal{J}(\varphi) \otimes_{\mathcal{O}_X} \omega_X) \iff \mu^* \omega \in \Gamma(\mu^{-1}(U), \mathcal{J}(\varphi \circ \mu) \otimes_{\mathcal{O}_Y} \omega_Y).
\]

**Proposition 2.6.22.** For every effective $\mathbb{Q}$-divisor $D$ on $X$ we have
\[
\mathcal{J}_{an}(D) = \mathcal{J}(D)^{an}.
\]

**Proof.** Since by Exercise 2.6.12 and Lemma 2.6.21 the two types of multiplier ideals satisfy the same birational transformation formula, by passing to a log resolution it is enough to check the statement when $D$ is assumed to have SNC support.

Let’s assume then that $D = \sum_{i=1}^k a_i D_i$, where $a_i \in \mathbb{Q}_{>0}$ and $\sum D_i$ is SNC. We need to show that
\[
\mathcal{J}_{an}(D) = \mathcal{O}_X(-[D])^{an}.
\]
Choosing local coordinates $x_i$ such that $D_i = (x_i = 0)$, this can be reinterpreted as saying that for a holomorphic function $g$ on such a neighborhood we have
\[
\frac{|g|^2}{\prod_{i=1}^k |x_i|^{2a_i}} \text{ is locally integrable } \iff x_1^{[a_1]} \cdots x_n^{[a_n]} | g.
\]
A standard reduction allows us to assume that $g$ is a monomial in the $x_i$, in which case by Fubini’s theorem we can separate the variables and reduce the statement to the fact that for a single variable $z$, on a ball $B$ around the origin, say of radius $\varepsilon$, we have
\[
\int_B \frac{1}{|z|^{2c}} < \infty \iff c < 1.
\]
In polar coordinates $z = re^{i\vartheta}$ the integral on the left is $2\pi \int_0^\varepsilon r^{1-2c} dr$, and this is easily checked.

**Corollary 2.6.23.** If $D$ is as in Definition 2.6.19, then
\[
\lct(D) = \sup \{ c > 0 \mid \frac{1}{\prod_{i=1}^k |f_i|^{2a_i}} \text{ is locally integrable around } x \}.
\]

**Proof.** We have seen that
\[
\lct_x(D) = \sup \{ c \in \mathbb{Q} \mid \mathcal{J}(cD)_x = \mathcal{O}_{X,x} \}.
\]
Thanks to Proposition 2.6.22 we then have
\[
\lct_x(D) = \sup \{ c \in \mathbb{Q} \mid 1 \in \mathcal{J}_{an}(cD)_x \},
\]
which is equivalent by definition to the assertion in the Corollary.
2.7. Log canonical threshold and jumping numbers as roots

Remark 2.6.24. Note that Corollary 2.6.23 (and the calculation in Proposition 2.6.22) shows that the invariant defined in terms of discrepancies in (2.6.4) is indeed independent of the choice of log resolution.

Example 2.6.25. Let $D = (x^a + y^b = 0) \subset \mathbb{C}^2$. We check using the interpretation in Corollary 2.6.23 that

$$\lct(D) = \frac{1}{a} + \frac{1}{b}.$$  

According to its statement, we have

$$\lct(D) = \sup \{ c > 0 \mid \int_B |x^a + y^b|^{-2c} < \infty \}$$

on any ball $B$ in the neighborhood of the origin. Consider now the unit ball $B = B(0; 1) \subset \mathbb{C}^2$, and the transformation

$$T: B \rightarrow B, \quad (x, y) \mapsto \left(\frac{x}{2^a}, \frac{y}{2^b}\right).$$

Denote $T^{(r)} = T \circ \ldots \circ T$, the $r$-fold composition, and set

$$\Omega_r := T^{(r-1)}(B) \setminus T^{(r)}(B).$$

We clearly have

$$\bigcup_{r \geq 1} \Omega_r = B \setminus \{0\}.$$  

Denoting $I_r = \int_{\Omega_r} |x^a + y^b|^{-2c}$, and using the change of variables $x \mapsto 2^{-b}x$, $y \mapsto 2^{-a}y$ (hence $dxdy \mapsto 2^{-(a+b)}dxdy$), we have

$$I_{r+1} = \int_{\Omega_r} 2^{2abc} |x^a + y^b|^{-2c} 2^{-(a+b)} = 2^{2(abc-(a+b))} \cdot I_r.$$  

Hence

$$\int_{B \setminus \{0\}} |x^a + y^b|^{-2c} = \sum_{r \geq 1} I_r = I_1 \cdot (1 + u + u^2 + \cdots),$$

where $u = 2^{(abc-(a+b))}$, and so this is finite if and only if $abc < a + b$.

2.7. Log canonical threshold and jumping numbers as roots

We go back to the discussion of the archimedean zeta function. Let $f$ be a regular function on $X$. It turns out that the greatest pole of the distribution $|f|^{2s}$ is the well-known invariant of the singularities of $f$ discussed in the previous section.

Proposition 2.7.1. The greatest pole of $|f|^{2s}$ is $-\lct(f)$.

Proof. We use the analytic interpretation of the log-canonical threshold discussed in Corollary 2.6.23, namely

$$c_0 := \lct(f) = \sup \{ c > 0 \mid |f|^{-2c} \text{ is locally integrable} \}.$$  

By the same argument as in Proposition 2.4.2, we have that $Z_\varphi(s) = \int_{\mathbb{C}^n} |f|^{2s} \varphi$ is analytic at $s = -c$ for any $c < c_0$ and any $\varphi \in C_\infty(\mathbb{C}^n)$. Therefore we need to show that there exists a $\varphi$ such that the continuation of $Z_\varphi(s)$ does have a pole at $s = -c_0$.  

Let $B$ be a ball in the neighborhood of a point $x$ in the zero locus of $f$ such that
\[ \int_B |f|^{-2c_0} = \infty, \]
and let $\varphi$ be a bump function in a neighborhood of $x$ such that $\varphi$ is identically 1 on $B$. If we assumed that the continuation of $Z_{\varphi}(s)$ were analytic at $s = -c_0$, and so the limit of $|Z_{\varphi}(-c)|$ would be finite as $c \to c_0$ from the left. On the other hand, by Fatou’s Lemma we have that
\[ \liminf_{c \to c_0} \int_{C^n} |f|^{-2c_0} \varphi \geq \int_{C^n} |f|^{-2c_0} \varphi \geq \int_B |f|^{-2c_0} = \infty, \]
which is a contradiction. \qed

According to Theorem 2.4.3 and Proposition 2.7.1, the log canonical threshold of $f$ should then be an integral shift of a root of $b_f(s)$. The best possible scenario does in fact happen, according to the following theorem appearing in works of Yano, Lichtin, and Kollár. We follow Kollár’s approach in [Ko], similar to the methods discussed above.

**Theorem 2.7.2.** Let $f \in \mathbb{C}[X_1, \ldots, X_n]$ be a nontrivial polynomial (or germ of analytic function), and let $\alpha_f$ be the negative of the greatest root of the Bernstein-Sato polynomial $b_f(s)$. Then
\[ \alpha_f = \text{lct}(f). \]

**Proof.** Step 1. In this step we only show that $-\text{lct}(f)$ is a root of $b(s) = b_f(s)$. We use the analytic interpretation of the log canonical threshold, namely
\[ c_0 := \text{lct}(f) = \sup \{ c > 0 \mid \frac{1}{|f|^{2c}} \text{ is locally } L^1 \}. \]
Therefore for some point $x$ in the zero locus of $f$ and some small ball $B$ around $x$, the function $\frac{1}{|f|^{2c}}$ with $c = c_0 - \varepsilon$ for $0 < \varepsilon \ll 1$ is integrable on $B$, but $\frac{1}{|f|^{2c_0}}$ is not integrable on some compact ball $B'$ strictly contained in $B$.

Let us now take $s = -c$ in Lemma 2.4.4, so that
\[ b(-c)^2|f|^{-2c} = (P(-c)\bar{P}(-c))|f|^{2(-c+1)}. \]
We can think of this as being an equality of distributions, since both sides are integrable on $B$. Therefore, for any smooth positive test function $\varphi$ supported on $B$, we have
\[ \int_B b(-c)^2|f|^{-2c} \varphi = \int_B (P(-c)\bar{P}(-c))|f|^{2(-c+1)} \varphi. \]

We can in fact take $\varphi$ to be a bump function with support in $B$, identically equal to 1 on $B'$, in which case we obtain that the left-hand side of (2.7.1) is at least
\[ b(-c)^2 \int_{B'} |f|^{-2c}. \]
Using integration by parts in a way similar to Exercise 2.4.5, we see that the right-hand side of (2.7.1) is equal to
\[ \int_B |f|^{2(-c+1)}(P(-c)\bar{P}(-c)\varphi), \]
and therefore if $\varepsilon$ is in a fixed interval $(0, \delta]$, then it is bounded above by some $M > 0$ depending only on $\varphi$ (as $c - 1$ belongs to a closed interval of values for which the integral is finite and depends continuously on $c$). We deduce that

$$b(-c)^2 \int_{B'} |f|^{-2c} \leq M < \infty$$

for every such $c$. On the other hand, $\frac{1}{|f|^{2c_0}}$ is not integrable on $B'$, hence by Fatou’s Lemma we have

$$\int_{B'} |f|^{-2c} \to \infty \quad \text{as} \quad c \to c_0.$$

The only way this can happen is if $b(-c_0) = 0$.

**Step 2.** To show in addition that $-c_0$ is the greatest root of $b(s)$, we need to use in addition Lichtin’s refinement of Kashiwara’s result on the rationality of the roots of $b(s)$, Theorem 3.5.1 below. If $f : Y \to \mathbb{C}^n$ is log resolution of $D = (f = 0)$ with the property that it is an isomorphism away from $D$ and $\tilde{D}$ smooth (which can always be achieved by a few extra blow-ups), then writing

$$f^* D = \sum_{i=1}^m a_i E_i \quad \text{and} \quad K_{Y/X} = \sum_{i=1}^m b_i E_i,$$

Lichtin’s theorem tells us that all the roots of $b(s)$ are of the form

$$-\frac{b_i + 1 + \ell}{a_i} \quad \text{with} \quad 1 \leq i \leq m,$$

where $\ell \geq 0$ is an integer. Since on the other hand we know by the proof of Lemma 2.6.6 that

$$c_0 = \min_{E_i \text{ exceptional}} \frac{b_i + 1}{a_i},$$

and $c_0 \leq 1$, it is then clear that no root can exceed $-c_0$. $\square$

In [ELSV], Theorem 2.7.2 was extended along similar lines to the following statement:

**Theorem 2.7.3.** With the same hypothesis as in Theorem 2.7.2, let $\xi$ be a jumping coefficient of $f$ in the interval $(0, 1]$. Then $-\xi$ is a root of $b_f(s)$.

**Proof.** Let $\xi'$ be the previous jumping coefficient (taken by convention to be equal to 0 if $\xi$ is the first jumping coefficient, i.e. the log canonical threshold). Recall that this means that for every $c \in [\xi', \xi)$ we have

$$\mathcal{I}(\xi' \cdot f) = \mathcal{I}(c \cdot f),$$

but there exists $x \in Z(f)$ such that

$$\mathcal{I}(\xi \cdot f)_x \subsetneq \mathcal{I}(c \cdot f)_x.$$

Recall now that the analytic interpretation of multiplier ideals gives

$$\mathcal{I}(c \cdot f) = \{ g \in \mathcal{O}_{\mathbb{C}^n} \mid \frac{|g|^2}{|f|^{2c}} \text{ is locally } L^1 \}.$$
Therefore for any $c \in [\xi', \xi)$, there exists a function $g$ and a small ball $B$ around $x$ such that
\[
\int_B \frac{|g|^2}{|f|^2 c} < \infty \quad \text{but} \quad \int_B \frac{|g|^2}{|f|^2} = \infty.
\]
The argument then goes through exactly as in the proof of Theorem 2.7.2, after multiplying the two integrands in (2.7.1) by $|g|^2$. \hfill \Box

**Remark 2.7.4.** The converse of the Theorem above is not true. For instance, Saito [Sa3, Example 3.5] shows that if $f(x, y) = x^5 + y^4 + x^3 y^2$ then $b_f(-s)$ has roots in $(0, 1]$ that are not jumping numbers for $f$; cf. also [ELSV, Example 2.5]. Despite this, we nevertheless have the following behavior, similar to that of jumping numbers:

**Corollary 2.7.5.** Let

\[-1 = \alpha_m < \alpha_{m-1} < \cdots < \alpha_1 < 0\]

be the distinct roots of $b_{f,x}(s)$ in the interval $[-1, 0)$, for some $x \in X$. Then
\[
\alpha_i + \alpha_1 \leq \alpha_{i+1}, \quad \text{for all} \quad 1 \leq i < m.
\]

**Proof.** Set $\beta_i = -\alpha_i$, so that the inequality to be shown is
\[
\beta_{i+1} \leq \beta_i + \beta_1.
\]
Denote by $c_j$ and $c_{j+1}$ the two consecutive jumping numbers of $f$ at $x$ such that $c_j \leq \beta_i < c_{j+1}$. Note also that the first nontrivial jumping number is $c_1 = \lct_x(f) = \beta_1$, according to Theorem 2.7.2. Now according to Lemma 2.6.18, we have
\[
c_{j+1} \leq c_j + c_1 \leq \beta_i + \beta_1.
\]
On the other hand, clearly $c_j < 1$, hence $c_{j+1} \leq 1$. Therefore by Theorem 2.7.3 there exists some $k > i$ such that $c_{j+1} = \beta_k \geq \beta_{i+1}$, and the result follows. \hfill \Box

**Note.** A detailed account of the developments leading to Theorems 2.7.2 and 2.7.3 can be found in [ELSV]. These theorems are also consequences of the stronger Budur-Saito theorem [BS], comparing multiplier ideals with the $V$-filtration, explained in §3.6. Another generalization, giving a criterion for jumping numbers of higher Hodge ideals to be roots of the Bernstein-Sato polynomial, is proved in [MP4, Proposition 6.14].
CHAPTER 3

The $V$-filtration, and more on Bernstein-Sato polynomials

The aim of this chapter is to introduce and study the $V$-filtration on $\mathcal{D}$-modules along hypersurfaces, according to Kashiwara, Malgrange, and Saito. Based on this notion, we introduce in passing nearby and vanishing cycles for $\mathcal{D}$-modules, which will be needed in later chapters. We then look at generalized Bernstein-Sato polynomials for arbitrary elements in certain $\mathcal{D}$-modules, and connect both the roots of these polynomials and the $V$-filtration to objects in birational geometry.

3.1. $V$-filtration: the smooth case

Let $Y$ be a smooth complex variety, of dimension $n$. We consider a hypersurface $X$ in $Y$ which we assume to be defined globally by a function $f \in \mathcal{O}_Y$. In this section $X$ is always assumed to be smooth.

**Definition 3.1.1.** For any $k \in \mathbb{Z}$ we define

$$V_k \mathcal{D}_Y := \{ P \in \mathcal{D}_Y \mid P \cdot (f)^i \subseteq (f)^{i-k} \},$$

with the convention that $(f)^i = \mathcal{O}_Y$ for $i < 0$.

Since $X$ is smooth, we can consider local algebraic coordinates $x_1, \ldots, x_{n-1}, t = f$ on $Y$. The following exercise summarizes the main properties of this filtration on $\mathcal{D}_Y$, some of them described in these local coordinates.

**Exercise 3.1.2.** The filtration $V \mathcal{D}_Y$ satisfies the following properties:

1. $V_0 \mathcal{D}_Y = \{ P \in \mathcal{D}_Y \mid P = \sum_{\alpha,k} a_{\alpha,k,t}(x)\partial_x^\alpha t^{k} \partial_t^k, \; \alpha \in \mathbb{N}^{n-1}, \; k \geq \ell \} = \mathcal{D}_Y[t, t\partial_t]$.
2. For all $k \in \mathbb{N}$, we have
   $$P \in V_{-k} \mathcal{D}_Y \iff P = t^k \cdot Q, \; \; Q \in V_0 \mathcal{D}_Y.$$
3. For all $k \in \mathbb{N}$, we have
   $$P \in V_k \mathcal{D}_Y \iff P = \sum_{j=0}^k Q_j \cdot \partial_t^j, \; \; Q_j \in V_0 \mathcal{D}_Y \; \; \text{for all } j.$$
4. $V \mathcal{D}_Y$ is increasing and exhaustive.
5. For all $k, \ell \in \mathbb{Z}$ we have
   $$V_k \mathcal{D}_Y \cdot V_\ell \mathcal{D}_Y \subseteq V_{k+\ell} \mathcal{D}_Y,$$
   with equality if $k, \ell \leq 0$ or $k, \ell \geq 0$.
6. $\bigcap_{k \in \mathbb{Z}} V_k \mathcal{D}_Y = \{0\}$. 

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(7) \( \text{gr}_0^V \mathcal{D}_Y \simeq \mathcal{D}_Y [t \partial] \).

(8) The class of \( t \partial_t \) in \( \text{gr}_0^V \mathcal{D}_Y \) does not depend on the choice of local coordinates.

Fix now a coherent left \( \mathcal{D} \)-module \( M \) on \( Y \).

**Definition 3.1.3.** A \( V \)-filtration on \( M \) along \( X \) is a rational filtration \( (V^\gamma = V^\gamma M)_{\gamma \in \mathbb{Q}} \) that is exhaustive, decreasing, discrete, and left continuous,\(^1\) such that the following conditions are satisfied:

1. Each \( V^\gamma \) is a coherent module over \( V_0 \mathcal{D}_Y = \mathcal{D}_X [t, \partial_t] \).
2. For every \( \gamma \in \mathbb{Q} \), we have an inclusion \( t \cdot V^\gamma \subseteq V^{\gamma + 1} \), with equality if \( \gamma > 0 \).
3. For every \( \gamma \in \mathbb{Q} \), we have \( \partial_t \cdot V^\gamma \subseteq V^{\gamma - 1} \).
4. For every \( \gamma \in \mathbb{Q} \), if we set \( V^{> \gamma} = \bigcup_{\gamma' > \gamma} V^{\gamma'} \), then \( \partial_t - \gamma \) acts nilpotently on \( \text{gr}_V^\gamma := V^\gamma / V^{> \gamma} \).

**Example 3.1.4.** (1) Let \( M = \mathcal{O}_Y \) and \( X = (t = 0) \). We can then consider the \( t \)-adic filtration, more precisely

\[
V^m \mathcal{O}_Y = (t)^{m-1}, \quad \forall m \in \mathbb{Z}
\]

(with the convention that \( (t)^j = \mathcal{O}_Y \) for \( j < 0 \)) and \( V^\alpha \mathcal{O}_Y = V^{[\alpha]} \mathcal{O}_Y \) for \( \alpha \in \mathbb{Q} \). This is easily seen to be a \( V \)-filtration. For instance, since \( \partial_t (t^{m-1}) = mt^{m-1} \), we have that \( \partial_t - m \) is identically zero on \( \text{gr}_V^m \mathcal{O}_Y \).

(2) More generally, let \( E \) be a vector bundle with flat connection on \( Y \). We can then consider the filtration on \( E \) where \( V^m E \) is the subsheaf generated by \( t^{m-1} \cdot E \) for \( m \in \mathbb{Z} \), with the same convention as above, and \( V^\alpha E = V^{[\alpha]} E \) for \( \alpha \in \mathbb{Q} \). Again, this is easily seen to be a \( V \)-filtration. For instance, for every local section \( s \) of \( E \) we have

\[
\partial_t (t^{m-1} \cdot s) = (m-1)t^{m-2} \cdot s + t^{m-1} \partial_t s \in t^{m-2} \cdot E,
\]

which verifies (3), and

\[
(\partial_t - m)(t^{m-1} \cdot s) = t^m \partial_t s,
\]

which shows that \( \partial_t - m \) is identically zero on \( \text{gr}_V^m E \) and hence verifies (4).

(3) Let \( M \) be an arbitrary coherent \( \mathcal{D}_Y \)-module supported on \( X \). In this case an explicit \( V \)-filtration on \( M \) is provided by the proof of Kashiwara’s theorem. Indeed, for every \( j \in \mathbb{Z} \) consider the eigenspace

\[
M^j := \{ s \in M \mid (\partial_t - j) \cdot s = 0 \}
\]

\(^1\)More precisely, the filtration has the property that there is a positive integer \( \ell \) such that \( V^\gamma \) takes constant value in each interval \( \left( \frac{i}{\ell}, \frac{i+1}{\ell} \right] \), for all \( i \in \mathbb{Z} \).
for the operator $\partial_t$. In the course of the proof of Kashiwara’s theorem we showed that

$$\mathcal{M} = \bigoplus_{j \leq 0} \mathcal{M}^j.$$  

We define

$$V^m \mathcal{M} := \bigoplus_{j \geq m} \mathcal{M}^j, \quad \forall m \in \mathbb{Z}$$

and $V^\alpha \mathcal{M} = V^{[\alpha]} \mathcal{M}$ for $\alpha \in \mathbb{Q}$. This is a $V$-filtration; all the required properties were shown in the course of the proof of Kashiwara’s theorem. Note that (4) follows by definition, since $\text{gr}_V^m \mathcal{M} = M^m$. Note also that $V^\alpha \mathcal{M} = 0$ for all $\alpha > 0$.

Also as part of the proof of Kashiwara’s theorem we have seen that

$$\mathcal{M} \simeq \mathcal{M}^0 \otimes \mathbb{C}[\partial_t] \quad \text{and} \quad \mathcal{M}^{-k} = \partial_t^k \cdot \mathcal{M}^0.$$  

Hence we have the alternative interpretation

$$V^{-m} \mathcal{M} = \bigoplus_{j=0}^m \mathcal{M}^0 \otimes \partial_t^j.$$  

**Proposition 3.1.5.** If it exists, a $V$-filtration on $\mathcal{M}$ along $X$ is unique.

**Proof.** Let $W^* = W^* \mathcal{M}$ be another filtration satisfying all the properties in Definition 3.1.3. It suffices to show

$$V^\alpha \subseteq W^\alpha, \quad \forall \alpha \in \mathbb{Q},$$

since then the process can be reversed. We first claim that for all rational numbers $\alpha \neq \beta$ we have

$$V^\alpha \cap W^\beta = V^{\alpha+\beta} \cap W^\beta + V^\alpha \cap W^{\beta+\alpha}.$$  

Indeed, consider the quotient module

$$U_{\alpha,\beta} := \frac{V^\alpha \cap W^\beta}{V^{\alpha+\beta} \cap W^\beta + V^\alpha \cap W^{\beta+\alpha}}.$$  

It is clear that $U_{\alpha,\beta}$ is a sub-quotient of both $\text{gr}_V^\alpha$ and $\text{gr}_V^\beta$, and hence both operators $\partial_t - \alpha$ and $\partial_t - \beta$ act nilpotently on it by property (4) in the definition of the $V$-filtration. It follows that so does multiplication by $\alpha - \beta$, hence $U_{\alpha,\beta} = 0$.

We next claim that for every $\alpha \in \mathbb{Q}$ we have

$$V^\alpha \subseteq V^{\alpha+\beta} + W^\alpha.$$  

To this end, fix $v \in V^\alpha$. Since $W^*$ is decreasing and exhaustive, there exists $\beta < \alpha$ such that $v \in W^\beta$. From (3.1.1) we deduce that we can write

$$v = v_1 + v_2, \quad v_1 \in V^{\alpha+\beta}, \quad v_2 \in V^\alpha \cap W^{\beta+\alpha}.$$  

We can therefore replace $v$ by $v_2$; note that

$$\hat{v} = \hat{v}_2 \in \text{gr}_V^\alpha,$$
but for $v_2$ can choose a larger $\beta$. We can now repeat this process as long as $\beta < \alpha$. Since $W^\bullet$ is discrete, after finitely many jumps we can then reach the situation where $\beta \geq \alpha$. Therefore $W^\beta \subseteq W^\alpha$, and the claim is proved.

The next claim is that in fact

\begin{equation}
V^\alpha \subseteq V^\beta + W^\alpha, \quad \forall \beta \geq \alpha.
\end{equation}

Indeed, let $\beta_0 > \alpha$ such that $V^{>\alpha} = V^{>\beta_0}$. Then by (3.1.2) we have

\[ V^\alpha \subseteq V^{>\beta_0} + W^\alpha \subseteq V^{>\beta_0} + W^{>\beta_0} + W^\alpha = V^{>\beta_0} + W^\alpha. \]

We can then repeat the process, and by the discreteness of $V^\bullet$ after finitely many steps we reach (3.1.3).

Finally from (3.1.3) and property (2) in the definition of the $V^\bullet$-filtration we deduce that for $\beta \gg 0$ we have

\begin{equation}
V^\alpha \subseteq t^q \cdot V^\beta + W^\alpha, \quad \forall q > 0.
\end{equation}

On the other hand, $V^\beta$ is finitely generated over $\mathcal{D}_X[t, t\partial_t]$, and $W^\bullet$ is exhaustive, so there exists $\gamma \in \mathbb{Q}$ such that $V^\beta \subseteq W^\gamma$. Choosing $q$ such that $q + \gamma \geq \alpha$, using (3.1.4) we obtain

\[ V^\alpha \subseteq t^q \cdot W^\gamma + W^\alpha \subseteq W^{q+\gamma} + W^\alpha = W^\alpha. \]

\[ \square \]

**Remark 3.1.6.** What we have in fact shown above is the more general fact that if the filtration $W^\bullet$ satisfies properties (2) and (4) in Definition 3.1.3, without being assumed to be finitely generated, then $V^\alpha \subseteq W^\alpha$ for all $\alpha \in \mathbb{Q}$.

Before stating the existence theorem, we list a few other basic properties of the $V$-filtration.

**Exercise 3.1.7.** Let $u: \mathcal{M} \to \mathcal{N}$ be a morphism of $\mathcal{D}_Y$-modules endowed with a $V$-filtration along $X$. Then $u$ is compatible with the $V$-filtration, and in fact strictly compatible, in the sense that

\[ u(V^\alpha \mathcal{M}) = u(\mathcal{M}) \cap V^\alpha \mathcal{N} \]

for all $\alpha$. In particular, the category of $\mathcal{D}_X$-modules endowed with $V$-filtration is abelian, and its morphisms are strict.

**Lemma 3.1.8.** Let $u: \mathcal{M} \to \mathcal{N}$ be a morphism of $\mathcal{D}_Y$-modules such that $u|_U$ is an isomorphism, where $U = Y \setminus X$. Then the induced morphisms

\[ u: V^\alpha \mathcal{M} \longrightarrow V^\alpha \mathcal{N} \]

are isomorphisms for all $\alpha > 0$. In particular $V^{>0} \mathcal{M}$ depends only on the restriction of $\mathcal{M}$ to $U$.

**Proof.** Denoting by $\mathcal{K}$ and $\mathcal{Q}$ the kernel and cokernel of $u$, by Exercise 3.1.7 we obtain, for each $\alpha$, an exact sequence

\[ 0 \longrightarrow V^\alpha \mathcal{K} \longrightarrow V^\alpha \mathcal{M} \longrightarrow V^\alpha \mathcal{N} \longrightarrow V^\alpha \mathcal{Q} \longrightarrow 0. \]
But since $K$ and $Q$ are supported on $D$, we have $V^\alpha K = V^\alpha Q = 0$ for all $\alpha > 0$ by Example 3.1.4(3).

The main existence theorem for the $V$-filtration is the following:

**Theorem 3.1.9 (Malgrange, Kashiwara).** Let $\mathcal{M}$ be a regular, holonomic, coherent $\mathcal{D}_X$-module, with quasi-unipotent monodromy around $X$. Then $\mathcal{M}$ admits a $V$-filtration along $X$.

**Remark 3.1.10.** (1) The quasi-unipotence hypothesis is necessary in order to obtain a rationally indexed $V$-filtration. It is a condition that is automatically satisfied in Hodge theory, in particular for all Hodge $\mathcal{D}$-modules. It can be removed if one is willing to work with a $\mathbb{C}$-indexed filtration instead.

(2) The more precise condition that implies the existence of the $V$-filtration is that $\mathcal{M}$ is a holonomic $\mathcal{D}$-module such that every element $u \in \mathcal{M}$ admits a Bernstein-Sato polynomial whose roots are rational numbers. We will use this approach in the next section to prove Theorem 3.1.9 for our main $\mathcal{D}$-modules of interest, following Malgrange.

### 3.2. $V$-filtration: the general case

Let now $X$ be a smooth complex variety of dimension $n$, and $f \in \mathcal{O}_X$ an arbitrary nontrivial function. One reduces to the smooth case using the graph embedding

$$\iota: X \hookrightarrow X \times \mathbb{C}, \quad x \mapsto (x, f(x))$$

described in §2.1. Denote $Y = X \times \mathbb{C}$, and let $t$ be the coordinate on the second factor $\mathbb{C}$. For a left $\mathcal{D}_X$-module $\mathcal{M}$, according to the previous section we can consider the notion of a $V$-filtration on $\iota_+ \mathcal{M}$ along the hypersurface ($t = 0$). We recall its definition for convenience.

**Definition 3.2.1.** A $V$-filtration on $\iota_+ \mathcal{M}$ is a rational filtration $(V^\gamma = V^\gamma \iota_+ \mathcal{M})_{\gamma \in \mathbb{Q}}$ that is exhaustive, decreasing, discrete, and left continuous such that the following conditions are satisfied:

1. Each $V^\gamma$ is a coherent module over $\mathcal{D}_X[t, \partial_t]$.
2. For every $\gamma \in \mathbb{Q}$, we have an inclusion

$$t \cdot V^\gamma \subseteq V^{\gamma+1},$$

with equality if $\gamma > 0$.
3. For every $\gamma \in \mathbb{Q}$, we have

$$\partial_t \cdot V^\gamma \subseteq V^{\gamma-1}.$$
4. For every $\gamma \in \mathbb{Q}$, if we set $V^{>\gamma} = \bigcup_{\gamma' > \gamma} V^{\gamma'}$, then $\partial_t t - \gamma$ acts nilpotently on $\text{gr}_{V^\gamma} := V^\gamma / V^{>\gamma}$.

**Exercise 3.2.2.** Show that the definition depends only on the hypersurface $D = (f = 0)$ and not on the particular $f$ chosen. Hence it can be extended to the global setting.
Remark 3.2.3. If a $V$-filtration on $\mathcal{M}$ exists, then it induces a filtration on $\mathcal{M}$ as well, by indentifying it with $\mathcal{M} \otimes 1 \subset \mathcal{M}$. In other words

$$V^\alpha \mathcal{M} := V^\alpha \mathcal{M} \cap (\mathcal{M} \otimes 1), \quad \forall \alpha \in \mathbb{Q}.$$ 

In the case $\mathcal{M} = \mathcal{O}_X$, this is intimately linked to important objects in birational geometry.

The main point we want to study here is the existence of the $V$-filtration on the $\mathcal{D}_Y$-module $\mathcal{M}$. To this end, it will be convenient to work in the larger $\mathcal{D}_X$-module $\mathcal{M} \otimes (\mathcal{D}_X)$, where $\mathcal{D}_X$ is the hypersurface ($f = 0$) in $X$. We will use the description of these two $\mathcal{D}$-modules in §2.1. More generally, we will consider any $\mathcal{D}_X$-module $\mathcal{M}$ on which multiplication by $f$ is bijective, as it is the case with $\mathcal{O}_X$. (EXPLAIN; we are only using injectivity below, which is clear.)

Suppose now that $\alpha \leq 0$ and $u = tw \in V^{\alpha+1}$. Let $\delta \ll 0$ be such that $w \in V^{\delta}$. If $\delta \geq \alpha$, then we are done. On the other hand, if $\delta < \alpha$, then $\delta \neq 0$ since $\alpha \leq 0$; since $tw \in V^{\delta+1}$, we conclude that $w \in V^{\delta}$. After repeating this argument finitely many times, we obtain $w \in V^\alpha$.

Remark 3.2.5. In light of Remarks 2.1.5 and 3.2.4, if $\mathcal{M}$ is a $\mathcal{D}_X$-module on which multiplication by $f$ is bijective, a $V$-filtration on $\mathcal{M}$ can be characterized as an exhaustive, decreasing, discrete, left continuous, rational filtration $\left(V^\gamma = V^\gamma \mathcal{M}\right)_{\gamma \in \mathbb{Q}}$, that satisfies the following conditions:

1. Each $V^\gamma$ is a coherent module over $\mathcal{D}_X(t, t^{-1}, s)$.
2. For every $\gamma \in \mathbb{Q}$, we have $t \cdot V^\gamma = V^{\gamma+1}$.
3. For every $\gamma \in \mathbb{Q}$, we have $\partial_t \cdot V^\gamma \subseteq V^{\gamma-1}$.
4. For every $\gamma \in \mathbb{Q}$, the operator $s + \gamma$ acts nilpotently on $\text{Gr}_V^\gamma$.

We now address the main result of this section. The proof makes use of the rationality of the roots of the Bernstein-Sato polynomial $b_f(s)$, and of the $\mathcal{D}_X[s]$ module $\mathcal{N}_f = \mathcal{D}_X[s] f^*$ studied in §2.5.

Theorem 3.2.6. There exists a $V$-filtration on $\mathcal{M}$ and $\mathcal{M} \otimes (\mathcal{D}_X)$.

Proof. The proof will be divided into a few steps:

Step 1. In this step we construct a $\mathcal{D}_X$-submodule $\mathcal{N}'$ of $\mathcal{M}$ with the following two properties:
(1) $t^k \mathcal{N}_f \subseteq \mathcal{N}' \subseteq t^m \mathcal{N}_f$ for some $k, m \in \mathbb{Z}$.

(2) $t \mathcal{N}' \subseteq \mathcal{N}'$, and the action of $-s = \partial_t t$ on $\mathcal{N}'/t \mathcal{N}'$ has a minimal polynomial whose roots belong to $\mathbb{Q} \cap (0, 1]$.

We start by noting that since the action of $t$ on $t_+ \mathcal{O}_X(*D)$ is bijective, for every $k \in \mathbb{Z}$ we have an induced isomorphism of $\mathcal{D}_X$-modules

$$t^k: \mathcal{N}_f/t \mathcal{N}_f \to t^k \mathcal{N}_f/t^{k+1} \mathcal{N}_f.$$ 

There is an induced action of $-s$ on $t^k \mathcal{N}_f/t^{k+1} \mathcal{N}_f$, and note that its minimal polynomial is equal to $b_f(-s + k)$. This follows from the identities

$$b_f(s + k) \cdot t^k \cdot f^s = t^k \cdot b_f(s) \cdot f^s = t^k \cdot P(s) \cdot f \cdot f^s \in t^{k+1} \mathcal{N}_f,$$

where the first equality uses (2.1.2), and from the fact that we can also go in reverse.

Let now $\alpha_1 < \cdots < \alpha_r$ be the distinct roots of the polynomial $b_f(-s)$. By Kashiwara’s Theorem 2.5.1 we know that they are positive rational numbers. We also choose $k, m \in \mathbb{Z}$ such that

$$k + \alpha_1 > 1 \quad \text{and} \quad m + \alpha_r \leq 1.$$ 

In particular we have $k > m$. Using the observation in the previous paragraph, we deduce that the action of $-s$ on $t^m \mathcal{N}_f/t^k \mathcal{N}_f$ has a minimal polynomial whose roots are of the form

$$\lambda_{ij} := \alpha_i + j, \quad \text{for} \quad i = 1, \ldots, r \quad \text{and} \quad j = m, \ldots, k - 1.$$ 

In what follows we will denote any of the $\lambda_{ij}$ by $\lambda$.

We now write

$$t^m \mathcal{N}_f/t^k \mathcal{N}_f = \bigoplus_{\lambda} P_\lambda,$$

where $P_\lambda$ is the submodule on which $s + \lambda$ acts nilpotently. Note that since the action of $-s$ is $\mathcal{D}_X$-linear, each $P_\lambda$ is a $\mathcal{D}_X$-submodule. We now consider the submodule

$$t^k \mathcal{N}_f \subseteq \mathcal{N}' \subseteq t^m \mathcal{N}_f$$

satisfying

$$\mathcal{N}'/t^k \mathcal{N}_f = \bigoplus_{\lambda > 0} P_\lambda.$$ 

We first claim that

$$t \cdot \mathcal{N}' \subseteq \mathcal{N}'.$$ 

To this end, note that if $u \in P_\lambda$, then $(s + \lambda)^e \cdot u = 0$ for some $e \geq 1$. Hence using (2.1.2) we have

$$t \cdot (s + \lambda)^e \cdot u = (s + \lambda + 1)^e \cdot tu = 0,$$

so the action of $t$ on $t^m \mathcal{N}_f/t^k \mathcal{N}_f$ maps $P_\lambda$ to $P_{\lambda+1}$, which gives the claim.

The next claim is that the operator

$$\sigma := \prod_{0 < \lambda \leq 1} (s + \lambda)$$
acts nilpotently on $\mathcal{N}'/t\mathcal{N}'$, which would finish the construction described at the beginning of this step. To check this, let $v \in \mathcal{N}'$. By definition, there exists a positive integer $N$, depending only on $b_f(s)$, such that if $w = \sigma^N(v)$, then

$$\hat{w} \in \bigoplus_{\lambda > 1} P_\lambda \subseteq \mathcal{N}'/t^k\mathcal{N}_f.$$  

We next show that $w \in t\mathcal{N}'$; this means that $\sigma^N(\hat{v}) = 0$, which proves the claim. Recall that the minimal polynomial of the action of $-s$ on $t^m\mathcal{N}_f/t^{m+1}\mathcal{N}_f$ is equal to $b_f(-s + m)$, hence by definition $b_f(-s + m)$ divides a power of

$$\prod_{j=1}^r (s - (\alpha_j + m)).$$

On the other hand we have $\alpha_j + m \leq 1$ for all $j$, hence the projection of (the preimage of) $P_\lambda$ to $t^m\mathcal{N}_f/t^{m+1}\mathcal{N}_f$ is 0 if $\lambda > 1$. Hence we have

$$w \in t^{m+1}\mathcal{N}_f.$$  

The proof of the claim will then be finished by showing that there exists an integer $N_1 > 0$ such that

$$(3.2.2) \quad \prod_{\lambda > 0} (s + \lambda)^{N_1} \cdot t^{-1}w \in t^{k}\mathcal{N}_f.$$  

Indeed, we would then have that $t^{-1}w \in \mathcal{N}'$, or equivalently $w \in t\mathcal{N}'$. Before proving this, let’s also record the fact that, again using (2.1.2), we have

$$(3.2.3) \quad \prod_{\lambda > 0} (s + \lambda)^{N_1} \cdot t^{-1}w = t^{-1} \cdot \prod_{\lambda > 0} (s + \lambda + 1)^{N_1} w$$

To see (3.2.2), first we know that by definition there exists an integer $N_2 > 0$ such that

$$w' := \prod_{\lambda > 0} (s + \lambda)^{N_2} w \in t^{k}\mathcal{N}_f.$$  

Now the minimal polynomial of the action of $-s$ on $t^k\mathcal{N}_f/t^{k+1}\mathcal{N}_f$ is $b_f(-s + k)$, hence there also exists an integer $N_3 > 0$ such that

$$\prod_{i=1}^r (s + \alpha_i + k)^{N_3} w' \in t^{k+1}\mathcal{N}_f.$$  

Recall that all $\lambda$ are of the form $\alpha_i + j$ where $j = m, \ldots, k - 1$, and so if $\lambda > 1$, then $\lambda - 1$ is also of this form. Thus all the factors in the product in (3.2.2) appear also on the right hand side in (3.2.3). Similarly, all the factors $(s + \alpha_i + k)$ appear on the right hand side in (3.2.3). Hence it simply suffices to take

$$N_1 \geq \max\{N_2, N_3\}.$$  

This concludes the first step.

Step 2. In this step we define a filtration $W^\alpha$ on $i_+ \mathcal{O}_X(*D)$, with $\alpha \in \mathbb{Q}$, and show that it is a $V$-filtration.
Recall from the previous step that $-s$ acts on $N'/tN'$ with a minimal polynomial with roots in $Q \cap (0, 1]$. Let’s denote these roots
\[ \beta_1 < \cdots < \beta_\ell. \]
We also denote by $P_i \subseteq N'/tN'$ the submodule on which $s + \beta_i$ acts nilpotently. For $i = 1, \ldots, \ell$, we consider the submodule $tN' \subseteq W^{\beta_i} \subseteq N'$ such that
\[ W^{\beta_i}/tN' = \bigoplus_{j \geq i} P_j. \]
In particular, $W^{\beta_1} = N'$. For every $q \in \mathbb{Z}$ and every $i$, we also define
\[ W^{\beta_i + q} := t^q \cdot W^{\beta_i} \subseteq t_+ \mathcal{O}_X(*D). \]
Finally, for an arbitrary $\alpha \in Q$, we can write $\gamma_1 < \alpha \leq \gamma_2$, where $\gamma_1$ and $\gamma_2$ are consecutive numbers in the set $\{\beta_i + q \mid i = 1, \ldots, \ell; q \in \mathbb{Z}\}$, and define
\[ W^{\alpha} := W^{\gamma_2}. \]
The claim is that this gives a $V$-filtration on $t_+ \mathcal{O}_X(*D)$. It is clear that $W^{\alpha}$ is finitely generated, decreasing, discrete and left continuous, and also that $t \cdot W^{\alpha} = W^{\alpha+1}$ for all $\alpha$.

To check that $\partial_t \cdot W^{\alpha} \subseteq W^{\alpha-1}$ for all $\alpha$, note first that by construction we have $\partial_t \cdot W^{\beta_i} \subseteq W^{\beta_i}$ for all $i$. On the other hand, for all $q$ we have $\partial_t \partial_q = t^{q-1}(\partial_t q + 1)$. We obtain
\[ \partial_t \partial_q \cdot W^{\beta_i} \subseteq t^{q-1} \cdot W^{\beta_i}, \]
which gives the assertion.

To check that $W^\bullet$ is exhaustive, let $v \in t_+ \mathcal{O}_X(*D)$. Then by Lemma 2.5.5 there exists $r \geq 0$ such that $t^r v \in N'_j$, so
\[ t^{k+r} v \in t^k N'_j \subseteq N' = W^{\beta_1}, \]
which implies that $v \in W^{\beta_1-q}$.

Finally, we check that $s + \alpha$ is nilpotent on $\text{gr}_W^\alpha$. Take $\gamma_2 = \beta_i + q$ as in the definition of $W^{\alpha}$, so that $\alpha \leq \gamma_2$ and $W^{\alpha} = W^{\gamma_2}$. If $\alpha < \gamma_2$, then $\text{gr}_W^\alpha = 0$ and there is nothing to prove. Hence we can assume $\alpha = \beta_i + q$, and so we are looking at the action of $s + \beta_i + q$ on $\text{gr}_W^{\beta_i+q} = t^q \cdot \text{gr}_W^{\beta_i}$. But
\[ (s + \beta_i + q)t^q = t^q(s + \beta_i), \]
and we know that $s + \beta_i$ is nilpotent on (the lift of) $P_i$, which is precisely $\text{gr}_W^{\beta_i}$. This concludes the proof.

**Step 3.** In this final step we define a filtration $V^{\alpha}$ on $t_+ \mathcal{O}_X$, with $\alpha \in Q$, and show that it is a $V$-filtration. This is in fact straightforward: taking into account Exercise 2.1.1, we simply define it as
\[ V^{\alpha} := W^{\alpha} \cap t_+ \mathcal{O}_X, \quad \forall \alpha \in Q. \]
The fact that $V^\bullet$ is a $V$-filtration follows immediately from the properties of $W^\bullet$ studied in the previous step.
We record further basic properties of this $V$-filtration. For the first, we need the following:

**Lemma 3.2.7.** In the language of Step 1, we have $N' \subseteq \iota_+ \mathcal{O}_X$.

**Proof.** Using Exercise 2.1.1, it is enough to show that for every $u \in N'$ and $m \in \mathbb{N}$ such that $t^m u \in \iota_+ \mathcal{O}_X$, we have $u \in \iota_+ \mathcal{O}_X$. By induction on $m$, it is also enough to assume $m = 1$. By the definition of $N'$, there exist $\lambda_1, \ldots, \lambda_\ell > 0$ (perhaps non-distinct) such that

$$
\prod_{i=1}^\ell (s + \lambda_i) \cdot u \in t^k N_f \subseteq \iota_+ \mathcal{O}_X.
$$

Note also that $s \cdot \iota_+ \mathcal{O}_X \subseteq \iota_+ \mathcal{O}_X$, and applying this inductively to the formula above, we obtain

$$
\prod_{i=1}^\ell (s + \lambda_i) \cdot u = w + (-1)^\ell \prod_{i=1}^\ell \lambda_i \cdot u \in \iota_+ \mathcal{O}_X,
$$

with $w \in \iota_+ \mathcal{O}_X$. But all $\lambda_i > 0$, so $u \in \iota_+ \mathcal{O}_X$. □

Having this lemma at our disposal we observe that

(3.2.4) \hspace{1cm} V^\alpha = W^\alpha, \quad \forall \alpha > 0.

Indeed, this amounts to checking that $W^\alpha \subset \iota_+ \mathcal{O}_X$ for all $\alpha > 0$. But by definition, for such $\alpha$ we have $W^\alpha = t^q \cdot W^{\beta_i}$ for some $i$ and some $q \geq 0$. Now by construction $W^{\beta_i} \subset N'$, and so using (3.2.1) and Lemma 3.2.7 we have

$$
W^\alpha \subset N' \subset \iota_+ \mathcal{O}_X.
$$

Another thing to note is that

(3.2.5) \hspace{1cm} \bigcap_{\alpha \in \mathbb{Q}} V^\alpha = 0.

To see this, note that if $v$ is in this intersection, then since $t^q \cdot V^\alpha = V^{\alpha+q}$ for $\alpha > 0$ and $q \geq 0$, we have

$$
v \in \bigcap_{q \in \mathbb{N}} t^q \cdot \iota_+ \mathcal{O}_X.
$$

But if $w = \sum_{j=0}^p g_j \partial_j^\delta \mathcal{O}_X$, then

$$
t^q \cdot w = \sum_{j=0}^p h_j \partial_j^\delta
$$

with $h_p = f^q \cdot g_p$, and from this it follows easily that $v = 0$. □

The proof of Theorem 3.2.6 also leads to the following statement regarding the nontrivial jumps in the $V$-filtration.

**Corollary 3.2.8.** Let $\alpha > 0$ be a rational number such that the $V$-filtration on $\iota_+ \mathcal{O}_X$ jumps at $\alpha$, i.e. $V^{\alpha+\varepsilon} \subset V^\alpha$ for all $\varepsilon > 0$. Then there exists a root $\lambda$ of $b_f(s)$ and $k \in \mathbb{Z}$ such that $\alpha = \lambda + k$. 

3.3. NEARBY AND VANISHING CYCLES

**Proof.** We have seen that each eigenvalue of the action of $-s$ on $N'/tN'$ is one of the $\beta_i$, which is in turn congruent to some $\lambda$ modulo $\mathbb{Z}$. One the other hand, by construction $V^\alpha$ jumps at rational numbers of the form $\beta_i + \mathbb{Z}$ for all $i$. □

3.3. Nearby and vanishing cycles

The standard reference for the material in this section is [Sa1], especially Sections 3.1 and 5.1.

**First take.** Let first $\mathcal{M}$ be a $\mathscr{D}_X$ module endowed with a $V$-filtration along a smooth hypersurface defined by a function $t$. Note that by definition, for each $\alpha \in \mathbb{Q}$ we have induced operators

$$t: \text{gr}^\alpha V \mathcal{M} \to \text{gr}^{\alpha+1} V \mathcal{M}$$

and

$$\partial_t: \text{gr}^\alpha V \mathcal{M} \to \text{gr}^{\alpha-1} V \mathcal{M}.$$  

**Exercise 3.3.1.** The operator $t$ is an isomorphism for all $\alpha \neq 0$, while the operator $\partial_t$ is an isomorphism for all $\alpha \neq 1$.

This leaves us with two interesting homomorphisms, which are crucial in what follows, namely

(3.3.1) \hspace{1cm} (\text{var :=}) \hspace{0.5cm} t: \text{gr}^0 V \mathcal{M} \to \text{gr}^1 V \mathcal{M}  

and

(3.3.2) \hspace{1cm} (\text{can :=}) \hspace{0.5cm} \partial_t: \text{gr}^1 V \mathcal{M} \to \text{gr}^0 V \mathcal{M}.

The names of these morphisms are motivated by analogous maps in the theory of perverse sheaves. This theory (and results of Kashiwara and Malgrange on the Riemann-Hilbert correspondence) also motivates the following:

**Definition 3.3.2.** The **unipotent nearby cycles** $\mathscr{D}$-module of $\mathcal{M}$ along $t$ is defined as

$$\psi_{t,1} \mathcal{M} := \text{gr}^1 V \mathcal{M}.$$  

The **vanishing cycles** $\mathscr{D}$-module of $\mathcal{M}$ along $t$ is

$$\varphi_{t,1} \mathcal{M} := \text{gr}^0 V \mathcal{M}.$$  

Note that these are $\mathscr{D}$-modules supported on the hypersurface ($t = 0$).

In the general case we consider a left $\mathscr{D}_X$-module $\mathcal{M}$ on the smooth variety $X$, and a nontrivial function $f \in \mathscr{O}_X(X)$. We denote as always by

$$\iota: X \hookrightarrow Y = X \times \mathbb{C}, \quad x \mapsto (x, f(x))$$

the closed embedding given by the graph of $f$, and by $t$ be the coordinate on the second factor $\mathbb{C}$, so that ($t = 0$) is the smooth hypersurface $X \times \{0\}$ in $Y$. We assume that there exists a rational $V$-filtration on $\iota_* \mathcal{M}$ along $t$, for instance as in Theorem 3.1.9 and Remark 3.1.10. We define the unipotent nearby cycles and the vanishing cycles as

$$\psi_{f,1} \iota_* \mathcal{M} := \text{gr}^1 V \iota_* \mathcal{M} \quad \text{and} \quad \varphi_{f,1} \iota_* \mathcal{M} := \text{gr}^0 V \iota_* \mathcal{M}.$$
Relationship with strict support decomposition. One of the uses of the nearby and vanishing cycle is to provide a criterion for when the \( \mathcal{D} \)-module \( \mathcal{M} \) endowed with a \( V \)-filtration along a hypersurface has no nontrivial sub-objects or quotient objects supported on that hypersurface.

**Proposition 3.3.3.** Assume that \( D \) is a smooth hypersurface defined by a function \( t \), and let \( \mathcal{M}' \subseteq \mathcal{M} \) be the smallest submodule such that \( \mathcal{M}'|_U = \mathcal{M}|_U \), where \( U = X \setminus D \). Denote by \( i : D \to X \) the inclusion map. Then:

1. \( \mathcal{M}' = \mathcal{D}_X \cdot (V^0 \mathcal{M}) = \mathcal{D}_X \cdot (V^\alpha \mathcal{M}) \) for any \( \alpha > 0 \).
2. \( \mathcal{M}/\mathcal{M}' = i_+ \text{Coker}(\partial : \text{gr}_V^1 \mathcal{M} \to \text{gr}_V^0 \mathcal{M}) \).
3. \( \mathcal{H}^0_D \mathcal{M} = i_+ \text{Ker}(t : \text{gr}_V^0 \mathcal{M} \to \text{gr}_V^1 \mathcal{M}) \), where \( \mathcal{H}^0_D \mathcal{M} \) is the sub-object of \( \mathcal{M} \) generated by sections whose support is contained in \( D \). (The notation is meant to suggest local cohomology.)

**Proof.** (1) Since \( \mathcal{M}' \) and \( \mathcal{M} \) coincide on \( U \), we have

\[ V^\alpha \mathcal{M}' = V^\alpha \mathcal{M} \quad \text{for all} \quad \alpha > 0 \]

by Lemma 3.1.8. Thus

\[ (V^\alpha \mathcal{M}) \cdot \mathcal{D}_X \subseteq \mathcal{M}' \quad \text{for all} \quad \alpha > 0. \]

Note also that

\[ ((V^\alpha \mathcal{M}) \cdot \mathcal{D}_X)|_U = \mathcal{M}|_U \quad \text{for all} \quad \alpha > 0, \]

since the \( V \)-filtration is trivial outside of \( D \). By the definition of \( \mathcal{M}' \), this gives the opposite inclusion

\[ \mathcal{M}' \subseteq (V^\alpha \mathcal{M}) \cdot \mathcal{D}_X. \]

(2) Denote \( Q = \mathcal{M}/\mathcal{M}' \). According to Lemma 3.1.7, we have

\[ \text{gr}_V^0 Q = \text{Coker}(\text{gr}_V^0 \mathcal{M}' \to \text{gr}_V^0 \mathcal{M}). \]

The claim is that we also have

\[ \text{gr}_V^0 Q = \text{Coker}(\text{gr}_V^1 \mathcal{M} \xrightarrow{\partial} \text{gr}_V^0 \mathcal{M}). \]

Let’s grant this for the moment. Now obviously \( Q \) has support contained in \( D \), and therefore by Example 3.1.4(3) we have \( Q = Q_0 \otimes_C \mathbb{C}[\partial_t] = i_+ Q_0 \) and \( V^0 Q = 0 \). We therefore obtain \( \text{gr}_V^0 Q = V^0 Q = Q_0 \), and so

\[ Q = i_+ \text{gr}_V^0 Q. \]

We are left with proving (3.3.3). To this end, note that \( \text{gr}_V^0 \mathcal{M}' \subseteq \text{gr}_V^0 \mathcal{M} \), and so it suffices to show that

\[ \text{gr}_V^0 \mathcal{M}' = \text{Im} \partial_t = \frac{\partial_t \cdot V^1 \mathcal{M} + V^0 \mathcal{M}}{V^0 \mathcal{M}}. \]

Now

\[ \text{gr}_V^0 \mathcal{M}' = \frac{V^0 \mathcal{M}'}{V^0 \mathcal{M}'}, \]
where we used Exercise 3.1.7 and Lemma 3.1.8. We conclude that it is enough to show the identity
\[ \mathcal{M}' \cap V^0\mathcal{M} = \partial_t \cdot V^1\mathcal{M} + V^{>0}\mathcal{M}. \]
This follows in turn from the inclusion
(3.3.4) \[ \mathcal{M}' \cap V^0\mathcal{M} \subseteq \partial_t \cdot V^1\mathcal{M} + V^1\mathcal{M}. \]
Indeed, the right hand side is contained in \( \partial_t \cdot V^1\mathcal{M} + V^{>0}\mathcal{M} \), which in turn is contained in \( \mathcal{M}' \cap V^0\mathcal{M} \) by part (1).

To establish (3.3.4), we need the following two claims:

(i) \[ \mathcal{M}' = \sum_{i \geq 0} \partial_t^i \cdot V^1\mathcal{M}. \]
(ii) \[ \partial_t^i : V^0\mathcal{M}/V^1\mathcal{M} \to V^{-i}\mathcal{M}/V^{-i+1}\mathcal{M}. \]

Claim (i) is clear, since we’ve seen in (1) that \( \mathcal{M}' = D_X \cdot (V^1\mathcal{M}) \), and the action of functions and other \( \partial_{x_i} \) leaves all \( V^0\mathcal{M} \) fixed. Claim (ii) follows from Exercise 3.3.1.

Let’s finally deduce (3.3.4) from the two claims. Using (i), we see that it is enough to show that for all \( i \geq 2 \) we have
\[ \partial_t^i \cdot V^1\mathcal{M} \cap V^0\mathcal{M} \subseteq \partial_t^{i-1} \cdot V^1\mathcal{M} + V^0\mathcal{M}, \]

since then, continuing inductively, we eventually get that the left hand side is contained in \( \partial_t \cdot V^1\mathcal{M} \). But now
\[ \partial_t^i \cdot V^1\mathcal{M} \cap V^0\mathcal{M} \subseteq \partial_t^{i-1} \cdot V^0\mathcal{M} \cap V^0\mathcal{M} \subseteq V^{-i+1}\mathcal{M}, \]

If we pick an element \( x \in \partial_t^i \cdot V^1\mathcal{M} \cap V^0\mathcal{M} \), via these inclusions we can write \( x = \partial_t^{i-1} \cdot y \) for some \( y \in V^0\mathcal{M} \). If \( y \in V^1\mathcal{M} \), we are done. Otherwise we have \( 0 \neq y \in V^0\mathcal{M}/V^1\mathcal{M} \), and so by (ii) we obtain that \( \partial_t \cdot x = \partial_t^i \cdot y \notin V^{-i+1}\mathcal{M} \). But \( \partial_t \cdot x \in V^1\mathcal{M} \), which gives a contradiction.

(3) We know that
\[ \mathcal{H}^0_{D_M} = i_* \text{Ker}(t : \mathcal{M} \to \mathcal{M}), \]
and so we will show the coincidence between this kernel, denoted by \( K \), and \[ K' := \text{Ker}(t : \text{gr}^0_V\mathcal{M} \to \text{gr}^1_V\mathcal{M}). \]

We first show that
\[ K \subseteq V^0\mathcal{M}. \]
Indeed, consider \( u \in K \), so that \( tu = 0 \), and say \( u \in V^0\mathcal{M} \). If \( \alpha > 0 \), then we are done, so let’s assume \( \alpha \leq 0 \). We then have
\[ (\partial_t t - \alpha)u = -\alpha u \in V^0\mathcal{M}, \]
and we take its image in \( \text{gr}^\alpha_V\mathcal{M} \), where the action of \( \partial_t t - \alpha \) is nilpotent. It follows that there exists \( p > 0 \) such that \( \alpha^p u = 0 \), i.e. \( u = 0 \) in \( \text{gr}^\alpha_V\mathcal{M} \), and so \( u \in V^{>0}\mathcal{M} \). Proceeding inductively, we eventually obtain that \( u \in V^0\mathcal{M} \).
With this in mind, note that we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & V^{>0}\mathcal{M} & \longrightarrow & V^0\mathcal{M} & \longrightarrow & \text{gr}_V^0\mathcal{M} & \longrightarrow & 0 \\
& & \downarrow t & & \downarrow t & & \downarrow t & & \\
0 & \longrightarrow & V^{>1}\mathcal{M} & \longrightarrow & V^1\mathcal{M} & \longrightarrow & \text{gr}_V^1\mathcal{M} & \longrightarrow & 0,
\end{array}
\]

Recalling that \( t \) acts bijectively on \( V^{>0}\mathcal{M} \), the Snake Lemma then implies \( K \simeq K' \).  

**Corollary 3.3.4.** Let \( f \) be a nonconstant function on \( X \), and \( \mathcal{M} \) a \( \mathcal{D}_X \)-module endowed with a \( V \)-filtration along \( f \). Then

1. \( \mathcal{M} \) has no nonzero sub-object supported on \( D = f^{-1}(0) \) \( \iff \) \( \text{var: } \varphi_{f,1t+}\mathcal{M} \longrightarrow \psi_{f,1t+}\mathcal{M} \) is injective.
2. \( \mathcal{M} \) no nonzero quotient object supported on \( D = f^{-1}(0) \) \( \iff \) \( \text{can: } \psi_{f,1t+}\mathcal{M} \longrightarrow \varphi_{f,1t+}\mathcal{M} \) is surjective.

**Proof.** This follows from Proposition 3.3.3 applied to \( \iota_+\mathcal{M} \) along \( t \), noting that the sub-objects or quotient objects of \( \mathcal{M} \) supported on \( D \) correspond via Kashiwara’s equivalence to those of \( \iota_+\mathcal{M} \) supported on \( t^{-1}(0) \).  

We have in fact the following more precise statement:

**Proposition 3.3.5.** Under the same hypotheses, the following are equivalent:

1. \( \varphi_{f,1t+}\mathcal{M} = \text{Ker( var: } \varphi_{f,1t+}\mathcal{M} \longrightarrow \psi_{f,1t+}\mathcal{M} \) \( \oplus \) \( \text{Im( can: } \psi_{f,1t+}\mathcal{M} \longrightarrow \varphi_{f,1t+}\mathcal{M} \).}
2. There is a decomposition \( \mathcal{M} = \mathcal{M}' \oplus \mathcal{M}'' \), with \( \text{Supp } \mathcal{M}' \subseteq D = (f = 0) \), and \( \mathcal{M}'' \) with no nontrivial sub-objects or quotient objects supported on \( D \).

**Proof.** Assume first that there is a decomposition as in (2). We then have

\[
\varphi_{f,1t+}\mathcal{M} = \varphi_{f,1t+}\mathcal{M}' \oplus \varphi_{f,1t+}\mathcal{M}''.
\]

Moreover, in the case of \( \mathcal{M}'' \), by Corollary 3.3.4, we have that var is injective and can is surjective. On the other hand, by the discussion in the proof of Proposition 3.3.3, in the case of \( \mathcal{M}' \) the action of var and can is trivial, as it is supported on \( D \). This implies that

\[
\text{Ker( var) = } \varphi_{f,1t+}\mathcal{M}' \text{ and } \text{Im( can) = } \varphi_{f,1t+}\mathcal{M}'',
\]

and so gives (1).

We now assume (1). We define

\[
\widetilde{\mathcal{M}}' := \mathcal{D}_{X \times \mathbb{C}} \cdot V^{>0}_{\iota+}\mathcal{M}
\]

and

\[
\widetilde{\mathcal{M}}'' := \text{Ker( var: } \varphi_{f,1t+}\mathcal{M} \longrightarrow \psi_{f,1t+}\mathcal{M}),
\]

so that as in Proposition 3.3.3, \( \iota_+\mathcal{M}/\widetilde{\mathcal{M}}' \) and \( \widetilde{\mathcal{M}}'' \) are the largest quotient object, respectively sub-object, of \( \iota_+\mathcal{M} \) with support in \( t^{-1}(0) \) (so in fact in \( t^{-1}(0) \cap \iota(X) = D \)). Property (1) implies that we have a direct sum decomposition

\[
\iota_+\mathcal{M} = \widetilde{\mathcal{M}}' \oplus \widetilde{\mathcal{M}}''
\]
and we obtain (2) by Kashiwara’s equivalence. □

**Definition 3.3.6.** Let \( Z \subset X \) be an irreducible closed subset. A \( \mathcal{D}_X \)-module \( \mathcal{M} \) has **strict support** \( Z \) if \( \mathcal{M} \) is supported on \( Z \) and has no nontrivial quotients or sub-objects supported on a proper subset of \( Z \).\(^2\)

In the following two important corollaries of the results above, \( \mathcal{M} \) is a \( \mathcal{D}_X \)-module admitting a rational \( V \)-filtration along any hypersurface, for instance a regular holonomic \( \mathcal{D}_X \)-module with quasi-unipotent monodromy; see Theorem 3.1.9 and Remark 3.1.10.

**Corollary 3.3.7.** \( \mathcal{M} \) has strict support \( X \) if and only if \( \varphi_{f,1+t} \mathcal{M} \rightarrow \psi_{f,1+t} \mathcal{M} \) is injective and \( \psi_{f,1+t} \mathcal{M} \rightarrow \varphi_{f,1+t} \mathcal{M} \) is surjective for all nonconstant \( f \in \mathcal{O}_X(X) \).

**Corollary 3.3.8.** \( \mathcal{M} \) has a strict support decomposition \( \mathcal{M} = \bigoplus_{Z \subset X} \mathcal{M}_Z \) (i.e. a direct sum over a finite collection of irreducible closed subsets \( Z \) such that each \( \mathcal{M}_Z \) is a \( \mathcal{D}_X \)-module with strict support \( Z \)) if and only if the decomposition
\[
\varphi_{f,1+t} \mathcal{M} = \text{Ker}(\varphi) \oplus \text{Im}(\psi)
\]
in Proposition 3.3.5 holds for all \( f \in \mathcal{O}_X \).

**Remark 3.3.9.** Note that a direct sum decomposition as the Corollary above is necessarily **unique**, since there can be no nontrivial morphisms between \( \mathcal{D} \)-modules with different strict support.

### 3.4. Sabbah’s description of the \( V \)-filtration

Keeping the notation of the previous sections, here we extend the notion of Bernstein-Sato polynomial to arbitrary elements of \( \iota_+ \mathcal{O}_X \), and use it in order to give an alternative description of the \( V \)-filtration.

**Proposition 3.4.1.** Let \( w \in \iota_+ \mathcal{O}_X \). Then there exists a non-zero polynomial \( b(s) \in \mathcal{C}[s] \) and \( P \in \mathcal{D}_X[s] \) such that
\[
b(s) \cdot w = P \cdot tw.
\]

**Proof.** Let us consider, by analogy with the \( \mathcal{D}_X[s] \)-module \( \mathcal{N}_f = \mathcal{D}_X[s]f^s \) which played a key role earlier, the submodule
\[
\mathcal{N}_w := \mathcal{D}_X[s] \cdot w \subseteq \iota_+ \mathcal{O}_X.
\]
We need to show that there exists \( b(s) \in \mathcal{C}[s] \) such that
\[
b(-\partial t) \cdot w \in \mathcal{N}_w = \mathcal{D}_X[\partial t] \cdot tw.
\]
Fix now \( \alpha \in \mathbb{Q} \) such that \( w \in V^\alpha \iota_+ \mathcal{O}_X \), and consider any \( \beta \in \mathbb{Q} \).

**Claim.** There exists a polynomial \( b(s) \in \mathcal{C}[s] \), all of whose roots are rational and \( \leq -\alpha \), such that
\[
b(s) \cdot w \in V^{\beta} \iota_+ \mathcal{O}_X.
\]
\(^2\)These should be thought of as the analogues of the simple objects (intersection complexes) in the theory of perverse sheaves.
To see this, let $\alpha' \in \mathbb{Q}$ such that $V^{>\alpha} = V^{\alpha'}$. Hence we have

$$(s + \alpha)^N \cdot w \in V^{\alpha'} \quad \text{for some } N' \geq 1.$$ 

If $\beta \leq \alpha'$, then $V^{\alpha'} \subseteq V^{\beta}$ and we are done. If $\beta > \alpha'$, then we consider $\alpha'' > \alpha'$ and $N'' \geq 1$ such that

$$(s + \alpha)^{N''} (s + \alpha)^{N'} \cdot w \in V^{\alpha''} = V^{>\alpha'}.$$ 

We can continue in this fashion, and by the discreteness of $V^\bullet$, after a finite number of steps we reach an index which is greater than $\beta$.

Continuing with the proof of the Proposition, due to the Claim above it suffices to show that there exists $\beta \in \mathbb{Q}$ such that

$$V^\beta \cap \mathcal{N}_w \subseteq t \mathcal{N}_w.$$ 

To this end, first note that there exists $\beta_0 > 0$ such that $V^{\beta_0} \subseteq \mathcal{N}_f$, and so for every $q \in \mathbb{N}$ we have

$$t^q \cdot V^{\beta_0} = V^{\beta_0+q} \subseteq t^q \cdot \mathcal{N}_f.$$ 

It therefore suffices in turn to show that there exists $q \in \mathbb{N}$ such that

$$t^q \cdot \mathcal{N}_f \cap \mathcal{N}_w \subseteq t \mathcal{N}_w.$$ 

Note also that by definition $\mathcal{N}_tw = t \mathcal{N}_w$, hence we can replace $w$ by $t^p w$ with $p \gg 0$. Thus by Lemma 2.5.5, finally we can assume in (3.4.1) that $w \in \mathcal{N}_f$. In particular $\mathcal{N}_w \subseteq \mathcal{N}_f$.

We now proceed to proving (3.4.1) under this assumption. We consider the sheaf of rings

$$\mathcal{R} := \mathcal{D}_X[s] = \mathcal{D}_X[\partial_t t] \subseteq \mathcal{D}_Y.$$ 

Exercise 2.5.6 implies that for all $p \geq 0$ we have

$$t^p \cdot \mathcal{R} = \mathcal{R} \cdot t^p = \bigoplus_{i-j=p} \mathcal{D}_X \cdot t^i \partial_t^j.$$ 

Using this and (2.1.2), we deduce that

$$\mathcal{S} := \bigoplus_{i \geq j} \mathcal{D}_X \cdot t^i \partial_t^j$$

is a subring of $\mathcal{D}_Y$, with a decreasing filtration

$$F_p \mathcal{S} := \bigoplus_{i-j \geq p} \mathcal{D}_X \cdot t^i \partial_t^j$$

for $p \geq 0$. Denoting $t \partial_t$ by $u$ and $t$ by $v$, they satisfy the relation $uv = v(u+1)$, and we obtain

$$\text{gr}_F^* \mathcal{S} \simeq \mathcal{D}_X[u,v]/(uv - v(u+1)).$$

This ring is Noetherian; see Exercise 3.4.2 below.

Consider now on $\mathcal{N}_f$ the decreasing filtration

$$G_p \mathcal{N}_f := t^p \cdot \mathcal{N}_f, \quad p \geq 0.$$
It is easily checked that $G_\bullet$ is compatible with the filtration $F_\bullet$ on $S$, and that the total associated graded $gr^G S_f$ is generated over $gr^F S$ by the class of $\delta$. Since $N_w \subseteq N_f$, we can consider on $N_w$ the filtration

$$G_p N_w := G_p N_f \cap N_w, \quad p \geq 0.$$ 

As $gr^G N_f$ is finitely generated over the (locally) Noetherian $gr^F S$, we deduce that $gr^G N_w$ is also finitely generated. It follows that for $q \gg 0$ we have (3.4.1), which concludes the proof. □

**Exercise 3.4.2.** Show that $D_X[u,v]/(uv - v(u+1))$ is a sheaf of Noetherian rings. (Hint: define on it locally an analogue of the Bernstein filtration, and check that its associated graded is a polynomial ring in $2n$ variables.)

**Definition 3.4.3.** The set of all polynomials satisfying the conclusion of Proposition 3.4.1 forms an ideal in $C[s]$. The unique monic generator $b_w(s)$ of this ideal is called the Bernstein-Sato polynomial of $w$. (Note that the usual Bernstein-Sato polynomial of $f$ is the special case $b_f(s) = b_\delta(s)$ of this construction.)

**Remark 3.4.4.** The proof of Proposition 3.4.1 shows that for every $w \in \iota_+ \O_X$, the roots of $b_w(s)$ are rational numbers. Moreover, if $w \in V^\alpha$ then they are all $\leq -\alpha$.

The result above leads to a useful alternative description of the $V$-filtration due to Sabbah [Sab]:

**Theorem 3.4.5.** For every $\alpha \in \Q$ we have

$$V^\alpha \iota_+ \O_X = \{ w \in \iota_+ \O_X \mid \text{all the roots of } b_w(s) \text{ are } \leq -\alpha \}.$$ 

**Proof.** The inclusion from left to right follows from the Remark above. In order to prove the opposite inclusion, it suffices to show that if $w \in V^\alpha \setminus V^{\alpha_+}$, then $b_w(-\alpha) = 0$.

Recall that $s + \alpha$ is nilpotent on $gr^\alpha V$, and therefore for every $\beta \neq \alpha$ we have that $s + \beta$ is invertible on $gr^\alpha V$. On the other hand, we know that

$$b_w(s) \cdot w \in \O_X[\partial t] \cdot tw \in V^{\alpha_+}.$$ 

Since $\hat{w} \neq 0$ in $gr^\alpha V$, this would be impossible if $-\alpha$ were not among the roots of $b_w(s)$. □

An important consequence regards the greatest root of the Bernstein-Sato polynomial of $f$. Consider in fact

$$\alpha_f := -\left( \text{greatest root of } b_f(s) \right).$$ 

Recall also that we can define a decreasing $V$-filtration on $\O_X$, by considering it embedded in $\iota_+ \O_X$ as $\O_X \otimes 1$, and taking

$$V^\alpha \O_X = V^\alpha \iota_+ \O_X \cap \O_X.$$ 

For every $\alpha \in \Q$, $V^\alpha \O_X$ is a coherent ideal sheaf in $\O_X$.

---

3Note however that this does not recover Kashiwara’s theorem on the rationality of the roots of $b_f(s)$, since this theorem was used in the proof of the existence of the $V$-filtration.
Corollary 3.4.6. We have the following equivalences:

\[ V^\alpha \mathcal{O}_X = \mathcal{O}_X \iff \delta \in V^\alpha \mathcal{O}_X \iff \text{all the roots of } b_f(s) \text{ are } \leq -\alpha. \]

In particular,

\[ \alpha_f = \max \{ \beta \in \mathbb{Q} \mid V^\beta \mathcal{O}_X = \mathcal{O}_X \}. \]

We will see later that \( \alpha_f \) coincides with the log canonical threshold of \( f \). We also have another interesting consequence about the jumps of \( V^\alpha \mathcal{O}_X \) for \( \alpha > 0 \).

Corollary 3.4.7. If \( \alpha \in (0,1) \cap \mathbb{Q} \) corresponds to a jump in \( V^\bullet \mathcal{O}_X \), i.e. \( V^{\alpha+\varepsilon} \mathcal{O}_X \not\subseteq V^\alpha \mathcal{O}_X \) for all \( \varepsilon > 0 \), then \( -\alpha \) is a root of \( b_f(s) \).

Proof. We continue to denote \( V^\alpha = V^\alpha \mathcal{O}_X \). Let’s first note that for every \( \beta \in (0,1) \cap \mathbb{Q} \) we have \( f\delta \in V^\beta \). Indeed, we know that \( \delta \in V^{\alpha_f} \), so \( f\delta = t\delta \in V^{\alpha_f+1} \). But since \( \alpha_f > 0 \), we have \( V^{\alpha_f+1} \subseteq V^\beta \).

We know that

\[ b_f(s)\delta \in \mathcal{D}_X[s]f\delta \subseteq V^{>\alpha}, \]

where the inclusion follows from the discussion above. Hence we also have

\[ b_f(s)h\delta \in V^{>\alpha}. \]

On the other hand, the condition on \( \alpha \) is equivalent to the existence of an \( h \in \mathcal{O}_X \) such that \( h\delta \in V^\alpha \setminus V^{>\alpha} \). By the definition of the \( V \)-filtration, for \( N \gg 0 \) we also have

\[ (s + \alpha)^Nh\delta \in V^{>\alpha}. \]

If the two polynomials \( b_f(s) \) and \( (s + \alpha)^N \) were coprime, we would infer that \( h\delta \in V^{>\alpha} \), which is a contradiction. Thus we deduce that \( b_f(-\alpha) = 0 \).

Example 3.4.8. (1) The converse of the statement in Corollary 3.4.7 does not necessarily hold, even for isolated singularities. For instance, Saito shows in [Sa3, Example 3.5] that if \( f = x^5 + y^4 + x^3y^2 \in \mathbb{C}[x,y] \), then \( b_f(s) \) has roots in \((-1,0)\) that do not correspond to jumps of the filtration \( V^\bullet \mathcal{O}_X \).

(2) In Corollary 3.4.7 one cannot replace the set of jumping exponents of the \( V \)-filtration on \( \mathcal{O}_X \) by the analogous (but often larger) set for \( \mathcal{O}_X \). For example, let \( f = x^2 + y^3 \in \mathbb{C}[x,y] \) be a cusp. Then the \( V \)-filtration on \( \mathcal{O}_X \) is known to jump at \( \alpha = 1/6 \); see for instance the combinatorial calculation of the microlocal \( V \)-filtration for isolated quasi-homogeneous singularities in [Sa6, (4.1.2)], combined with the fact that for \( \alpha \leq 1 \) it coincides with \( V^{\bullet}\mathcal{O}_X \). On the other hand, the only negative of a root of \( b_f(s) \) that is less than 1 is \( 5/6 \); see Example 2.2.11(2).

Going back to the statement of Corollary 3.4.6, we will later see that the condition

\[ \partial_P^\alpha \delta \in V^\alpha \mathcal{O}_X \]

for \( p \geq 1 \) is also significant; it is equivalent to the triviality the Hodge ideal \( I_p(D) \), where \( D \) is the \( \mathbb{Q} \)-divisor \( D = \alpha \cdot \text{div}(f) \) (when \( \text{div}(f) \) is reduced).
Example 3.4.9. In view of the paragraph above, it is interesting to have some understanding of $b_{Qf}\delta(s)$ for $p \geq 1$, at least in terms of the standard $b_f(s) = b_\delta(s)$. Recall that for all nontrivial $f$, $(s + 1)$ divides $b_f(s)$, so we can also consider the reduced Bernstein-Sato polynomial

$$\tilde{b}_f(s) := \frac{b_f(s)}{s + 1}.$$  

We claim that we have the divisibility relation

$$b_{Qf}\delta(s)|(s + 1)\tilde{b}_f(s - p).$$

We begin by noting that for every polynomial $Q(s)$, we have

$$\partial_t \cdot Q(\partial_t) = Q(\partial_t + 1) \cdot \partial_t \quad \text{and} \quad t \cdot Q(\partial_t) = Q(\partial_t - 1) \cdot t.$$  

Indeed, it is enough to check this when $Q(s) = s^q$ is a monomial, and in this case both equalities can be easily verified by induction on $q$; the second relation is simply (2.1.2).

By the definition of $b_f(s)$, there exists $P \in \mathcal{D}_X[s]$ such that

$$b_f(-\partial_t)\delta = P(-\partial_t)t\delta.$$  

Using (3.4.2), we obtain

$$P(-\partial_t)t\delta = tP(-\partial_t - 1)\delta \quad \text{and} \quad b_f(-\partial_t) = (1 - \partial_t)t\tilde{b}_f(-\partial_t)t\partial_t = -t \cdot \tilde{b}_f(-\partial_t - 1)\partial_t.$$  

Since the action of $t$ on $\mathcal{O}_X$ is injective, we deduce that

$$\tilde{b}_f(-\partial_t - 1)\partial_t\delta = R(-\partial_t)\delta, \quad \text{where} \quad R(s) = -P(s - 1).$$  

Again using (3.4.2), we also obtain

$$\tilde{b}_f(-\partial_t - p)\partial_t^p\delta = \partial_t^{p-1} \cdot \tilde{b}_f(-\partial_t - 1)\partial_t\delta = \partial_t^{p-1} \cdot R(-\partial_t)\delta$$

$$= R(-\partial_t - p + 1)\partial_t^{p-1}\delta,$$

hence

$$(1 - \partial_t)t\tilde{b}_f(-\partial_t - p)\partial_t^p\delta = R(-\partial_t - p + 1) \cdot (1 - \partial_t)\partial_t^{p-1}\delta$$

$$= -R(-\partial_t - p + 1) \cdot t\partial_t^p\delta \in \mathcal{D}_X[-\partial_t] \cdot t\partial_t^p\delta.$$  

By the definition of $b_{Qf}\delta(s)$, we thus conclude that

$$b_{Qf}\delta(s)|(s + 1)\tilde{b}_f(s - p).$$  

It is also the case that

$$\tilde{b}_f(s - p)b_{Qf}\delta(s),$$  

but this requires further background; see [MP4, Proposition 6.12].
3.5. Lichtin’s theorem and generalizations

In this section we discuss a refinement of Kashiwara’s theorem on the rationality of the roots of the Bernstein-Sato polynomial. This is due to Lichtin [Li], and relates the roots of Bernstein-Sato polynomials with invariants appearing on resolutions of singularities. We will in fact prove a more general recent theorem of Dirks-Mustată [DM], extending Lichtin’s theorem to the Bernstein-Sato polynomials of certain elements in the $\mathcal{D}$-module $\iota_+ \mathcal{O}_X$ as in Proposition 3.4.1.

We start by stating Lichtin’s theorem. We fix a nonzero regular function $f$ on $X$, and set $D = (f = 0)$. We fix a log log resolution $\mu: Y \to X$, with the property that it is an isomorphism away from $\text{Supp}(D)$, and that the proper transform $\tilde{D}$ is smooth. We write $K_{Y/X}$ and $\mu^* D$ as in (2.6.1) and (2.6.2), and we have that all $a_i \neq 0$. The following is [Li, Theorem 5].

**Theorem 3.5.1.** With the notation above, all the roots of the Bernstein-Sato polynomial $b_f(s)$ are of the form

$$-\frac{b_i + 1 + \ell}{a_i}$$

for some $1 \leq i \leq m$ and some integer $\ell \geq 0$.

The proof is a refinement of Kashiwara’s arguments described in the previous section. A recent result of Dirks-Mustată uses arguments similar to Lichtin’s (and Kashiwara’s) to extend this further to other Bernstein-Sato polynomials related to $f$. The following is [DM, Theorem 1.2].

**Theorem 3.5.2.** Let $g \in \mathcal{O}_X(X)$, and let $u = g \partial^p f^s \in \iota_+ \mathcal{O}_X$. With the same notation as in Theorem 3.5.1, we set $k_i = \text{ord}_{E_i}(g)$. Then the following hold:

1. The greatest root of $b_u$ is at most $\max\{-1, p - \min_{1 \leq i \leq m} \frac{b_i + 1 + k_i}{a_i}\}$.

2. If $p = 0$, then the greatest root of $b_u$ is at most $-\min_{1 \leq i \leq m} \frac{b_i + 1 + k_i}{a_i}$.

3. If $g = 1$, then every root of $b_u$ is either a negative integer or of the form

$$p - \frac{b_i + 1 + \ell}{a_i}$$

for some integers $1 \leq i \leq m$ and $\ell \geq 0$.

If we assume in addition that $D = (f = 0)$ is reduced and the proper transform $\tilde{D}$ is smooth, then we may consider only those $i$ such that $E_i$ is exceptional.

Lichtin’s theorem above is then the special case $g = 1$ and $p = 0$ of this result. (Note that in this case all negative integers are also of the second type described in (3).) The extra ingredient introduced by Lichtin compared to the proof of Kashiwara’s theorem is to pass from left to right $\mathcal{D}$-modules, and work with a slightly modified $\mathcal{D}$-module on the resolution in order to absorb the relative canonical divisor $K_{Y/X}$ in the calculations.

4Note that the proof of Kashiwara’s result, Theorem 2.5.1, only shows that the roots of $b_f(s)$ are all of the form $-\frac{1 + \ell}{a_i}$ for some $1 \leq i \leq m$ and some integer $\ell \geq 0$. 

We now introduce the necessary ingredients for the proof. We use the notation introduced in §2.5 and §3.4.

**Generalities of Bernstein-Sato polynomials.** A useful point for later is the following:

**Lemma 3.5.3.** Let \( u \) be a section of \( \mathcal{O}_X \), and \( h \in \mathcal{O}_X \). Then the greatest root of \( b_{hu}(s) \) is at most equal to the greatest root of \( b_u(s) \).

**Proof.** Let \(-\alpha\) be the greatest root of \( u \). By Sabbah’s description of the \( V \)-filtration, Theorem 3.4.5, we then have \( u \in V^\gamma \mathcal{O}_X \). But \( V^\gamma \) is an \( \mathcal{O}_X \)-module, and therefore \( hu \in V^\gamma \mathcal{O}_X \) as well. Applying Sabbah’s result again, we obtain that the greatest root of \( b_{hu}(s) \) is at most \(-\alpha\). \( \square \)

The following lemma is a generalization of Exercise 2.2.8:

**Lemma 3.5.4.** Let \( p \) and \( q \) be invertible functions on \( X \), and let \( u = g\partial_t^m f^s \) for some regular functions \( f \) and \( g \). If \( v = (qg)\partial_t^m (pf)^s \), then \( b_u(s) = b_v(s) \).

**Proof.** We consider the sheaf

\[ \mathcal{O}_X(\ast D) \simeq \mathcal{O}_X[s, \frac{1}{f}]f^s \]

as an \( \mathcal{O}_X \)-module; it already has the standard \( \mathcal{D}_X(t, \partial_t) \)-action, but we endow it with a new one, denoted \( \ast \) and given by:

- \( D \ast w = Dw + swD(p)p^{-1} \), for all \( D \in \text{Der}_C(\mathcal{O}_X) \).
- \( t \ast w = (pt)w \).
- \( \partial_t \ast w = (p^{-1}\partial_t)w \).

We denote this \( \mathcal{D}_X(t, \partial_t) \)-module by \( \mathcal{O}_X(\ast D)^\ast \). Note that the new action of \( s = -\partial_t \) coincides with the old one.

A simple calculation shows that the map

\[ \nu: \mathcal{O}_X(\ast D) \rightarrow \mathcal{O}_X(\ast D)^\ast, \quad P(s)(pf)^s \mapsto P(s)f^s \]

is an isomorphism of \( \mathcal{D}_X(t, \partial_t) \)-modules, mapping \( v \) to \( qp^{-m}g\partial_t^m f^s \). Recalling that \( b_v(s) \) is the monic polynomial of minimal degree such that \( b_v(s)v \in \mathcal{D}_X(t,s)tv \), it follows that it is also the monic polynomial of minimal degree satisfying

\[ b_v(s)qp^{-m}g\partial_t^m f^s \in \mathcal{D}_X(t,s)t \ast qp^{-m}g\partial_t^m f^s. \]

On the other hand, for every section \( w \) of \( \mathcal{O}_X(\ast D) \) and every invertible function \( \varphi \in \mathcal{O}_X \) we have

\[ \mathcal{D}_X(t,s)t \ast w = \mathcal{D}_X(t,s)tw \quad \text{and} \quad \mathcal{D}_X(t,s)\varphi w = \mathcal{D}_X(t,s)w. \]

Hence we deduce that \( b_v(s) = b_u(s) \). \( \square \)

**Remark 3.5.5.** If \( g \in \mathcal{O}_X \) is such that \( g/f \) is not a regular function, then

\[ (s+1) \mid b_{gf^s}(s). \]

Indeed, we have that

\[ b_{gf^s}(s)gf^s = P(s)gf^{s+1} \quad \text{with} \quad P \in \mathbb{D}_X[s], \]
and taking \( s = -1 \) we obtain that \( b_{gf^s}(-1) \cdot (g/f) \) is a regular function, which cannot happen unless \( b_{gf^s}(-1) = 0 \).

**Lemma 3.5.6.** In local algebraic coordinates \( x_1, \ldots, x_n \), let \( u = g\partial^p_t f^s \) be the section of \( \mathcal{O}_X \) given by \( f = x_1^{c_1} \cdots x_n^{c_n} \) and \( g = x_1^{d_1} \cdots x_n^{d_n} \), for some non-negative integers \( c_i, d_i \) and \( p \). Then the following hold:

1. \( b_u(s) \) divides \( (s+1) \prod_{i=1}^n \prod_{j=1}^{c_i} \left( s - p + \frac{d_i + j}{c_i} \right) \). (Here the second product is taken to be 1 if \( c_i = 0 \).)

2. If \( p = 0 \), then \( b_u(s) \) divides \( \prod_{i=1}^n \prod_{j=1}^{c_i} \left( s + \frac{d_i + j}{c_i} \right) \).

3. If \( c_1 = 1 \) and \( d_1 = 0 \), then \( b_u(s) \) divides \( (s+1) \prod_{i=2}^n \prod_{j=1}^{c_i} \left( s - p + \frac{d_i + j}{c_i} \right) \).

**Proof.** We use the notation \( g\partial^p_t f = x^{cs+d} := \prod_{i=1}^n x_i^{c_i s + d_i} \) and \( t g\partial^p_t f = x^{c(s+1)+d} := \prod_{i=1}^n x_i^{c_i(s+1)+d_i} \).

Note that we can rewrite \( u = \partial^p_t x^{cs+d} \).

Consider now the polynomial

\[ c(s) := \prod_{i=1}^n \prod_{j=1}^{c_i} (c_i(s - p) + d_i + j). \]

Using the identity \( \partial_t P(s) = P(s - 1)\partial_t \) for all \( P \in C[s] \), we derive

\[ \partial_{x_1}^{c_1} \cdots \partial_{x_n}^{c_n} x^{c(s+1)+d} = \partial_t^p \prod_{i=1}^n \prod_{j=1}^{c_i} (c_i s + d_i + j) x^{cs+d} = c(s) \partial_t^p x^{cs+d} = c(s)u. \]

This immediately implies (2), since for \( p = 0 \) it gives

\[ c(s)u \in \mathcal{O}_X[s] \cdot tu. \]

By repeatedly applying the formula \( \partial_t t = t\partial_t + 1 \), and noting that \( s = -\partial_t t \), we obtain

\[ \partial^p_t t = -(s - p + 1)\partial_t^{p-1}, \]

and therefore we have

\[ (s+1)\partial^p_t t = (s - p + 1)t\partial^p_t. \]

We obtain

\[ (s+1)\partial^p_t x^{c(s+1)+d} = (s - p + 1)tu. \]

Hence for arbitrary \( p \geq 1 \), going back to (3.5.1), this gives

\[ (s+1)c(s)u = (s - p + 1)\partial_{x_1}^{c_1} \cdots \partial_{x_n}^{c_n} tu, \]

which implies (1).
For (3), note first that under the assumption $c_1 = 1$ and $d_1 = 0$, $g/f$ cannot be a regular function. By Remark 3.5.5 we then know that $(s + 1)$ divides $\tilde{b}_{gf^s}$, so we can talk about the reduced version $\tilde{b}_{gf^s}$. Moreover, exactly as in Example 3.4.9, we have that

$$b_u(s) \mid (s + 1)\tilde{b}_{gf^s}(s - p).$$

It therefore suffices to show that

$$\tilde{b}_{gf^s}(s) \mid \prod_{i=2}^{n} \prod_{j=1}^{c_i} \left(s + \frac{d_i + j}{c_i}\right).$$

But by the assumption on $c_1$ and $d_1$, this is the same as what we saw in (2). □

**Left to right correspondence.** Recall that there is an equivalence of categories between left and right $\mathcal{D}_X$-modules, taking a left $\mathcal{D}_X$-module $M$ to the right $\mathcal{D}_X$-module $\omega_X \otimes_{\mathcal{O}_X} M$. In local coordinates $x_1, \ldots, x_n$, this is given by an involution

$$\mathcal{D}_X \to \mathcal{D}_X, \quad P \mapsto P^*,$$

where $P^*$ is the adjoint of $P$, determined uniquely by the rules: $(PQ)^* = Q^*P^*$, $f^* = f$ for $f \in \mathcal{O}_X$, and $\partial_{x_i}^* = -\partial_{x_i}$. For a section $u$ of $\mathcal{M}$, we define the section

$$u^* = dx \otimes u$$

of $\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$, where $dx := dx_1 \wedge \cdots \wedge dx_n$. We then have

$$(Pu)^* = u^*P^* \quad \text{for all} \quad P \in \mathcal{D}_X.$$

We now extend this to $\mathcal{D}_X(s,t)$-modules. The same rule

$$\mathcal{M} \mapsto \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

takes a left $\mathcal{D}_X(s,t)$-module to a right one as follows. The involution of $\mathcal{D}_X$ we just described extends to one of $\mathcal{D}_X(t,\partial_t)$ by mapping $t \mapsto t$ and $\partial_t \mapsto -\partial_t$, hence mapping

$$s = -\partial_t \mapsto t\partial_t = -\partial_t - 1 = -s - 1.$$

We again have $(Pu)^* = u^*P^*$ for all sections $u$ of $\mathcal{M}$ and $P$ of $\mathcal{D}_X(s,t)$.

Just as with $\iota_+ \mathcal{O}_X(*D)$, we can also consider Bernstein-Sato polynomials for sections of the right $\mathcal{D}_X(s,t)$-module $\omega_X \otimes_{\mathcal{O}_X} \iota_+ \mathcal{O}_X(*D)$. More precisely, for a section $u$ of $\iota_+ \mathcal{O}_X(*D)$, a Bernstein-Sato relation $b_u(s)u = P(tu)$ becomes, after passing to adjoints,

$$u^*b_u(-s - 1) = (u^*t)P^*,$$

hence we have

$$b_{u^*}(s) = b_u(-s - 1).$$

**Main construction, and proof of the theorem.** Fix $g \in \mathcal{O}_X(X)$ and $p \geq 0$, and consider the $\mathcal{D}_X(s,t)$-module

$$\mathcal{N}_{f,p}(g) := \mathcal{D}_X(s,t) \cdot g\partial_t^p f^s \subseteq \iota_+ \mathcal{O}_X,$$

where $\iota: X \hookrightarrow Y = X \times \mathbb{C}$ is the graph embedding induced by $f$. Note that when $g = 1$ and $p = 0$ we have

$$\mathcal{N}_{f,0}(1) = \mathcal{N}_f,$$
the $\mathcal{D}$-module used in the proof of Kashiwara’s theorem. Just as in that case, we have:

**Exercise 3.5.7.** The action of $t$ preserves $N_{f,p}(g)$, so that $tN_{f,p}(g)$ is a $\mathcal{D}_X(t,s)$-submodule of $N_{f,p}(g)$. Furthermore, if $u = g\partial^p_t f^s$, then the Bernstein-Sato polynomial $b_u(s)$ is the minimal polynomial of the action of $s$ on the quotient $N_{f,p}(g)/tN_{f,p}(g)$.

As in the proof of Theorem 2.5.1, by restricting to an open neighborhood of the zero locus $Z(f)$ we can make the harmless extra assumption that

$$Z(\text{Jac}(f)) \subseteq Z(f),$$

where $\text{Jac}(f)$ is the Jacobian ideal of $f$. Recall that we denote by $W_f$ the closure of the subset

$$\{(x, sdf(x)) \mid f(x) \neq 0, s \in \mathbb{C}\} \subseteq T^*X.$$

This is an irreducible subvariety of $T^*X$, of dimension $n + 1$, which dominates $X$. A result that Kashiwara proves at the same time as Theorem 2.5.14 is the following:

**Theorem 3.5.8 ([Ka1, Theorem 5.3]).** The $\mathcal{D}_X$-module $N_f$ is coherent, and $\text{Ch}(N_f) = W_f$. In particular $N_f$ is subholonomic.

Based on this, one can show the following:

**Proposition 3.5.9.** If $f$ defines a divisor with SNC support and satisfies (3.5.2), denoting $N_{f,p} = N_{f,p}(1)$, for every $p \geq 0$ we have:

1. As a $\mathcal{D}_X$-module, $N_{f,p}$ is generated by $\partial^j_t f^s$, with $0 \leq j \leq p$.
2. $\text{Ch}(N_{f,p}) = W_f$.

**Proof.** Recall that for every $j \geq 0$ we have

$$t\partial^j_t f^s = f\partial^j_t f^s - j\partial^{j-1}_t f^s,$$

from which descending induction on $j$ shows that $\partial^j_t f^s \in N_{f,p}$, for $0 \leq j \leq p$. The formula also shows that

$$N'_{f,p} := \sum_{j=0}^p \mathcal{D}_X \cdot \partial^j_t f^s$$

is a $\mathcal{D}_X[t]$-submodule of $N_{f,p}$. Thus by the definition of $N_{f,p}$, to deduce (1) it suffices to show that

$$s\partial^j_t f^s \in N'_{f,p}, \quad \text{for all} \quad 0 \leq j \leq p.$$

Recall now that $\partial_t s = (s - 1)\partial_t$, and so

$$s\partial^j_t f^s = \partial^j_t (s + j)f^s.$$

It suffices thus to show that $sf^s \in \mathcal{D}_X f^s$, which is left as Exercise 3.5.10 below.

We show (2) by induction on $p$; the base case $p = 0$ is Theorem 3.5.8. Assuming that $p \geq 1$, part (1) implies that $N_{f,p-1} \subseteq N_{p,f}$, and the quotient $N_{f,p}/N_{f,p-1}$ is generated over $\mathcal{D}_X$ by the class of $\partial^p_t f^s$. In particular, there is a surjective $\mathcal{D}_X$-module homomorphism

$$N_{f,0} = \mathcal{D}_X f^s \rightarrow N_{f,p}/N_{f,p-1}, \quad Pf^s \mapsto P\partial^p_t f^s,$$
which implies that \( \text{Ch}(\mathcal{N}_{f,p}/\mathcal{N}_{f,p-1}) \subseteq W_f \). The fact that \( \text{Ch}(\mathcal{N}_{f,p}) = W_f \) follows then from the chain of inclusions

\[
W_f = \text{Ch}(\mathcal{N}_{f,p-1}) \subseteq \text{Ch}(\mathcal{N}_{f,p}) \subseteq \text{Ch}(\mathcal{N}_{f,p-1}) \cup \text{Ch}(\mathcal{N}_{f,p}/\mathcal{N}_{f,p-1}) \subseteq W_f.
\]

\[\square\]

**Exercise 3.5.10.** Show that if \( f \) defines a divisor with SNC support and satisfies (3.5.2), then \( sf^a \in \mathcal{D}_X f^a \).

**Proof of Theorem 3.5.2. Step 1.** Using Proposition 2.2.6 and Remark 2.2.7, we see that the statement of the theorem is local on \( X \). We may therefore assume that \( X \) is affine, with algebraic coordinates \( x_1, \ldots, x_n \). We may also assume that \( f \) is not an invertible function on \( X \), hence after passing to an open neighborhood of the zero locus of \( f \), that condition (3.5.2) is satisfied.

For each \( u \) as in the statement, we consider the section

\[
u^* = dx \otimes u\]

of the right \( \mathcal{D}_X(s,t) \)-module \( \mathcal{N}_u := \omega_X \otimes \mathcal{O}_X \mathcal{N}_{f,p}(g) \). The right \( \mathcal{D} \)-module version of Exercise 3.5.7 tells us that \( b_u^*(s) \) is equal to \( b_{\mathcal{N}_u}(s) \), the minimal polynomial of the action of \( s \) on \( \mathcal{N}_u/\mathcal{N}_u t \). By the discussion above we also know that \( b_{u^*}(s) = b_u(-s - 1) \).

Recalling that \( \mu : Y \to X \) is the fixed log resolution, we denote

\[
f' := f \circ \mu \quad \text{and} \quad g' = g \circ \mu.
\]

By analogy with the construction on \( X \), on \( Y \) we consider the \( \mathcal{D}_Y(s,t) \)-module

\[
\mathcal{N}_v := \omega_Y \otimes \mathcal{O}_Y \mathcal{N}_{f',p}(g'),
\]

where \( u' = g' \partial_t^p f'' \in \mathcal{D}_Y \) and \( \mathcal{N}_{f',p}(g') = \mathcal{D}_Y(s,t) \cdot u' \). We also consider its submodule

\[
\mathcal{N}_v := v \cdot \mathcal{D}_Y(s,t), \quad \text{with} \quad v := \mu^* dx \otimes u'.
\]

We fix an open set \( V \subseteq Y \) on which we have algebraic coordinates \( y_1, \ldots, y_n \) in which

\[
f' = h_1 y_1^{a_1} \cdots y_n^{a_n}, \quad \mu^* dx = h_2 y_1^{b_1} \cdots y_n^{b_n} dy, \quad \text{and} \quad g' = h y_1^{k_1} \cdots y_n^{k_n},
\]

according to our usual notation (the only \( a_i \) which are non-zero correspond to \( 1 \leq i \leq m \)), where \( h_1 \) and \( h_2 \) are invertible functions on \( V \), and \( h \) is a regular function on \( V \). Note that \( b_v(s) \) (in the sense of right \( \mathcal{D} \)-modules) is the same as \( b_{\mathcal{N}_v}(s) \), the minimal polynomial of the action of \( s \) on \( \mathcal{N}_v/\mathcal{N}_v t \). According to the right \( \mathcal{D} \)-module version of Lemma 3.5.4, we then have that

\[
b_{\mathcal{N}_v}(s) = b_{u^*}(s) = b_{u'}(-s - 1),
\]

where \( u' = hw \), with

\[
w := y_1^{b_1+k_1} \cdots y_n^{b_n+k_n} \partial_t^p (y_1^{a_1} \cdots y_n^{a_n})^s \in \mathcal{D}_Y \left[ \frac{1}{f_1} \right].
\]

\(^5\)Note that this is where things diverge from (the analogue of) Kashiwara’s argument, which would proceed using \( \mathcal{N}_{u'} \), while here this is replaced by \( \mathcal{N}_v \).
Using Lemma 3.5.3, we see that the greatest root of \( b_w(s) \) is at most equal to the greatest root of \( b_w(s) \). On the other hand, \( w \) is an element to which we can apply Lemma 3.5.6. Using it, and recalling that we have \( a_i \neq 0 \) only for \( 1 \leq i \leq m \), we obtain the following:

- The greatest root of \( b_w(s) \) is at most \( \max \{-1, p - \min_{1 \leq i \leq m} \frac{b_i + 1 + k_i}{a_i} \} \).
- If \( p = 0 \), then the greatest root of \( b_w(s) \) is at most \( -\min_{1 \leq i \leq m} \frac{b_i + 1 + k_i}{a_i} \).
- If \( g = 1 \) (so that \( h = 1 \) and we don’t need to use Lemma 3.5.3) every root of \( b_w(s) \) is either equal to \(-1\) or to some \( p - \frac{b_i + \ell}{a_i} \), with \( 1 \leq i \leq m \) and \( 1 \leq \ell \leq a_i \). If in addition \( D \) is reduced and its proper transform \( \tilde{D} \) is smooth, then we may assume that the divisor given by \((y_i = 0)\) on \( Y \) is exceptional. Indeed, note that in this case at most one \( y_i \) satisfies \( b_i = 0 \), i.e. it is not exceptional, and \( a_i > 0 \); in this case in fact \( a_i = 1 \).

Covering \( Y \) by open sets \( V \) on which we have such local coordinates, the global polynomial \( b_{N_v}(s) \) is the least common multiple of the respective polynomials on each \( V \), described above. Hence to conclude the proof, it suffices to show that each root of \( b_{N_v}(s) \) is of the form \( \alpha + \ell \), where \( \alpha \) is a root of \( b_{N_v}(s) \) and \( \ell \) is a non-negative integer.

**Step 2.** According to the paragraph above, it suffices to show that there exists an integer \( N \geq 0 \) such that

\[
(3.5.3) \quad b_{u^*}(s) \mid b_{N_v}(s)b_{N_v}(s-1) \cdots b_{N_v}(s-N).
\]

This is now very similar to the proof of Kashiwara’s theorem, only in the setting of right \( D \)-modules, and with \( N \) playing the role of \( N_f \) there. We define

\[
\mathcal{N} := \mathcal{H}^0_\mu_+ \mathcal{N}_v = R^0_\mu_+(\mathcal{N}_v \otimes \mathcal{O}_Y \mathcal{D}_{Y \rightarrow X}).
\]

Exactly as in Lemma 2.5.16, we have the divisibility

\[
(3.5.4) \quad b_{\mathcal{N}}(s) \mid b_{N_v}(s).
\]

We now perform Kashiwara’s main construction (see §2.5) in this setting, by constructing a distinguished section \( r \in \Gamma(X, \mathcal{N}) \). Note first that on \( Y \) we have a morphism of \( \mathcal{D} \)-modules

\[
\mathcal{D}_Y \rightarrow \mathcal{N}_v, \quad 1 \mapsto v.
\]

Taking the derived tensor product with \( \mathcal{D}_{Y \rightarrow X} \), we obtain a homomorphism

\[
\mathcal{D}_{Y \rightarrow X} \rightarrow \mathcal{N}_v \otimes \mathcal{O}_Y \mathcal{D}_{Y \rightarrow X}.
\]

On the other hand, the section \( 1 \in \mathcal{D}_X \) induces an \( \mathcal{O}_Y \)-module homomorphism \( \mathcal{O}_Y \rightarrow \mathcal{D}_{Y \rightarrow X} \), since \( \mathcal{D}_{Y \rightarrow X} \simeq \mu^* \mathcal{D}_X \) as an \( \mathcal{O}_Y \)-module. By composition we obtain a section

\[
\mathcal{O}_Y \rightarrow \mathcal{N}_v \otimes \mathcal{O}_Y \mathcal{D}_{Y \rightarrow X},
\]

and applying \( R^0_\mu \) we finally obtain a global section \( r \in \Gamma(X, \mathcal{N}) \). It is immediate from the construction that \( r = u^* \) on \( U \), the complement of the zero locus of \( f \). We define

\[
\mathcal{N}^\circ := r \cdot \mathcal{D}_X(s,t) \subseteq \mathcal{N},
\]
a right $\mathcal{D}_X(s,t)$-submodule of $\mathcal{N}$. We can also define a $\mathcal{D}_X(s,t)$-module homomorphism
\[ \mathcal{N}' \to \mathcal{N}, \quad r \mapsto u^*, \]
and precisely as in the proof of Lemma 2.5.7, this is well defined, and obviously surjective. In conclusion, we have a diagram of $\mathcal{D}_X(s,t)$-modules
\[ \begin{array}{ccc}
\mathcal{N}' & \xrightarrow{i} & \mathcal{N} \\
\downarrow{g} & & \\
\mathcal{N}_u & & 
\end{array} \]
where $i$ is the inclusion map, and $g$ is surjective.

As in the proof of Theorem 2.5.1, we see that the $\mathcal{D}_X(s,t)$-module $\mathcal{M} := \mathcal{N}/\mathcal{N}'$ is holonomic as a $\mathcal{D}_X$-module. Indeed, as in the paragraph after (2.5.1), it suffices to have the analogue of Theorem 2.5.14, namely
\[ \text{Ch}(\mathcal{N}) = W_f \cup \Lambda, \]
where $\Lambda$ is a Lagrangian subvariety of $T^*X$. As $\mathcal{N}_v \subseteq \mathcal{N}_{f,p}(g')$, this in turn follows by using Proposition 3.5.9(2) and the obvious fact that $\mathcal{N}_{f,p}(g') \subseteq \mathcal{N}_{f,p}$ (together with a standard results on the characteristic variety of a direct image, that I have not explained yet; this is also used for proving Theorem 2.5.14, and it will be added eventually).

The right $\mathcal{D}$-module analogue of Lemma 2.5.12 then implies that
\[ b_{\mathcal{N}'}(s) \mid b_{\mathcal{N}}(s) b_{\mathcal{N}}(s-1) \cdots b_{\mathcal{N}}(s-N). \]
for some integer $N \geq 0$. (Indeed the signs change, since now we are using the identity $b(s)t = tb(s-1)$ multiplying from the right.) In view of (3.5.4), it suffices then to show that
\[ b_{u^*}(s) \mid b_{\mathcal{N}'}(s). \]
But this follows immediately from the surjection $\mathcal{N}' \to \mathcal{N}_u$, since $b_{u^*}(s) = b_{\mathcal{N}_u}(s)$. \qed

\section{3.6. Multiplier ideals vs. $V$-filtration on $\mathcal{O}_X$}

We fix a non-invertible function $f$ on $X$, and denote $D = (f = 0)$. We have seen in §2.6 and §3.2 that certain aspects of the behavior of the multiplier ideals $\mathcal{J}(\alpha D)$ and the filtration $V^\alpha \mathcal{O}_X$ induced on $\mathcal{O}_X$ by the $V$-filtration $V^\bullet t_+ \mathcal{O}_X$ are very similar. For instance the threshold where they both become trivial is $\text{lc}(f)$, and more generally they both change at the jumping coefficients of the pair $(X,D)$ in the interval $[0,1]$ (as by Theorem 2.7.3 these are roots of $b_f(s)$), though with different semicontinuity behavior.

Budur and Saito [BS, Theorem 0.1] have noted that this is not an accident, enhancing these numerical properties to the following statement, proved using the theory of mixed Hodge modules:

\textbf{Theorem 3.6.1.} For every $\alpha \in \mathbb{Q}_{>0}$, we have
\[ V^\alpha \mathcal{O}_X = \mathcal{J}((\alpha - \varepsilon)D) \quad \text{for} \quad 0 < \varepsilon \ll 1. \]
In this section, following [DM] we give a different proof of this result, which does not make use of the Hodge filtration, but only of more elementary statements discussed up to now.

Recall that in analytic terms we have, for every \( c > 0 \), that

\[
\mathcal{J}(cD)^{an} = \{ g \in \mathcal{O}_X \mid |g|^2 / |f|^{2c} \text{ is locally integrable} \}.
\]

For a function \( g \in \mathcal{O}_X(X) \), by analogy with the usual definition we denote

\[
\lct_g(f) := \sup\{ c > 0 \mid g \in \mathcal{J}(cD) \}.
\]

Fixing a log resolution \( \mu : Y \to X \) which is an isomorphism away from the support of \( D \), recall that we write

\[
\mu^* D = \sum_{i=1}^m a_i E_i \quad \text{and} \quad K_{Y/X} = \sum_{i=1}^m b_i E_i.
\]

For each \( i \), we also denote \( k_i = \text{ord}_{E_i}(g) \). By analogy with (2.6.4), we have:

**Exercise 3.6.2.** The threshold \( \lct_g(f) \) is computed on \( Y \) by the formula

\[
\lct_g(f) = \min_{1 \leq i \leq m} \frac{b_i + 1 + k_i}{a_i}.
\]

On the other hand, as in §3.4 we denote

\[
u = gf^s \in \mathcal{O}_X,
\]

and according to Sabbah’s description of the \( V \)-filtration we have

\[
g \in V^\alpha \mathcal{O}_X \iff u \in V^\alpha \mu_+ \mathcal{O}_X \iff c \leq -\alpha \quad \text{for all } c \text{ such that } b_u(c) = 0.
\]

Therefore Theorem 3.6.1 is equivalent to the following analogue of Theorem 2.7.2:

**Theorem 3.6.3.** The greatest root of \( b_u(s) \) is \( -\lct_g(f) \).

**Proof.** Just as with Theorem 2.7.2, first we show that \( -\lct_g(f) \) is a root of \( b_u(s) \). Let us first recall that by definition we have

\[
b_u(s) \cdot gf^s = P(s) t(gf^s), \quad P(s) \in \mathcal{D}_X[s].
\]

Now the action of \( t \) on this element is \( t(gf^s) = gf^{s+1} \) (recall that we are identifying \( f^s \) with \( \delta \), and see Exercise B.3 in the notes on the \( V \)-filtration), where as always \( f^{s+1} := f \cdot f^s \). Hence this can be rewritten as

\[
b_u(s) \cdot gf^s = P(s)gf^{s+1}.
\]

We now proceed precisely as in the proof of Theorem 2.7.2; we repeat the argument for convenience. We denote \( c_0 = \lct_g(f) \). Therefore for some point \( x \) in the zero locus of \( f \) and some small ball \( B \) around \( x \), the function \( \frac{|g|^2}{|f|^{2c_0}} \) with \( c = c_0 - \varepsilon \) for \( 0 < \varepsilon \ll 1 \) is integrable on \( B \), but \( \frac{|g|^2}{|f|^{2c_0}} \) is not integrable on some compact ball \( B' \) strictly contained in \( B \).
3.6. Multiplier Ideals vs. V-Filtration on \( \mathcal{O}_X \)

Arguing as in Lemma 2.4.4, and taking \( s = -c \), we obtain the identity 
\[
\begin{align*}
    b_u(-c)^2|g|^2|f|^{-2c} &= \left(P(-c) \bar{P}(-c)\right)|g|^2|f|^{2(-c+1)}.
\end{align*}
\]
Both sides are integrable on \( B \), and so for any smooth positive test function \( \varphi \) supported on \( B \) we have
\[
\int_B b_u(-c)^2|g|^2|f|^{-2c} \varphi = \int_B \left(P(-c) \bar{P}(-c)\right)|g|^2|f|^{2(-c+1)} \varphi.
\]
We can in fact take \( \varphi \) to be a bump function with support in \( B \), identically equal to 1 on \( B' \), in which case we obtain that the left-hand side of (3.6.1) is at least
\[
\begin{align*}
    b_u(-c)^2 \int_{B'} |g|^2|f|^{-2c} \varphi.
\end{align*}
\]
Using integration by parts in a way similar to Exercise 2.4.5, we see that the right-hand side of (3.6.1) is equal to
\[
\int_B |g|^2|f|^{2(-c+1)} \left(P(-c) \bar{P}(-c)\right) \varphi,
\]
and therefore if \( \varepsilon \) is in a fixed interval \( (0, \delta] \), then it is bounded above by some \( M > 0 \) depending only on \( \varphi \) (as \( c - 1 \) belongs to a closed interval of values for which the integral is finite and depends continuously on \( c \)). We deduce that
\[
\begin{align*}
    b_u(-c)^2 \int_{B'} |g|^2|f|^{-2c} &\leq M < \infty
\end{align*}
\]
for every such \( c \). On the other hand, \( \frac{|g|^2}{|f|^{2c}} \) is not integrable on \( B' \), hence by Fatou’s Lemma we have
\[
\begin{align*}
    \int_{B'} |g|^2|f|^{-2c} \to \infty &\quad \text{as} \quad c \to c_0.
\end{align*}
\]
The only way this can happen is if \( b_u(-c_0) = 0 \).

Having established that \(-\text{lct}_g(f)\) is a root of \( b_u(s) \), the full statement now follows from Theorem 3.5.2(2), which thanks to Exercise 3.6.2 can be rephrased as saying that the greatest root of \( b_u(s) \) is at most equal to \(-\text{lct}_g(f)\).

It makes sense to wonder whether \(-\text{lct}_g(f)\) is also related to the poles of an archimedean zeta function as in §2.4, and this is indeed the case. Concretely, this time we can consider the distribution \(|g|^2|f|^{2s}\), which on any \( \varphi \in C_\infty^\infty(\mathbb{C}^n) \) is defined by
\[
\begin{align*}
    (|g|^2|f|^{2s}, \varphi) := \int_{\mathbb{C}^n} |g|^2|f(x)|^{2s} \varphi(x) = Z_\varphi^g(s).
\end{align*}
\]
Arguments completely analogous to those in Proposition 2.4.2, Theorem 2.4.3, and Proposition 2.7.1 show the following:

**Theorem 3.6.4.** With the notation above, for every smooth complex-valued function with compact support \( \varphi \in C_\infty^\infty(\mathbb{C}^n) \), \( Z_\varphi^g \) admits an analytic continuation to \( \mathbb{C} \) as a meromorphic function whose poles are of the form \( \alpha - m \), where \( \alpha \) is a root of the
Bernstein-Sato polynomial \(b_u(s)\) and \(m \in \mathbb{N}\). Moreover, the greatest pole of the distribution \([g^2|f|^2]s\) (meaning the maximum over the poles of \(Z_\varphi^g\) for all \(\varphi\)) is equal to \(-\text{lct}_g(f)\), hence to the greatest root of \(b_u(s)\).

### 3.7. Minimal exponent

Let \(X\) be a smooth variety of dimension \(n\), and \(f \in \mathcal{O}_X(X)\) a non-invertible function. We have seen that \(-1\) is always a root of \(b_f(s)\). We can therefore consider the polynomial

\[
\tilde{b}_f(s) = \frac{b_f(s)}{s+1},
\]

called the reduced Bernstein-Sato polynomial of \(f\).

**Definition 3.7.1.** The negative \(\tilde{\alpha}_f\) of the greatest root of the reduced of \(\tilde{b}_f(s)\) is called the minimal exponent of \(f\).\(^6\)

According to Theorem 2.7.2, the log canonical threshold \(\text{lct}(f)\) is equal to \(\alpha_f\), the greatest root of \(b_f(s)\). Therefore if \(\tilde{\alpha}_f \leq 1\), then it coincides with \(\text{lct}(f)\); more precisely

\[
\alpha_f = \min\{1, \tilde{\alpha}_f\}.
\]

Thus \(\tilde{\alpha}_f\) is a refinement of the log canonical threshold, and it provides a new interesting invariant precisely when the pair \((X, D)\) is log canonical.

**Remark 3.7.2 (Local version).** Recall that we also have a local version \(b_{f,x}\) of the Bernstein-Sato polynomial, around a point \(x \in X\); see Definition 2.2.5. If \(f\) is not invertible around \(x\), then \((s+1) \mid b_{f,x}(s)\), and we define \(\tilde{\alpha}_{f,x}\) to be the negative of the greatest root of \(\tilde{b}_{f,x}(s) = b_{f,x}(s)/(s+1)\).

**Remark 3.7.3 (Global version).** We can also define a global version of the minimal exponent. For each non-trivial effective divisor \(D\) on \(X\), there is an associated Bernstein-Sato polynomial \(b_D(s)\) such that \((s+1) \mid b_D(s)\); see Remark 2.2.9. We have

\[
b_D(s) = \text{lcm}_{x \in D} b_{D,x}(s),
\]

where \(b_{D,x}(s) := b_{f,x}\) for any locally defining equation \(f\) for \(D\) in a neighborhood of \(x\). (Hence we can also write \(\tilde{\alpha}_{D,x} := \tilde{\alpha}_{f,x}\) for any such \(f\).) As above, the minimal exponent \(\tilde{\alpha}_D\) is the negative of the greatest root of \(\tilde{b}_D(s) = b_D(s)/(s+1)\). The description above implies that

\[
\tilde{\alpha}_D = \min_{x \in D} \tilde{\alpha}_{D,x}.
\]

**Example 3.7.4 (Quasi-homogeneous isolated singularities).** Let \(f \in \mathbb{C}[X_1, \ldots, X_n]\) be a quasi-homogeneous polynomial, with weights \(w_1, \ldots, w_n\) (see §2.3), having an isolated singularity. Since in the notation of that section \(\rho(1) = 0\), Theorem 2.3.4 implies that

\[
\tilde{\alpha}_f = |w| := w_1 + \cdots + w_n.
\]

\(^6\)This is also called the microlocal log canonical threshold of \(f\) in [Sa9].
In particular, for a diagonal hypersurface $f = X_1^{a_1} + \cdots + X_n^{a_n}$ we have the celebrated
\[
\tilde{\alpha}_f = \frac{1}{a_1} + \cdots + \frac{1}{a_n}.
\]

Just like $\alpha_f$, the minimal exponent it is known to be related to standard types of singularities due the following results of Saito:

**THEOREM 3.7.5.** Assume that $D$ is reduced around a point $x$. Then

1. [Sa3, Theorem 0.4] $D$ has rational singularities at $x$ if and only if $\tilde{\alpha}_{D,x} > 1$.
2. [Sa6, Theorem 0.5] $D$ has Du Bois singularities at $x$ if and only if $\tilde{\alpha}_{D,x} \geq 1$.

Note that this implies that rational hypersurface singularities are du Bois; this is in fact known to be true for arbitrary varieties, by work of Kovács [Kov] and Saito [Sa5]. For a first look at du Bois singularities, see for instance [KS].

The proof of Theorem 3.7.5 requires the Hodge filtration, and therefore will be given later. For (2) we may alternatively not worry right now about what du Bois means; clearly $\tilde{\alpha}_D \geq 1$ is equivalent to the pair $(X, D)$ having log canonical singularities, while on the other hand using birational geometry arguments it is shown in [KS, Corollary 6.6] that

**PROPOSITION 3.7.6.** The pair $(X, D)$ has log canonical singularities if and only if the divisor $D$ has du Bois singularities.

Let now $D$ be a reduced effective divisor. One of the main questions about the minimal exponent of $D$ is whether we can express it explicitly in terms of discrepancies on a log resolution, like in the case of the log canonical threshold as in Remark 2.6.8. We use again the notation introduced in §3.5, and denote
\[
\gamma := \min_{E_i \text{ exceptional}} \frac{b_i + 1}{a_i}.
\]

We have noted in Remark 2.6.8 that $\alpha_D = \min\{1, \gamma\}$, while on the other hand by definition $\alpha_D = \min\{1, \tilde{\alpha}_D\}$.

It is natural then to ask whether $\tilde{\alpha}_D = \gamma$, and Lichtin [Li, Remark 2, p.303] did indeed pose this question. This would provide a very simple description, but as noted by Kollár [Ko, Remark 10.8] in general the answer is negative, since $\gamma$ usually depends on the choice of log resolution. Nevertheless, at least if we assume that the proper transform $\widetilde{D}$ is smooth,\(^7\) one inequality does hold:

**THEOREM 3.7.7 ([MP4, Corollary D]).** We always have $\tilde{\alpha}_D \geq \gamma$.

It is worth noting that the inequality follows easily from Lichtin’s result, Theorem 3.5.1, if $\tilde{\alpha}_D$ is not an integer; however, it is not clear how to use it otherwise. The original proof of the theorem in [MP4] relies on the theory of Hodge ideals. However a more elementary proof due to Dirks-Mustață can be given using Theorem 3.5.2 discussed in this chapter, and we present this next.

\(^7\)This can always be achieved by performing a few more blow-ups, if needed.
**Proof of Theorem 3.7.7.** Let’s assume for simplicity that $D$ is defined globally by a function $f$. We write $\gamma = p + \alpha$, where $p$ is a non-negative integer, and $\alpha \in (0, 1]$. Recall now from Example 0.33 in the notes on the $V$-filtration (see also [MP4, Proposition 6.12]) that

$$b_{\partial f^*}(s) | (s + 1)\tilde{b}_f(s - p) \quad \text{and} \quad \tilde{b}_f(s - p) | b_{\partial f^*}(s).$$

We deduce that all roots of $\tilde{b}_f(s)$ are $\leq -\gamma$ (which is what we want) if and only if all roots of $b_{\partial f^*}(s)$ are $\leq -\alpha$.

On the other hand, by Theorem 3.5.2(3) we know that for every root $\beta$ of $b_{\partial f^*}(s)$ we either have that $\beta$ is a negative integer, in which case we clearly have $\beta \leq -1 \leq -\alpha$, or we have

$$\beta = p - \frac{b_i + 1 + \ell}{a_i}$$

for some exceptional divisor $E_i$ and some non-negative integer $\ell$. Now by definition

$$\frac{b_i + 1 + \ell}{a_i} \geq \gamma = p + \alpha,$$

and therefore $\beta \leq -\alpha$. \qed

Mustaţă and I expect a substantially stronger statement to hold; we have formulated the following:

**Conjecture 3.7.8.** On every log resolution of $(X, D)$, there exists an exceptional divisor $E_i$ for which $\tilde{\alpha}_D = \frac{b_i + 1}{a_i}$.

The log canonical threshold is well known to satisfy a few fundamental semicontinuity and restriction properties, as well as numerical bounds in terms of the multiplicity; see for instance [Ko, §8]. It turns out that the same can be said about the minimal exponent; however the proofs are more complicated, and go beyond what we have studied up to this point (they depend on the theory of Hodge ideals). I am nevertheless including some statements below for completeness.

**Theorem 3.7.9 ([MP4, Theorem E]).** Let $X$ be a smooth $n$-dimensional complex variety, and $D$ an effective divisor on $X$.

1. If $Y$ is a smooth subvariety of $X$ such that $Y \nsubseteq D$, then for every $x \in D \cap Y$, we have

$$\tilde{\alpha}_{D|_{Y,x}} \leq \tilde{\alpha}_{D,x}.$$

2. Consider a smooth morphism $\pi : X \to T$, together with a section $s : T \to X$ such that $s(T) \subseteq D$. If $D$ does not contain any fiber of $\pi$, so that for every $t \in T$ the divisor $D_t = D|_{\pi^{-1}(t)}$ is defined, then the function

$$T \ni t \to \tilde{\alpha}_{D_t,s(t)}$$

is lower semicontinuous.

---

8This conjecture is also heuristically motivated by what is called Igusa’s Strong Monodromy Conjecture for the local zeta function associated to polynomials $f \in \mathbb{Z}[X_1, \ldots, X_n]$. 
(3) For every $x \in X$, if $m = \text{mult}_x(D) \geq 2$, then
\[ \frac{n - r - 1}{m} \leq \tilde{\alpha}_{D,x} \leq \frac{n}{m}, \]

where $r$ is the dimension of the singular locus of the projectivized tangent cone $\mathbb{P}(C_x D)$ of $D$ at $x$ (with the convention that $r = -1$ if $\mathbb{P}(C_x D)$ is smooth).

**Example 3.7.10 (Ordinary singularities).** In the case of a Fermat hypersurface $f = X_1^m + \cdots + X_n^m$, Example 3.7.4 gives $\tilde{\alpha}_f = n/m$. This is however also an example of an ordinary singular point of multiplicity $m$ (see Example 2.6.3(3)), and for these we always have $\tilde{\alpha}_f = \frac{n}{m}$.

This is the highest one can go: it turns out that for any $f$ and any singular point $x$ of multiplicity $m$ we have $\tilde{\alpha}_{f,x} \leq \frac{n}{m}$; see Theorem 3.7.9(3). The equality above follows from work of Saito. Alternatively, one can again use Theorem 3.7.9(3), noting that ordinary singularities are precisely the case $r = -1$. Note that the inequality $\tilde{\alpha}_f \geq \frac{n}{m}$ follows also from Theorem 3.7.7, since for an ordinary singularity $\gamma = n/m$ (just take the resolution given by blowing up the singular point).

**Remark 3.7.11.** Another general fact worth mentioning is the following result due to Saito [Sa4, Theorem 0.4]: the negative of every root of $\tilde{b}_f$ is in the interval $[\tilde{\alpha}_f, n - \tilde{\alpha}_f]$.

---

9This gives another way of seeing that we always have $\tilde{\alpha}_f \leq n/2$. 
CHAPTER 4

Filtered $\mathcal{D}$-modules and Hodge $\mathcal{D}$-modules

The aim of this chapter is to introduce some basic notions in the study of filtered $\mathcal{D}$-modules, focus on the interaction between the $F$-filtration and the $V$-filtration, and provide a first definition of Hodge modules. To get to this notion as quickly as possible, at first I will not develop things in the most systematic possible way. Later on I will rearrange the notes in a better order.

4.1. $F$-filtration and $V$-filtration

Let $X$ be a smooth complex variety of dimension $n$, and let $(\mathcal{M}, F)$ be a coherent $\mathcal{D}_X$-module endowed with a good filtration. We will see here that the nearby and vanishing cycles constructed in §3.3, and so implicitly the $V$-filtration, are useful for imposing restrictions on the filtration $F$.

Assume first that $D$ is a smooth divisor on $X$, given by $t = 0$, and consider the $V$-filtration on $\mathcal{M}$ induced by $t$. Assume that $\mathcal{M}$ has strict support $Z$ which is not contained in $D$.

For every $p \in \mathbb{Z}$ and $\alpha \in \mathbb{Q}$ we define

$$F_p V^\alpha \mathcal{M} := F_p \mathcal{M} \cap V^\alpha \mathcal{M}$$

and

$$F_p \text{gr}^V_\alpha \mathcal{M} := \frac{F_p \mathcal{M} \cap V^\alpha \mathcal{M}}{F_p \mathcal{M} \cap V^{\alpha+1} \mathcal{M}}.$$

Recall that in the proof of Proposition ?? we saw that

$$\mathcal{M} = \sum_{i \geq 0} \partial^i \cdot (V^0 \mathcal{M}).$$

On the other hand, we know that $V^0 \mathcal{M}$ is determined by $\mathcal{M}|_U$ by Lemma 3.1.8. A situation in which we can also recover $F_p V^0 \mathcal{M}$ from its restriction to $U$ is provided by the following:

**Lemma 4.1.** Denoting by $j : U = X \setminus D \hookrightarrow X$ the inclusion map, we have the identity

$$F_p V^0 \mathcal{M} = V^0 \mathcal{M} \cap j_* j^* F_p \mathcal{M}$$

if and only if

$$t : F_p V^\alpha \mathcal{M} \rightarrow F_p V^{\alpha+1} \mathcal{M}$$

is surjective for all $\alpha > 0$. 

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PROOF. First assume the identity. Consider \( y \in F_p^\alpha V^{\alpha+1}M \), for \( \alpha > 0 \). In particular we have \( y \in F_p V^{>0}M \), and so by hypothesis \( y \in V^{\alpha+1}M \cap j_* j^* F_p M \). We conclude the surjectivity of \( t \) due to the fact that \( t: V^\alpha M \to V^{\alpha+1}M \) is an isomorphism, combined with the fact that the action of \( t \) on \( j_* j^* F_p M \) is bijective.

Assume now that \( t \) acts surjectively on \( F_p^\alpha V \) for every \( \alpha > 0 \). Note that the inclusion from left to right in the identity is clear. Consider now

\[
x \in V^\alpha M \cap j_* j^* F_p M = \{ x \in V^\alpha M \mid x|_U \in F_p M|_U \},
\]

with \( \alpha > 0 \). Thus locally there exists \( i \geq 0 \) such that \( t^i x \in F_p M \). Since \( x \in V^\alpha M \), we have \( t^i x \in V^{\alpha+i}M \), so \( t^i x \in F_p V^{\alpha+i}M \). By hypothesis and the injectivity of the action of \( t \), we then have that \( x \in F_p^\alpha V \subseteq F_p V^{>0}M \).

\( \square \)

REMARK 4.1.2. Note that since \( t: V^\alpha M \to V^{\alpha+1}M \) is an isomorphism for \( \alpha > 0 \), we have that in this range

\[
t: F_p^\alpha M \longrightarrow F_p V^{\alpha+1}M
\]

is in any case injective, so the condition in the Lemma is that it is in fact an isomorphism.

Lemma 4.1.1 can be combined with the following result in order to obtain a criterion for recovering the full \( F_p M \) from its restriction to \( U \). Note first that we always have

\[
\sum_{i \geq 0} \partial_i \cdot (F_{p-i} V^{>0}M) \subseteq F_p M.
\]

LEMMA 4.1.3. Assume that \( \partial_i: \text{gr}_V^1 M \to \text{gr}_V^0 M \) is surjective.\(^1\) Then the following are equivalent:

1. \( F_p M = \sum_{i \geq 0} \partial_i \cdot (F_{p-i} V^{>0}M) \).
2. \( \partial_i: F_p \text{gr}_V^\alpha M \to F_{p+1} \text{gr}_V^{\alpha-1} M \) is surjective for all \( \alpha \leq 1 \).

PROOF. We define a new filtration \( F'_p M \) by

\[
F'_p M = \sum_{i \geq 0} \partial_i \cdot (F_{p-i} V^{>0}M).
\]

We consider the following two claims:

Claim 1. For all \( j \geq 0 \) we have

\[
F'_p V^{>j} M = \sum_{0 \leq i \leq j} \partial_i \cdot (F_{p-i} V^{>0}M).
\]

Claim 2. For all \( \alpha \leq 0 \) we have

\[
F'_p V^\alpha M = \sum_{0 \leq i \leq [-\alpha]} \partial_i \cdot (F_{p-i} V^\alpha M) + \partial_{[-\alpha]+1} \cdot (F_{p-[-\alpha]-1} V^{\alpha+[-\alpha]+1} M).
\]

\(^1\) By Proposition ??, this is equivalent to \( M = \mathcal{D}_X \cdot V^{>0}M \), so to the fact that there are no proper submodules of \( M \) that agree with \( M \) on \( U = X \setminus D \).
Let's first conclude the argument assuming these two claims. Note that the first implies
\[ F'_pV^{>0}\mathcal{M} = F_pV^{>0}\mathcal{M}, \]
while the second implies
\[ F'_p\gr^\alpha\mathcal{M} = \partial^\alpha_{t}[\mathcal{M}] \cap (F_{p-1}\gr^\alpha_{\mathcal{M}}). \]
We clearly have \( F'_p\mathcal{M} \subseteq F_p\mathcal{M} \). Since the \( V \)-filtration is discrete, descending gradually through the values of \( \alpha \) for which there are jumps, we therefore conclude that \( F'_p\mathcal{M} = F_p\mathcal{M} \) if and only if
\[ F'_p\gr^\alpha\mathcal{M} = \partial^\alpha_{t}[\mathcal{M}] \cap (F_{p-1}\gr^\alpha_{\mathcal{M}}), \]
which is equivalent to the condition in (2).

It remains to prove the two claims. Note first that Claim 2 reduces to Claim 1. Indeed the mapping
\[ \partial^j_{t}: \gr^{\alpha+j}\mathcal{M} \to \gr^{\alpha}\mathcal{M} \]
is injective for \( \alpha + j < 1 \), and so
\[ \partial^\alpha_{t}[\mathcal{M}] \cap (F_{p-1}V^{>0}\mathcal{M}) \subseteq \partial^\alpha_{t}[\mathcal{M}] \cap (F_{p-1}V^{>0}\mathcal{M}). \]
for \( \alpha \leq 0 \).

Finally, to prove Claim 1, it suffices to show that
\[ \partial^\alpha_{t}[\mathcal{M}] \cap (F_{p-1}V^{>0}\mathcal{M}) \subseteq \partial^\alpha_{t}[\mathcal{M}] \cap (F_{p-1}V^{>0}\mathcal{M}). \]
Let \( u \in F_{p-1}V^{>0}\mathcal{M} \) such that \( \partial^\alpha_{t}[u] \in V^{>0}\mathcal{M} \). Since
\[ \partial^j_{t}: V^{>-1}\mathcal{M}/V^{>0}\mathcal{M} \to V^{>-j-1}\mathcal{M}/V^{>0}\mathcal{M} \]
injective, it follows that \( \partial_t u \in V^{>0}\mathcal{M} \). This in turn implies \( \partial^\alpha_{t}[u] \in \partial^\alpha_{t}[\mathcal{M}] \cap (F_{p-1}V^{>0}\mathcal{M}) \), which is what we want. \( \square \)

**Lemma 4.1.4.** If \( \text{Supp } \mathcal{M} \subseteq D \), define \( \mathcal{M}_0 = \text{Ker}(t: \mathcal{M} \to \mathcal{M}) \), and \( F_p\mathcal{M}_0 = F_p\mathcal{M} \cap \mathcal{M}_0 \). Then the following are equivalent:

1. \( \mathcal{M} = \mathcal{M}_0 \otimes_{C} C[\partial_t] \) and \( F_p\mathcal{M} = \sum_{i \geq 0} F_{p-i}\mathcal{M}_0 \otimes \partial^i_t \).
2. \( \partial_t: F_p\gr^\alpha\mathcal{M} \to F_{p+1}\gr^{\alpha-1}\mathcal{M} \) is surjective for all \( \alpha < 1 \).

**Proof.** The hypothesis means of course that \( \mathcal{M} = \iota_+\mathcal{M}_0 \), where \( \iota: D \to X \). Moreover, as in Example 3.1.4(3), based on Kashiwara’s equivalence, we know that for all \( \alpha \leq 0 \) we have
\[ V^\alpha\mathcal{M} = \sum_{0 \leq i \leq [-\alpha]} \mathcal{M}_0 \otimes \partial^i_t. \]
In particular it suffices to focus on integral \( \alpha \).

We define a new filtration \( F'_p\mathcal{M} \) by
\[ F'_p\mathcal{M} = \sum_{i \geq 0} F_{p-i}\mathcal{M}_0 \otimes \partial^i_t. \]
Note that we clearly have $F'_p\mathcal{M} \subseteq F_p\mathcal{M}$, and moreover by definition
\[ F'_p V^0 \mathcal{M} = F_p V^0 \mathcal{M} \quad \text{and} \quad F'_p \text{gr}^{-i} \mathcal{M} = \partial^i \cdot (F_{p-i} \text{gr}^0 V \mathcal{M}) \]
for all $p \in \mathbb{Z}$ and $i \geq 0$. This implies (proceeding inductively on $i$) that we have $F'_p\mathcal{M} = F_p\mathcal{M}$ if and only if
\[ F_p \text{gr}^{-i} \mathcal{M} = \partial^i \cdot (F_{p-i} \text{gr}^0 V \mathcal{M}) \]
for all such $p$ and $i$. This is easily seen to be equivalent to the condition in (2). \qed

We now move to the case of hypersurfaces defined by arbitrary functions $f \in \mathcal{O}_X(X)$. We are led to collecting the conditions in the lemmas above into the following definition proposed by Saito [Sa1, 3.2.1]. To simplify the notation, we write
\[ \mathcal{M}_f := \iota_+ \mathcal{M}, \]
where $\iota$ stands as always for the graph embedding of $X$ along $f$.

**Definition 4.1.5 (Regular and quasi-unipotent condition).** We say that $(\mathcal{M}, F)$ is quasi-unipotent along $f$ if $\mathcal{M}_f$ admits a rational $V$-filtration along the coordinate $t$ on $X \times \mathbb{C}$, and the following conditions are satisfied:

1. $t \cdot (F_p V^\alpha \mathcal{M}_f) = F_p V^{\alpha+1} \mathcal{M}_f$ for $\alpha > 0$.
2. $\partial_t \cdot (F_p \text{gr}^V \mathcal{M}_f) = F_{p+1} \text{gr}^{\alpha-1} V \mathcal{M}_f$ for $\alpha < 1$.

Moreover, $(\mathcal{M}, F)$ is called regular along $f$ if in addition the filtration $F_\bullet \text{gr}^\alpha V \mathcal{M}_f$ is a good filtration for $0 \leq \alpha \leq 1$.

**Remark 4.1.6.** (1) Given the properties of the $V$-filtration discussed before, we have in fact that the actions of $t$ and $\partial_t$, in the ranges in (1) and (2) respectively, will in fact be bijective.

(2) We do not include $\alpha = 1$ in (2) in order for this notion to make sense even in the case when the morphism $\partial_t : \text{gr}^1 V \mathcal{M} \to \text{gr}^0 V \mathcal{M}$ is not surjective, for more flexibility. In the case of Hodge modules however, this condition will be automatically satisfied, and therefore Lemma 4.1.3 will apply.

It is not so hard, but it is very useful in practice, to recognize when a filtered $\mathcal{D}$-module with support contained in $D$ is regular and quasi-unipotent with respect to a defining equation:

**Proposition 4.1.7.** Let $(\mathcal{M}, F)$ be a filtered coherent $\mathcal{D}$-module with support contained in $D = (f = 0)$, such that $\mathcal{M}_f$ admits a rational $V$-filtration along $t$. Denote by $\iota : X \hookrightarrow X \times \mathbb{C}$ the graph embedding of $X$ along $f$, and by $\iota_0 : X = X \times X \times \{0\} \hookrightarrow X \times \mathbb{C}$ the natural embedding. Then the following are equivalent:

1. $\mathcal{M}$ is regular and quasi-unipotent along $f$.
2. $F_k \mathcal{M} \subseteq f \cdot F_{k-1} \mathcal{M}$ for all $k$.
3. There is a canonical isomorphism $(\mathcal{M}_f, F) = \iota_+(\mathcal{M}, F) \simeq \iota_0+(\mathcal{M}, F)$. 

Proof. This is [Sa1, Lemma 3.2.6]. I will include a proof after more discussion of pushforward of filtered $\mathcal{D}$-modules.

\hfill \Box

**Conclusion.** The main point of this section can roughly be summarized as follows. Assume that $(\mathcal{M}, F)$ is a $\mathcal{D}_X$-module admitting a rational $V$-filtration, with strict support $Z$. If it is regular and quasi-unipotent along $D = (f = 0)$, such that $f|_Z$ is not constant, then most of the conditions in Lemmas 4.1.1 and 4.1.3 are satisfied. If we assume that the last needed condition holds as well, namely that 

$$\partial_t: F_p \text{gr}^1 \mathcal{M}_f \longrightarrow F_p \text{gr}^0 \mathcal{M}_f$$

is surjective, which will always hold in the case of Hodge modules, then we conclude that

\begin{equation}
F_p \mathcal{M}_f = \sum_{i=0}^{\infty} \partial^i \cdot (V^{>0} \mathcal{M}_f \cap j_* j^* F_{p-i} \mathcal{M}_f),
\end{equation}

so in particular $(\mathcal{M}, F)$ is uniquely determined by its restriction to $Z \setminus (Z \cap D)$.

### 4.2. $\mathcal{D}$-modules with $\mathbb{Q}$-structure

The following definition underlies the notion of a Hodge module, which will be introduced later. I will therefore start with the following:

**Note.** The definition that follows refers to two notions that we have not focused on in this course, namely that of a **regular** holonomic $\mathcal{D}$-module, and that of a **perverse sheaf**. If you are unfamiliar with them, the ideal thing to do would be to acquire some familiarity, by reading for example parts of Chapters 6-8 in [HTT]. In the meanwhile however, for temporarily following the theory, such familiarity is not really crucial. Specifically, on one hand we will not deal with the theory of perverse sheaves at all; on the other hand, in these notes regularity is only used to ensure the existence of a $V$-filtration (due to the Kashiwara-Malgrange theorem), and as an input for the Riemann-Hilbert correspondence.

**Definition.** 4.2.1. (1) A **filtered regular holonomic $\mathcal{D}_X$-module with $\mathbb{Q}$-structure** is a triple $M = (\mathcal{M}, F, P)$ consisting of:

- A perverse sheaf $P$ of $\mathbb{Q}$-vector spaces on $X$.
- A regular holonomic $\mathcal{D}_X$ module such that $\text{DR}(\mathcal{M}) \simeq P \otimes_{\mathbb{Q}} \mathbb{C}$.
- A good filtration $F_\bullet \mathcal{M}$ by coherent $\mathcal{O}_X$-modules.

(2) Given such a triple, for any $k \in \mathbb{Z}$, its **$k$-th Tate twist** is defined as

$$M(k) := (\mathcal{M}, F_\bullet - k, P(k)),$$

where $P(k) = P \otimes_{\mathbb{Q}} \mathbb{Q}(k)$, with $\mathbb{Q}(k) = (2\pi i)^k \cdot \mathbb{Q} \subset \mathbb{C}$.

(3) Given a filtered regular holonomic $\mathcal{D}_X$-module with $\mathbb{Q}$-structure $M = (\mathcal{M}, F, P)$ and a function $f: X \to \mathbb{C}$, and denoting $\lambda = e^{2\pi i \alpha}$ for $\alpha \in \mathbb{Q}$, we define

$$\psi_f M := \bigoplus_{0 < \alpha \leq 1} \left( \text{gr}_V^\alpha \mathcal{M}_f, F_{\bullet - 1} \text{gr}_V^\alpha \mathcal{M}_f, \psi_{f, \lambda} P \right).$$
• \( \psi_{f,1} M := (\text{gr}_V^1 M_f, F_{\leq -1} \text{gr}_V^1 M_f, p \psi_{f,1} P) \).
• \( \varphi_{f,1} M := (\text{gr}_V^0 M_f, F_{=0} \text{gr}_V^0 M_f, p \varphi_{f,1} P) \).

to be the total and unipotent nearby cycles, and the unipotent vanishing cycles of \( M \) along \( f \), respectively. Here the last term in the parenthesis denotes the corresponding topological construction on the perverse sheaf \( P \). Note that all of these objects have support contained in \( f^{-1}(0) \).

The criterion regarding decomposition by strict support in terms of nearby and vanishing cycles, Corollary 3.3.8, has an enhancement to the setting of filtered \( \mathcal{D} \)-modules with \( \mathbb{Q} \)-structure:

**Theorem 4.2.2.** Let \( M = (M, F, P) \) be a filtered regular holonomic \( \mathcal{D}_X \)-module with \( \mathbb{Q} \)-structure, and suppose that \((M,F)\) is regular and quasi-unipotent along \((f=0)\) for all locally defined functions \( f \in \mathcal{O}_X(U) \) on all open sets \( U \subset X \). Then there exists a decomposition

\[
M \cong \bigoplus_{Z \subseteq X} M_Z,
\]

with each \( M_Z = (M_Z, F, P_Z) \) a filtered regular holonomic \( \mathcal{D}_X \)-module with \( \mathbb{Q} \)-structure and strict support \( Z \), if and only if

\[
\varphi_{f,1} M = \text{Ker} (\text{var}: \varphi_{f,1} M \to \psi_{f,1} M(-1)) \oplus \text{Im} (\text{can}: \psi_{f,1} M \to \varphi_{f,1} M),
\]
where the filtration on \( \text{Im}(\text{can}) \) is induced by that on \( \psi_{f,1} M \).

**Exercise 4.2.3.** Prove Theorem 4.2.2.

**Monodromy weight filtration.** Before defining pure Hodge modules, we need a discussion of the monodromy weight filtration.

First, I review a number of well-known facts regarding nearby and vanishing cycles on perverse sheaves.\(^2\) Let \( P \) be a perverse sheaf on \( X \), and \( f: X \to \mathbb{C} \) a holomorphic function. We have associated perverse nearby and vanishing cycles \( p \psi_f P \) and \( p \varphi_f P \), together with natural morphisms

\[
\text{can}: p \psi_f P \to p \varphi_f P \quad \text{and} \quad \text{var}: p \varphi_f P \to p \psi_f P(-1),
\]

and an action

\[
T: p \psi_f P \to p \psi_f P
\]

of the monodromy operator.

Recall that perverse sheaves form an abelian category; therefore there is a generalized eigenspace decomposition

\[
p \psi_f P = \bigoplus_{\lambda \in \mathbb{C}} p \psi_{f,\lambda} P
\]

under the action of \( T \), and similarly for \( p \varphi_f P \). Concretely, we have

\[
p \psi_{f,\lambda} P = \text{Ker}(T - \lambda \cdot \text{Id})^m, \quad \text{for} \quad m \gg 0.
\]

\(^2\)It’s ok to skip if you are not familiar with this story. For more details and further references you can consult for instance D. Massey’s notes arXiv:math/9908107; see also [Sch, §8].
The unipotent nearby cycles are \( p_\psi f,1 P = \text{Ker}(\text{Id} - T)^m \), with \( m \gg 0 \).

Consider now the nilpotent operator
\[
N = \frac{1}{2\pi i} \cdot \log T_u,
\]
where \( T_u \) is the unipotent part of the monodromy \( T \). (Recall from linear algebra that there is always a decomposition \( T = T_u \cdot T_s = T_s \cdot T_u \), where \( T_u \) is unipotent and \( T_s \) is semisimple.) It turns out that we have
\[
N = \text{var} \circ \text{can}: p_\psi f,1 P \to p_\psi f,1 P.
\]
This operator, as any nilpotent endomorphism, induces a filtration which in this context is called the monodromy weight filtration; this is an increasing filtration \( W^k p_\psi f,1 P \), going from \( W^0 = 0 \) to \( W^{2k} = 0 \) if \( N^{k+1} = 0 \). Denoting \( W^\ell = W_\ell p_\psi f,1 P \), it has the properties
\[
\begin{align*}
(1) & \quad N(W^\ell) \subseteq W^{\ell-2}. \\
(2) & \quad N^p: \text{gr}^W_{\ell+p} \to \text{gr}^W_{\ell-p} \text{ is an isomorphism.}
\end{align*}
\]
Moreover, if we denote \( P_\ell = \text{Ker}(N^{\ell-k+1}) \subseteq \text{gr}^W_\ell \) for \( \ell > k \), and \( P_\ell = 0 \) for \( \ell < k \), then \( \text{gr}^W_\ell = \bigoplus_i N^i(P_{\ell+2i}) \).

Going back to filtered \( \mathcal{D} \)-modules \( (\mathcal{M}, F) \) with \( V \)-filtration along \( f \) an important result is that the two notions of nearby and vanishing cycles we have seen are related by the Riemann-Hilbert correspondence. (This is of course the reason for adopting this terminology in the world of \( \mathcal{D} \)-modules.) The following is a result of Kashiwara and Malgrange, enhanced by Saito to the case of the rational \( V \)-filtration.

**Theorem 4.2.4 ([Sa1, Proposition 3.4.12]).** If \( \mathcal{M} \) is holonomic, and is regular and quasi-unipotent along \( f \), then we have
\[
\text{DR}(\text{gr}^V_\lambda \mathcal{M}_f) \simeq \begin{cases} 
p_\varphi f,\lambda \text{DR}(\mathcal{M}) & \text{for } 0 \leq \lambda < 1 \\
p_\psi f,\lambda \text{DR}(\mathcal{M}) & \text{for } 0 < \lambda \leq 1 \end{cases}
\]
where \( \lambda = e^{-2\pi i \alpha} \). Moreover, via this identification the operators \( N, \text{can} \) and \( \text{var} \) at the level of perverse sheaves are identified with \( t \partial_t, \partial_t \) and \( t \) at the level of \( \mathcal{D} \)-modules.

To repeat, on \( \psi f,1 \mathcal{M} = \text{gr}^1_\psi \mathcal{M}_f \), the nilpotent operator \( N = \text{var} \circ \text{can} \) corresponds to the operator \( t \partial_t \). This respects the filtration \( F \); thus putting all of this together, if \( \mathcal{M} = (\mathcal{M}, F, P) \) is a filtered regular holonomic \( \mathcal{D}_X \)-module with \( \mathbb{Q} \)-structure, we get an induced nilpotent operator
\[
N: \psi f,1 \mathcal{M} \to \psi f,1 \mathcal{M},
\]
and therefore a monodromy weight filtration \( W \) with associated graded objects \( \text{gr}^W_k \psi f,1 \mathcal{M} \).

A similar construction can be made for the full nearby cycle functor \( \psi f \mathcal{M} \).

### 4.3. Pure Hodge modules

We can now define the category \( \text{HM}(X, w) \) of pure Hodge modules of weight \( w \) on a smooth complex variety \( X \). The definition is inductive on the dimension of the support.
of the Hodge module, so it makes sense to successively define the categories $\text{HM}_{\leq d}(X, w)$ of pure Hodge modules on $X$ of weight $w$ and support of dimension at most $d$.

In any case, the objects of $\text{HM}(X, w)$ are filtered regular holonomic $\mathcal{D}_X$-modules with $\mathbb{Q}$-structure $M = (\mathcal{M}, F, P)$ which are subject to certain conditions; namely:

1. $(\mathcal{M}, F)$ is regular and quasi-unipotent along every locally defined holomorphic function $f : U \to \mathbb{C}$, with $U \subseteq X$ open.

2. $M$ admits a strict support decomposition in the category of filtered regular holonomic $\mathcal{D}_X$-modules with $\mathbb{Q}$-structure. (Thus the formula in Theorem 4.2.2 holds.)

Pausing for a second from imposing conditions, note that (2) implies that it is enough to define pure Hodge modules with strict support. Therefore it is enough to define for each irreducible closed subset $Z \subseteq X$ the category $\text{HM}_Z(X, w)$ of pure Hodge modules on $X$ of weight $w$ and strict support $Z$.

Moreover, because of (1) and the discussion in the previous section, for every $f : U \to \mathbb{C}$ the nearby and vanishing cycles $\psi_f M$, $\psi_{f,1} M$, and $\varphi_{f,1} M$ do make sense as filtered regular holonomic $\mathcal{D}_X$-modules with $\mathbb{Q}$-structure, this time with support contained in $f = 0$. They will be used for the inductive definition.

Continuing with the definition, for the base case we impose the following:

3. If $Z = \{x\}$, where $x \in X$ is a point, embedded via $i : \{x\} \hookrightarrow X$, then $H_{\{x\}}(X, w) = \{i_* H \mid H \text{ is a } \mathbb{Q}-\text{Hodge structure of weight } w\}$.

4. For arbitrary $Z$ of dimension $d$, we say that $M$ belongs to $\text{HM}_Z(X, w)$ if the following is satisfied: for every function $f : U \to \mathbb{C}$ which does not vanish identically on $Z \cap U$, we require for all $k$ that $\text{gr}^W_k \psi_{f,1} M \in \text{HM}_{\leq d-1}(X, w - 1 + k)$, where $W$ is the monodromy weight filtration as in the previous section. Note that Theorem 4.2.2, which can be applied as discussed above, implies then that $\text{gr}^W_k \varphi_{f,1} M \in \text{HM}_{\leq d-1}(X, w + k)$ as well.

This completes inductively the definition of $\text{HM}(X, w)$ as $\bigcup_{d \geq 0} \text{HM}_{\leq d}(X, w)$, where the morphisms are morphisms of filtered regular holonomic $\mathcal{D}_X$-modules with $\mathbb{Q}$-structure. One can show that these are strictly compatible with the filtration $F$, so that in particular $\text{HM}(X, w)$ is an abelian category.

**Remark 4.3.1.** Strictly speaking the definition above is a definition for analytic pure Hodge modules, that can be given on any complex manifold. To have a category of algebraic Hodge modules on a smooth complex variety, one can check the properties above with respect to regular functions, but one also has to work with polarizable Hodge structures, and take into account in the definition of the notion of a polarization on a Hodge module. See for instance [Sa8, §2.2] for a discussion. See also the end of §4.4 below.

\footnote{For instance, when $Z = X$ is a curve, then $\psi_f M$ is locally supported on a point, and this data corresponds to a mixed Hodge structure.}
Remark 4.3.2. It is worth noting that the use of nearby cycles in the inductive
definition of Hodge modules is rather natural, due to an important result in Hodge theory
due to W. Schmid, called the SL$_2$-orbit theorem. A consequence of this result is that the
nearby cycles of a polarizable VHS on the punctured disk carry a mixed Hodge structure
whose Hodge filtration is the limit Hodge filtration of the variation, and whose weight
filtration is the monodromy weight filtration of N (up to a shift).

Example 4.3.3 (Variations of Hodge structure). Going back to Example 1.2.3, a
VHS is the most basic type of a pure Hodge module, namely what is considered to be the
“smooth” case. To a Q-VHS $V = (V, F^\bullet, V_Q)$ of weight $\ell$ on $X$ one associates the pure
Hodge module $M = (\mathcal{M}, F, P)$ of weight $n + \ell$, where:

$\cdot \mathcal{M} = V$
$\cdot F^p\mathcal{M} = F^{-p}V$
$\cdot P = V_Q[n]$.

Given the complicated inductive definition that depends on looking at all (locally defined)
functions on $X$, it is not at all obvious that this indeed defines a pure Hodge module.
This goes through several fundamental theorems in Hodge theory, or alternatively (and
somewhat hiding what lies behind it) follows from functorial properties of Hodge modules
established later on.

This applies even to the most basic example of this construction, namely the trivial
Hodge module on $X$, i.e. the object

$$Q_X^H[n] := (\mathcal{O}_X, F, Q_X[n])$$

associated to the trivial variation of Hodge structure $Q$ on $X$, where the filtration $F$ is
the trivial filtration on $\mathcal{O}_X$ defined in Example 1.2.2.

We will see other examples later on, but first we need to understand some functorial
properties of pure Hodge modules. On the other hand, just as with holonomic $\mathcal{D}$-modules
every pure Hodge with strict support is generically “smooth”.

Proposition 4.3.4. Let $X$ be a smooth complex variety, and $Z \subseteq X$ an irreducible
closed subset of dimension $m$. If $M$ is a pure Hodge module of weight $w$ on $X$, with strict
support $Z$, then there exists a Zariski open subset $U \subseteq Z$ such that $M|_U$ is a Q-VHS of
weight $w - m$ on $U$, and $M$ is uniquely determined by this VHS.

Proof. Let $M = (\mathcal{M}, F, P) \in HM_Z(X, w)$. Then $\mathcal{M}$ is holonomic, hence by a basic
result discussed last quarter there exists an open set $V \subseteq X$ intersecting $Z$ nontrivially,
such that if $V = Z \cap U$, then $\mathcal{M}|_V$ is an integrable connection. Similarly, a well-known fact
from the theory of perverse sheaves says that we can also take $V$ such that $P|_V = L[m]$, 
where $L$ is a local system on $V$ (and in fact $P$ is the intersection complex of $L$). Restricting
the filtration to $V$ as well, clearly gives us the data of a Q-VHS.

On the other hand, this VHS uniquely determines $M$ (meaning an element in
HM$_Z(X, w)$ which restricts to it on $V$). By possibly shrinking $U$, we can assume that
its complement in $X$ is a hypersurface. The assertion then follows from the property of
$\mathcal{M}, F$) of being regular and quasi-unipotent along this hypersurface, as explained in the paragraph around (4.1.1). (Again, the perverse sheaf $P$ is determined as the intersection complex of the local system underlying the VHS.)

One of the main results of Saito’s theory is that under the extra assumption that the VHS is polarizable (which is satisfied by all those of algebro-geometric origin), the converse of this statement is true.

**Theorem 4.3.5 ([Sa2]).** Every polarizable $\mathbb{Q}$-VHS of weight $w - m$ on a Zariski open set $V \subseteq Z$ extends uniquely to a (polarizable) pure Hodge module in $HM_Z(X, w)$.

A nice survey of the proof of this theorem is given in [Sch, §17-18].

### 4.4. Mixed Hodge modules

In this section we will record a few facts from the theory of mixed Hodge modules, to at least be able to place some of the objects we’ve been looking at, like the localization $\mathcal{O}_X(*D)$ along a hypersurface, in the proper context. The relevant reference is [Sa2], while [Sch, §20-22] provides a quick overview emphasizing useful details that I will skip here.

Again, the initial definition is an analytic one, that works on an arbitrary complex manifold $X$.

**Definition 4.4.1.** The category $\text{MHW}(X)$ of weakly mixed Hodge modules consists of objects $(M, W_\bullet)$, where $M = (\mathcal{M}, F, P)$ is a filtered regular holonomic $\mathcal{D}_X$-module with $\mathbb{Q}$-structure, and $W_\bullet$ is a finite decreasing filtration on $M$ by filtered regular holonomic $\mathcal{D}_X$-modules with $\mathbb{Q}$-structure, compatible with the isomorphism $\text{DR}(\mathcal{M}) \simeq P \otimes_{\mathbb{Q}} \mathbb{C}$, such that for all $\ell \in \mathbb{Z}$ we have

$$\text{gr}_W^\ell M \in \text{HM}(X, \ell).$$

The morphisms in this category and strict morphisms of filtered regular holonomic $\mathcal{D}_X$-module with $\mathbb{Q}$-structure that strictly respect the filtration $W_\bullet$. A weakly mixed Hodge module $M$ is graded-polarizable if each $\text{gr}_W^\ell M$ is polarizable. These form a subcategory of $\text{MHW}(X)$ denoted $\text{MHW}^p(X)$.

**Remark 4.4.2.** It is not hard to see that the category $\text{MHW}(X)$ is abelian; see [Sa2, Proposition 5.1.14].

Mixed Hodge modules form a subcategory of $\text{MHW}(X)$. The reason for imposing further restrictions is an issue that already appears in the theory of mixed Hodge structures, which can roughly be expressed as the fact that variations of Hodge structure can have bad singularities at the boundary, that do not allow for extensions in the category of regular $\mathcal{D}$-modules.

**Definition 4.4.3.** Let $(M, W_\bullet) \in \text{MHW}^p(X)$, and let $f: X \to \mathbb{C}$ be a non-constant holomorphic function on $X$. We say that $(M, W_\bullet)$ is admissible along $f$ if the following hold:
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(1) \((M, F)\) is regular and quasi-unipotent along \(f\).
(2) The three filtrations \(F_\bullet M_f, V_\bullet M_f\), and \(W_\bullet M_f\) are compatible, i.e. the order does not matter when we compute the associated graded.
(3) Consider the naive limit filtrations
\[
L_i(\psi f M) = \psi f(W_{i+1} M) \quad \text{and} \quad L_i(\varphi_{f,1} M) = \varphi_{f,1}(W_i M),
\]
preserved by the nilpotent endomorphism \(N = (2\pi i)^{-1}\log T_u\). Then, the relative monodromy filtrations
\[
W_\bullet(\psi f M) = W_\bullet(N, L_\bullet(\psi f M)) \quad \text{and} \quad W_\bullet(\varphi_{f,1} M) = W_\bullet(N, L_\bullet(\varphi_{f,1} M))
\]
for the action of \(N\) exist; see [Sa2, 1.1.3-4].

**Remark 4.4.4.** It is not hard to see that condition (1) above holds in fact automatically, since it holds for each of the pure Hodge modules \(\text{gr}_W^M\).

With this preparation, we can define mixed Hodge modules; the definition is again inductive on dimension.

**Definition 4.4.5.** A weakly mixed Hodge module \((M, W_\bullet) \in \text{MHW}(X)\) is a **mixed Hodge module** if for every locally defined holomorphic function \(f: U \to \mathbb{C}\) we have:

(1) The pair \((M, W_\bullet)\) is admissible along \(f\).
(2) Both \((\psi f M, W_\bullet)\) and \((\varphi f M, W_\bullet)\) are mixed Hodge modules, whenever \(f^{-1}(0)\) does not contain any irreducible components of \(U \cap \text{Supp}(M)\).

We denote by \(\text{MHM}(X)\) and \(\text{MHM}^p(X)\) the full subcategories of \(\text{MHW}(X)\) and \(\text{MHW}^p(X)\) the full subcategories of mixed Hodge modules and graded-polarizable mixed Hodge modules, respectively. Morphisms are given by morphisms of filtered regular holonomic \(\mathcal{D}_X\)-modules with \(\mathbb{Q}\)-structure that are compatible with \(W_\bullet\).

The following theorem summarizes some important properties and functors associated with mixed Hodge modules.

**Theorem 4.4.6.** [Sa2] (1) The categories \(\text{MHM}(X)\) and \(\text{MHM}^p(X)\) are abelian.
(2) The categories \(\text{MHM}(X)\) and \(\text{MHM}^p(X)\) are stable under applying the nearby and vanishing cycles functors.
(3) If \(f: X \to Y\) is a projective morphism, then there exist direct image functors
\[
\mathcal{H}^i f_* M : \text{MHM}^p(X) \to \text{MHM}^p(Y).
\]
(4) If \(f: X \to Y\) is an arbitrary morphism, then there exist inverse image functors
\[
\mathcal{H}^i f^* M, \mathcal{H}^i f^! M : \text{MHM}^p(Y) \to \text{MHM}^p(X).
\]

There is also an important analogue of Theorem 4.3.5:

**Theorem 4.4.7 ([Sa2, Theorem 3.27]).** Let \(Z\) be an irreducible closed subset of \(X\). Then a graded-polarizable variation of mixed Hodge structure on an open subset of \(Z\) can be extended to an object in \(\text{MHM}(X)\) supported on \(Z\) if and only if it is admissible relative to \(Z\).
Regarding the two theorems above, note that for an open embedding \( j: U \hookrightarrow X \) we cannot hope to have a general functor \( j_* \) on \( \text{MHM}^p(U) \), as the admissibility condition does not necessarily hold.

Our main interest however is to work on a smooth algebraic variety \( X \), and consider subcategory \( \text{MHM}_{\text{alg}}(X) \subseteq \text{MHM}^p(X_{\text{an}}) \) of algebraic mixed Hodge modules. (Note that these are automatically polarizable.) These were originally defined in \([Sa2]\) using a compactification of \( X \), and taking advantage of GAGA-type theorems. However in \([Sa7]\) Saito found an alternative intrinsic definition which can be summarized as follows:

**Theorem 4.4.8.** A weakly mixed Hodge module \((M, W_\bullet)\) belongs to the category \( \text{MHM}_{\text{alg}}(X) \) if and only if \( X \) can be covered by Zariski open subsets \( U \) such that:

1. There exists \( f \in \mathcal{O}_X(U) \) such that \( U \setminus f^{-1}(0) \) is smooth and dense in \( U \).
2. The restriction of \((M, W_\bullet)\) to the open subset \( U_{\text{an}} \setminus f^{-1}(0) \) is a graded-polarizable admissible variation of mixed Hodge structure.
3. The pair \((M, W_\bullet)\) is admissible along \( f \).
4. The pair \( (\psi_f M, W_\bullet(\psi_f M)) \) belongs to \( \text{MHM}_{\text{alg}}(f^{-1}(0)) \).

Unlike the story in the analytic setting, for an open embedding \( j: U \hookrightarrow X \) of smooth algebraic varieties the functors \( j_* \) and \( j! \) are defined on \( \text{MHM}_{\text{alg}}(U) \), and this allows one to define functors \( f_* \) and \( f! \) for an arbitrary morphism of algebraic varieties.

For all practical purposes, besides the existence of these functors, what we need to know concretely at this stage is the construction of the direct image via the open embedding from the complement of an SNC divisor. The full construction of direct image via an open embedding combines this with resolution of singularities; see \([Sa2, \text{Theorem 3.27}]\).

**Extension of a VHS across an SNC divisor.** Let \( D \) be an SNC divisor in a smooth variety \( X \), and denote \( j: U = X \setminus D \hookrightarrow X \). We consider a polarizable VHS \( \mathbf{V} = (\mathcal{V}, F_\bullet, \mathbf{V}_\mathbb{Q}) \) over \( U \), with quasii-unipotent local monodromies along the components \( D_i \) of \( D \). In particular the eigenvalues of all residues are rational numbers. We call \( M \) the associated pure Hodge module on \( U \).

For \( \alpha \in \mathbb{Z} \), we denote by \( \mathcal{V}_{\geq \alpha} \) (resp. \( \mathcal{V}^{>\alpha} \)) the Deligne extension with eigenvalues of residues along the \( D_i \) in \([\alpha, \alpha + 1)\) (resp. \((\alpha, \alpha + 1]\)). \(^4\) Recall that \( \mathcal{V}_{\geq \alpha} \) is filtered by

\[
F_p \mathcal{V}_{\geq \alpha} = \mathcal{V}_{\geq \alpha} \cap j_* F_p \mathbf{V},
\]

while the filtration on \( \mathcal{V}^{>\alpha} \) is defined similarly. The terms in the filtration are locally free by Schmid’s nilpotent orbit theorem (which can be extended to the quasi-unipotent case).

The mixed Hodge module \( j_* M \) on \( X \) is then defined as follows:

\[
j_* M = (\mathbf{V}(\ast D), F_\bullet, j_* \mathbf{V}_\mathbb{Q}).
\]

\(^4\)Unfortunately we haven’t discussed yet this important construction in this course; see for instance \([HTT, \text{Theorem 5.2.17}]\).
Here $\mathcal{V}(\ast D)$ is Deligne’s meromorphic connection extending $\mathcal{V}$. It has a lattice defined by $\mathcal{V}^{\geq \alpha}$ for any $\alpha \in \mathbb{Q}$, i.e. $\mathcal{V}(\ast D) = \mathcal{V}^{\geq \alpha} \otimes \mathcal{O}_X(\ast D)$, and its filtration is given by

$$F_p \mathcal{V}(\ast D) = \sum_i F_i \mathcal{D}_X \cdot F_{p-i} \mathcal{V}^{\geq -1}.$$ 

For the weight filtration see [Sa2, Proposition 2.8].

**Example 4.4.9 (Localization).** When $M = \mathbb{Q}_U^H[n]$ is the trivial Hodge module, then $\mathcal{M} = \mathcal{O}_U$, whose Deligne canonical extension is $\mathcal{O}_X$. In this case

$$j_* \mathcal{O}_U = (\mathcal{O}_X(\ast D), F_\bullet, j_* \mathbb{Q}).$$

Thus $\mathcal{O}_X(\ast D)$ underlies a mixed Hodge module, and the Hodge filtration can be described as follows. First, it is well known that $\mathcal{O}_U^{\geq -1} = \mathcal{O}_X(D)$, and its filtration is

$$F_k \mathcal{O}_X(D) = \mathcal{O}_X(D) \cap j_* F_k \mathcal{O}_U = \mathcal{O}_X(D) \cap j_* \mathcal{O}_U = \mathcal{O}_X(D)$$

for all $k \geq 0$, and 0 otherwise. The formula above gives

$$F_p \mathcal{O}_X(\ast D) = F_p \mathcal{O}_X(D)$$

for $p \geq 0$, and 0 otherwise.

Since we have introduced this terminology, it is worth noting that the pure Hodge module extension of $M$ with strict support $X$ (see Theorem 4.3.5) is given by the minimal extension functor, which we looked at briefly earlier in this course. More precisely, as shown in [Sa2, §3.b], it is

$$(\mathcal{D}_X \cdot \mathcal{V}^{\geq -1}, F_\bullet, j_* V_Q),$$

where the filtration is defined as

$$F_p (\mathcal{D}_X \cdot \mathcal{V}^{\geq -1}) = \sum_i F_i \mathcal{D}_X \cdot F_{p-i} \mathcal{V}^{\geq -1}.$$ 

Now that we have at least seen one of the main definitions in the theory of Hodge modules, we need to go back and try to understand in more explicit detail some of the special properties of the filtered $\mathcal{D}$-modules that underlie these objects, and how they behave under natural functors.

### 4.5. Induced $\mathcal{D}$-modules and filtered differential morphisms

In this section we show that complexes of filtered $\mathcal{D}_X$-modules are quasi-isomorphic to complexes whose entries are special types of $\mathcal{D}$-modules arising from plain old $\mathcal{O}_X$-modules. These can be further interpreted as complexes of $\mathcal{O}_X$-modules with a special type of $\mathcal{C}$-linear differentials. Since we will apply this to studying filtered push-forward, it is more convenient to work with right $\mathcal{D}_X$-modules.

Recall that when passing between a filtered left $\mathcal{D}_X$-module $(\mathcal{N}, F)$ and its right version $(\mathcal{M} = \omega_X \otimes_{\mathcal{O}_X} \mathcal{N}, F)$, the filtration rule is

$$F_{k-n} \mathcal{M} = F_k \mathcal{N} \otimes_{\mathcal{O}_X} \omega_X$$
for all $k$, where $n = \dim X$. For instance, the trivial filtration on the right $\mathcal{D}_X$-module $\omega_X$, the right analogue of Example 1.2.2, is given by $F_k\omega_X = 0$ for $k < -n$ and $F_k\omega_X = \omega_X$ for $k \geq -n$.

Note once and for all that all types of filtrations appearing below satisfy the property that $F_p = 0$ for $p \ll 0$.

We begin by denoting by $\text{FM}(\mathcal{D}_X)$ the category of filtered right $\mathcal{D}_X$-modules. A morphism in this category is a $\mathcal{D}_X$-module morphism $f : M \to N$ such that $f(F_kM) \subseteq F_kN$ for all $k$. Special objects in this category are those induced by filtered $\mathcal{O}_X$-modules. Concretely, consider a filtered $\mathcal{O}_X$-module $(G, F^\bullet G)$, with respect to the trivial filtration on $\mathcal{O}_X$ given by $F_k\mathcal{O}_X = 0$ for $k < 0$ and $F_k\mathcal{O}_X = \mathcal{O}_X$ for $k \geq 0$. We can associate to it an object in $\text{FM}(\mathcal{D}_X)$ given by

$$M := G \otimes_{\mathcal{O}_X} \mathcal{D}_X \text{ and } F_kM := \sum_{i=0}^{k} F_{k-i} G \otimes F_i \mathcal{D}_X.$$ 

**Definition 4.5.1.** An object in $\text{FM}(\mathcal{D}_X)$ is an induced filtered $\mathcal{D}_X$-module if it is isomorphic to one defined as above. We use the notation $\text{FM}_i(\mathcal{D}_X)$ for the full subcategory of $\text{FM}(\mathcal{D}_X)$ whose objects are induced filtered $\mathcal{D}_X$-modules.

**Remark 4.5.2.** Saito [Sa1, §2.2] calls the functor

$$M := G \otimes_{\mathcal{O}_X} \mathcal{D}_X \mapsto \text{DR}(M) := M \otimes_{\mathcal{O}_X} \mathcal{O}_X \in \text{Mod}(\mathcal{C}_X).$$

the de Rham functor on induced $\mathcal{D}$-modules. We have of course a canonical isomorphism $\text{DR}(M) \simeq G$.

**Lemma 4.5.3.** Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then there is a natural homomorphism

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \to \text{Hom}_{\mathcal{C}_X}(\mathcal{F}, \mathcal{G})$$

given by $\bullet \otimes_{\mathcal{D}_X} \mathcal{O}_X$, and this homomorphism is injective.

**Proof.** By adjunction we have an isomorphism

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X),$$

and therefore the homomorphism in the statement corresponds to sending $\varphi$ to $\psi$ in each diagram of the form

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\psi} & \mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\
\end{array}$$

where the $\mathcal{O}_X$-module structure on the top sheaf is obtained by restriction of scalars from its $\mathcal{D}_X$-module structure (hence in particular $\psi$ is not $\mathcal{O}_X$-linear), and the vertical arrow is obtained by sending $s \otimes P \mapsto P(1)s$, for a section $s$ of $\mathcal{G}$ and a differential operator $P$.

Assume now that $\varphi \neq 0$. Hence there is an open set $U$ with local coordinates $x_1, \ldots, x_n$, and a section $s \in \Gamma(U, \mathcal{F})$ such that $\varphi(s) \neq 0$. We write

$$\varphi(s) = \sum_{\alpha} t_{\alpha} \otimes \partial^\alpha,$$
where the sum is finite, over \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), and \( t_\alpha \in \Gamma(U, \mathscr{G}) \). We use the standard notation \( \partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \), and we will also use \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \), \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and \( \alpha! = \alpha_1! \cdots \alpha_n! \). Let

\[
k_0 := \min \{ |\alpha| \mid t_\alpha \neq 0 \},
\]

and consider \( \beta \in \mathbb{N}^n \) such that \( |\beta| = k_0 \) and \( t_\beta \neq 0 \). We then have

\[
\psi(s \cdot x^\beta) = \sum_\alpha t_\alpha \otimes \partial^\alpha (x^\beta) = \beta! \cdot t_\beta \neq 0,
\]

hence \( \psi \neq 0 \). \qed

**Definition 4.5.4.** Let \( \mathcal{F} \) and \( \mathcal{G} \) be \( \mathcal{O}_X \)-modules. The group of *differential morphisms* from \( \mathcal{F} \) to \( \mathcal{G} \) is defined as the image in \( \text{Hom}_{\mathcal{C}_X}(\mathcal{F}, \mathcal{G}) \) of the homomorphism in the Lemma above, and is denoted by \( \text{Hom}_{\text{Diff}}(\mathcal{F}, \mathcal{G}) \).

Note that \( \text{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \) admits a filtration whose \( p \)-th term is

\[
\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} F_p \mathcal{D}_X),
\]

and therefore we can define a filtration on differential morphisms where \( F_p \text{Hom}_{\text{Diff}}(\mathcal{F}, \mathcal{G}) \) is (the image of) this subgroup. We call these differential morphisms *of order* \( \leq p \).

We now give a filtered version of this construction.

**Definition 4.5.5.** Let \( (\mathcal{F}, \mathcal{F}) \) and \( (\mathcal{G}, \mathcal{F}) \) be filtered \( \mathcal{O}_X \)-modules. The group of *filtered differential morphisms*

\[
\text{Hom}_{\text{Diff}}((\mathcal{F}, \mathcal{F}), (\mathcal{G}, \mathcal{F}))
\]

is the subgroup of \( \text{Hom}_{\text{Diff}}(\mathcal{F}, \mathcal{G}) \) consisting of morphisms \( f \) satisfying, for every \( p \) and \( q \), the fact that the composition

\[
F_p \mathcal{F} \hookrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \rightarrow \mathcal{G} / F_{p-q-1} \mathcal{G}
\]

(which itself is a differential morphism) has order \( \leq q \).

**Exercise 4.5.6.** Restricting to filtered morphisms, the homomorphism in Lemma 4.5.3 induces an isomorphism

\[
\text{Hom}_{\mathcal{F}(\mathcal{D}_X)}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \text{Hom}_{\text{Diff}}((\mathcal{F}, \mathcal{F}), (\mathcal{G}, \mathcal{F})).
\]

**Definition 4.5.7.** We denote by \( \mathcal{F}(\mathcal{D}_X, \text{Diff}) \) the additive category whose objects are filtered \( \mathcal{O}_X \)-modules, and whose morphisms are filtered differential morphisms.

Putting together all of the above, we obtain the following interpretation of the category of induced filtered \( \mathcal{D} \)-modules.

**Proposition 4.5.8.** The functor

\[
\text{DR}^{-1} : \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X
\]

induces an equivalence of categories

\[
\text{DR}^{-1} : \mathcal{F}(\mathcal{O}_X, \text{Diff}) \xrightarrow{\simeq} \mathcal{F}(\mathcal{D}_X).
\]
It is not hard to see that this equivalence extends to an equivalence of triangulated categories
\[(4.5.1) \quad \text{DR}^{-1} : D(\mathcal{O}_X, \text{Diff}) \xrightarrow{\simeq} D(\mathcal{O}_X, \mathcal{D}_X)).\]
However, we need a brief discussion of these and other derived categories that will be used from now on, which is done in the next remark.

**Remark 4.5.9** (Definition of derived categories). By $D(\mathcal{O}_X, \mathcal{D}_X))$, $D(\mathcal{O}_X, \mathcal{D}_X))$ and all the others, we mean $D^\bullet$, where $\bullet$ can be either absent or any of $\bullet = -, +, b$. However, the definition of these derived categories needs some explanation. I will only do this for $D(\mathcal{O}_X, \mathcal{D}_X))$, as all the others are similar. First, note that $D(\mathcal{O}_X, \mathcal{D}_X))$ is an additive category which has (co)kernels and (co)images, but it is not in general an abelian category. Hence we are not looking at the derived category associated to an abelian category. We form $\mathcal{C}^\bullet(\mathcal{O}_X, \mathcal{D}_X))$, the category of complexes of objects in $\mathcal{O}_X, \mathcal{D}_X))$, and then the homotopy category $K^\bullet(\mathcal{O}_X, \mathcal{D}_X))$, where the homotopies are required to preserve the filtrations. It is not hard to see that $K^\bullet(\mathcal{O}_X, \mathcal{D}_X))$ has a natural structure of triangulated category. Finally, the filtered derived category $D^\bullet(\mathcal{O}_X, \mathcal{D}_X))$ is the localization of $K^\bullet(\mathcal{O}_X, \mathcal{D}_X))$ at the class of filtered quasi-isomorphisms. As with the derived category of an abelian category, one can show that there is a unique triangulated structure on $D^\bullet(\mathcal{O}_X, \mathcal{D}_X))$ such that the canonical localization functor $K^\bullet(\mathcal{O}_X, \mathcal{D}_X)) \xrightarrow{\text{DR}^{-1}} D^\bullet(\mathcal{O}_X, \mathcal{D}_X))$ is exact.

On the other hand, the derived category $D(\mathcal{O}_X, \mathcal{D}_X))$ is obtained by inverting $D$-quasi-isomorphisms in $\mathcal{O}_X, \mathcal{D}_X))$, meaning those morphisms that are mapped to (filtered) quasi-isomorphisms via the functor $\text{DR}^{-1}$.

We next observe that every object in $\mathcal{O}_X, \mathcal{D}_X))$ admits a finite resolution by induced filtered $\mathcal{D}_X$-modules, and use this to find a quasi-inverse for the equivalence above. To this end, recall the Spencer complex
\[
0 \to \mathcal{D}_X \otimes \mathcal{D}_X \wedge^n T_X \to \cdots \to \mathcal{D}_X \otimes \mathcal{D}_X T_X \to \mathcal{D}_X \to 0,
\]
placed in degrees $-n, \cdots, 0$. The differentials are such that this complex is isomorphic in local coordinates $x_1, \ldots, x_n$ to the Koszul complex $K(\mathcal{D}_X; \partial_1, \ldots, \partial_n)[n]$ associated to the (right) action of $\partial_1, \ldots, \partial_n$ on $\mathcal{D}_X$. We consider this to be a complex of filtered left $\mathcal{D}_X$-modules, where the filtration on $\mathcal{D}_X \otimes \mathcal{D}_X \wedge^i T_X$ is given by
\[
F_k(\mathcal{D}_X \otimes \mathcal{D}_X \wedge^i T_X) := F_{k+i} \mathcal{D}_X \otimes \mathcal{D}_X \wedge^i T_X.
\]
This complex is filtered quasi-isomorphic to the left $\mathcal{D}_X$-module $\mathcal{D}_X$ with the trivial filtration; see [HTT, Lemma 1.5.27].

Consider now an arbitrary $(M, F) \in \mathcal{O}_X, \mathcal{D}_X))$. Recall that $M \otimes_{\mathcal{D}_X} \mathcal{D}_X$ has a natural right $\mathcal{D}_X$-module structure (see e.g. [HTT, Proposition 1.2.9(ii)]). Applying $M \otimes_{\mathcal{D}_X} \bullet$ to

\[\text{Given a morphism } \varphi : M \to N \text{ in } \mathcal{O}_X, \mathcal{D}_X)) \text{, it is not necessarily the case that the induced morphism } \text{Coi}(\varphi) : \text{Im}(\varphi) \to \text{Im}(\varphi) \text{ is an isomorphism.}\]

\[\text{Essentially one has to note that the cone of a morphism in } \mathcal{O}_X, \mathcal{D}_X) \text{ carries a natural filtration, such that all the morphisms in the associated exact triangle are compatible with the filtrations.}\]
the complex above corresponds to the Spencer complex of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X$:

\[(4.5.2)\quad 0 \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^n T_X \to \cdots \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X T_X \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to 0.\]

**Proposition 4.5.10.** The complex in (4.5.2) is a complex of filtered induced right $\mathcal{D}_X$-modules, quasi-isomorphic to $(\mathcal{M}, F)$.

**Proof.** By the same [HTT, Proposition 1.2.9(ii)], all the terms $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^i T_X$ have a natural (filtered) right $\mathcal{D}_X$-module structure, and it is not hard to check that there is a filtered isomorphism

\[\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^i T_X \cong \mathcal{M} \otimes_{\mathcal{O}_X} \wedge^i T_X \otimes_{\mathcal{O}_X} \mathcal{D}_X,\]

with the obvious right $\mathcal{D}_X$-module structure on the right hand side (exercise!). This realizes our complex as a complex of induced $\mathcal{D}_X$-modules, using the natural $\mathcal{O}_X$-module structure on $\mathcal{M} \otimes_{\mathcal{O}_X} \wedge^i T_X$.

On the other hand, there is a natural map

\[\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \mathcal{M}, \quad m \otimes P \mapsto P(1)m, \]

and this is a surjective filtered right $\mathcal{D}_X$-module morphism. Placing this at the right end of the complex in (4.5.2), the claim is that it induces a filtered quasi-isomorphism between this complex and $(\mathcal{M}, F)$ (which then finishes the proof). This is of course equivalent to saying that the complex in (4.5.2) has no cohomology except at the right-most term.

Let’s reinterpret this for convenience in terms of the corresponding complex in $\text{FM}(\mathcal{O}_X, \text{Diff})$. Concretely, our complex is obtained by applying $\text{DR}^{-1}$ to the complex in $\text{FM}(\mathcal{O}_X, \text{Diff})$

\[\overline{\text{DR}}(\mathcal{M}, F) : \quad 0 \to \mathcal{M} \otimes_{\mathcal{O}_X} \wedge^n T_X \to \cdots \to \mathcal{M} \otimes_{\mathcal{O}_X} T_X \to \mathcal{M} \to 0.\]

(we are using the notation introduced in [Sa1, §2]), where the filtration on this complex obtained by setting

\[F_p(\mathcal{M} \otimes_{\mathcal{O}_X} \wedge^i T_X) = F_{p+i} \mathcal{M} \otimes_{\mathcal{O}_X} \wedge^i T_X.\]

Note that $\overline{\text{DR}}(\mathcal{M}, F)$ is obtained by applying $(\mathcal{M}, F) \otimes_{\mathcal{D}_X} \bullet$ to the Spencer complex resolving $\mathcal{O}_X$, and therefore in local coordinates $x_1, \ldots, x_n$ it is isomorphic to the Koszul complex $K(\mathcal{M}; \partial_1, \ldots, \partial_n)[n]$ associated to the elements $\partial_1, \ldots, \partial_n$ acting on $\mathcal{M}$, hence the assertion. \(\square\)

Restating the last part of the proof of the Proposition above, we have the following:

**Corollary 4.5.11.** For every $(\mathcal{M}, F) \in \text{FM}(\mathcal{O}_X)$ there is a natural quasi-isomorphism of filtered complexes of right $\mathcal{D}_X$-modules

\[\text{DR}^{-1}\overline{\text{DR}}(\mathcal{M}, F) \to (\mathcal{M}, F).\]

Via standard homological algebra, this discussion leads to the following equivalence of filtered derived categories:

\[\text{Note that this map is induced by tensoring with } \mathcal{M} \text{ over } \mathcal{O}_X \text{ the natural map } \mathcal{D}_X \to \mathcal{O}_X, \text{ taking an operator } P \text{ to } P(1) \text{ (and realizing the quasi-isomorphism between the Spencer complex and } \mathcal{O}_X).\]
**Proposition 4.5.12.** The natural functor \( D \left( FM_i(D_X) \right) \to D \left( FM(D_X) \right) \) is an equivalence of categories.

Moreover, restricted to filtered induced \( D \)-modules, the functor \( \widehat{DR} \) provides a quasi-inverse for the functor \( DR^{-1} \) in (4.5.1).

Together with the equivalence in (4.5.1), the Proposition above shows that in order to study operations on \( D \left( FM(D_X) \right) \) we may restrict to complexes of induced \( D \)-modules, or to filtered differential complexes. We will take advantage of this below, when defining the push-forward functor for filtered \( D \)-modules.

**Proposition 4.5.10** gives us a canonical approach to finding resolutions by induced \( D \)-modules. Other explicit resolutions may however be more meaningful and easier to work with. The following extended example is very important for applications.

**Example 4.5.13 (Localization along an SNC divisor).** Let \( E \) be a reduced simple normal crossing (SNC) divisor on a smooth \( n \)-dimensional variety \( Y \).\(^8\) Recall that \( \omega_Y(\ast E) \simeq \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(\ast E) \) stands for the right \( D \)-module version of the localization along the divisor \( E \), in other words the sheaf of \( n \)-forms with arbitrary poles along \( E \).

We endow \( \omega_Y(\ast E) \) with what we will call the Hodge filtration, namely
\[
F_k \omega_Y(\ast E) := \omega_Y(E) \cdot F_{k+n} \mathcal{D}_Y \quad \text{for} \quad k \geq -n.
\]
For instance, the first two nonzero terms are
\[
F_{-n} \omega_Y(\ast E) = \omega_Y(E) \quad \text{and} \quad F_{-n+1} \omega_Y(\ast E) = \omega_Y(2E) \cdot \text{Jac}(E),
\]
where \( \text{Jac}(E) \) is the Jacobian ideal of \( E \), i.e. \( F_1 \mathcal{D}_Y \cdot \mathcal{O}_Y(-E) \), whose zero locus is the singular locus of \( E \).

To gain intuition for what comes next, recall that the right \( D \)-module \( \omega_Y \) has a standard filtered resolution
\[
0 \to \mathcal{D}_Y \to \Omega^1_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \cdots \to \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \omega_Y \to 0
\]
by induced \( \mathcal{D}_Y \)-modules. This is simply the resolution of \( \omega_Y \) (with trivial filtration) described by the procedure in Proposition 4.5.10; see also [HTT, Lemma 1.2.57]. It is a simple check that \( \widehat{DR}(\mathcal{O}_X) \), i.e. the associated complex in \( \text{FM}(\mathcal{O}_X, \text{Diff}) \), is the standard de Rham complex
\[
0 \to \mathcal{O}_Y \xrightarrow{d} \Omega^1_Y \xrightarrow{d} \cdots \xrightarrow{d} \omega_Y \to 0.
\]

A similar type of resolution by right induced \( \mathcal{D}_Y \)-modules can be found for \( \omega_Y(\ast E) \), only this time it will correspond to the de Rham complex with log poles along \( E \).

**Proposition 4.5.14.** The right \( \mathcal{D}_Y \)-module \( \omega_Y(\ast E) \) has a filtered resolution by induced \( \mathcal{D}_Y \)-modules, given by
\[
0 \to \mathcal{D}_Y \to \Omega^1_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \cdots \to \omega_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \omega_Y(\ast E) \to 0.
\]

\(^8\)We use this notation since in practice we will consider this setting on a log resolution \( f: Y \to X \) of a pair \((X,D)\), with \( E = f^{-1}(D)_{\text{red}} \).
Here the morphism
\[ \omega_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \longrightarrow \omega_Y(*E) \]
is given by \( \omega \otimes P \rightarrow \omega \cdot P \) (the \( \mathcal{D} \)-module operation), and for each \( p \) the morphism
\[ \Omega^p_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \longrightarrow \Omega^{p+1}_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \]
is given by \( \omega \otimes P \mapsto d\omega \otimes P + \sum_{i=1}^n (dx_i \wedge \omega) \otimes \partial_i P \), in local coordinates \( x_1, \ldots, x_n \).

**Proof.** It is not hard to check that the expression in the statement is indeed a complex, which we call \( A^\bullet \). We consider on \( \Omega^p_Y(\log E) \) the filtration
\[ F_i\Omega^p_Y(\log E) = \begin{cases} \Omega^p_Y(\log E) & \text{if } i \geq -p \\ 0 & \text{if } i < -p, \end{cases} \]
and on \( \Omega^p_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \) the tensor product filtration. This filters \( A^\bullet \) by subcomplexes \( F_{k-n}A^\bullet \) given by
\[ \cdots \longrightarrow \Omega^{n-1}_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow F_k\omega_Y(*E) \rightarrow 0 \]
for each \( k \geq 0 \). Note that they can be rewritten as
\[ \cdots \longrightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{T}_Y(-\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow F_k\omega_Y(*E) \rightarrow 0, \]
where \( \mathcal{T}_Y(-\log E) \) is the dual of \( \Omega^1_Y(\log E) \), and we use the isomorphisms \( \omega_Y(E) \otimes \wedge^i \mathcal{T}_Y(-\log E) \simeq \Omega^{n-i}_Y(\log E) \).

It is clear directly from the definition that every such complex is exact at the term \( F_k\omega_Y(*E) \). We now check that they are exact at the term \( \omega_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \). Let us assume that, in the local coordinates \( x_1, \ldots, x_n \), the divisor \( E \) is given by \( x_1 \cdots x_r = 0 \). Using the notation \( \omega = dx_1 \wedge \cdots \wedge dx_n \), we consider an element
\[ u = \frac{\omega}{x_1 \cdots x_r} \otimes \sum_{|\alpha| \leq k} g_\alpha \partial^\alpha \]
mapping to 0 in \( F_k\omega_Y(*E) = \omega_Y(E) \cdot F_k\mathcal{D}_Y \). This means that
\[ \sum_{|\alpha| \leq k} \alpha_1! \cdots \alpha_r! \cdot g_\alpha \cdot x_1^{-\alpha_1} \cdots x_r^{-\alpha_r} = 0. \]
We show that \( u \) is in the image of the morphism \( \beta_k \) by using a descending induction on \( |\alpha| \). What we need to prove is the following claim: for each \( \alpha \) in the sum above, with \( |\alpha| = k \), there exists some \( i \) with \( \alpha_i > 0 \) such that \( x_i \) divides \( g_\alpha \). If so, an easy calculation shows that the term \( u_\alpha = \frac{\omega}{x_1 \cdots x_r} \otimes g_\alpha \partial^\alpha \) is in the image of \( \beta_k \), and hence it is enough to prove the statement for \( u - u_\alpha \). Repeating this a finite number of times, we can reduce to the case when all \( |\alpha| \leq k - 1 \). But the claim is clear: if \( x_i \) did not divide \( g_\alpha \) for all \( i \) with \( \alpha_i > 0 \), then the Laurent monomial \( x_1^{-\alpha_1} \cdots x_r^{-\alpha_r} \) would appear in the term \( g_\alpha \cdot x_1^{-\alpha_1} \cdots x_r^{-\alpha_r} \) of the sum above, but in none of the other terms.

To check the rest of the statement, note that after discarding the term on the right, the associated graded complexes
\[ \cdots \longrightarrow \omega_Y(E) \otimes \bigwedge^2 \mathcal{T}_Y(-\log E) \otimes_{\mathcal{O}_Y} \mathcal{S}^{k-2} \mathcal{T}_Y \longrightarrow \]
\[ \omega_Y(E) \otimes T_Y(-\log E) \otimes O_Y \rightarrow \omega_Y(E) \otimes T_Y S^k \rightarrow 0 \]

are acyclic. Indeed, each such complex is, up to a twist, an Eagon-Northcott complex associated to the inclusion of vector bundles of the same rank

\[ \varphi: T_Y(-\log E) \rightarrow T_Y. \]

Concretely, in the notation on [La, p.323], the complex above is \((EN_k)\) tensored by \(\omega_Y(E)\).

According to [La, Theorem B.2.2(iii)], \((EN_k)\) is acyclic provided that

\[ \text{codim } D_{n-\ell}(\varphi) \geq \ell \quad \text{for all } 1 \leq \ell \leq \min\{k, n\}, \]

where

\[ D_s(\varphi) = \{ y \in Y \mid \text{rk}(\varphi_y) \leq s \} \]

are the deneracy loci of \(\varphi\). But locally \(\varphi\) is given by the diagonal matrix

\[ \text{Diag}(x_1, \ldots, x_r, 1, \ldots, 1) \]

so this condition is verified by a simple calculation. \(\square\)

**Remark 4.5.15.** It is again an immediate check that the associated filtered differential complex \(\tilde{\mathcal{D}}R(\omega_Y(\ast E))\) is precisely the well-known de Rham complex of holomorphic forms with log poles along \(E\), namely

\[ 0 \rightarrow \mathcal{O}_Y \xrightarrow{d} \Omega^1_Y(\log E) \xrightarrow{d} \cdots \xrightarrow{d} \omega_Y(E) \rightarrow 0. \]

### 4.6. Push-forward of filtered \(\mathcal{D}\)-modules

We want to enhance the definition of push-forward of \(\mathcal{D}\)-modules to the filtered setting, following a construction due to Saito [Sa1, §2.1-2.3]. This is more natural in the setting of right \(\mathcal{D}\)-modules. The usual left-right transformation allows us to recover the corresponding construction for left \(\mathcal{D}\)-modules.

Let \(f: X \rightarrow Y\) be a morphism of smooth complex varieties. Recall that the associated transfer module

\[ \mathcal{D}_{X\rightarrow Y} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y \]

has the structure of a \((\mathcal{D}_X, f^{-1}\mathcal{D}_Y)\)-bimodule, and is used to define the push-forward functor at the level of derived categories by the formula

\[ f_+ : D(\mathcal{D}_X) \rightarrow D(\mathcal{D}_Y), \quad \mathcal{M}^\bullet \mapsto Rf_*(\mathcal{M}^\bullet \otimes_{\mathcal{D}_X} \mathcal{D}_{X\rightarrow Y}). \]

(For details, a good refresher is [HTT, §1.5], where this functor is denoted by \(\int_f\).) Here we loosely use the symbol \(D(\mathcal{D}_X)\) to stand for \(D^-_{\text{qcoh}}(\mathcal{D}_X)\), where \(\ast\) can be any either absent, or any of \(-, +\) or \(b\) for instance; recall that all the \(\mathcal{D}\)-modules we work with are assumed to be quasi-coherent. However if \(f\) is proper, which is our main focus, this induces a functor

\[ f_+ : D^b_{\text{coh}}(\mathcal{D}_X) \rightarrow D^b_{\text{coh}}(\mathcal{D}_Y) \]

between the bounded derived categories of coherent \(\mathcal{D}\)-modules.

Note furthermore that \(\mathcal{D}_{X\rightarrow Y}\) has a natural filtration given by \(f^*F_k\mathcal{D}_Y\). More precisely, the sheaf \(f^{-1}\mathcal{D}_Y\) carries a filtration induced by the standard filtration on \(\mathcal{D}_Y\).
By analogy with the previous section, we can also consider the categories \( \text{FM}(f^{-1}\mathcal{D}_Y) \) and \( \text{FM}_i(f^{-1}\mathcal{D}_Y) \) of filtered \( f^{-1}\mathcal{D}_Y \)-modules and filtered \textit{induced} \( f^{-1}\mathcal{D}_Y \)-modules respectively, where the latter are isomorphic to filtered \( f^{-1}\mathcal{D}_Y \)-modules of the form \( \mathcal{G} \otimes f^{-1}\mathcal{O}_Y \), with \( \mathcal{G} \) a filtered \( f^{-1}\mathcal{O}_Y \)-module.

We define the functor
\[
\text{DR}_{X/Y}: \text{FM}_i(\mathcal{D}_X) \to \text{FM}_i(f^{-1}\mathcal{D}_Y), \quad (\mathcal{M}, F) \to (\mathcal{M}, F) \otimes_{\mathcal{O}_X} \mathcal{D}_{X \to Y}.
\]
This is indeed well defined, since if \( \mathcal{M} = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X \), with \( \mathcal{G} \) a filtered \( \mathcal{O}_X \)-module, then we have an isomorphism of filtered \( f^{-1}\mathcal{D}_Y \)-modules
\[
\text{DR}_{X/Y}(\mathcal{M}) \simeq \mathcal{G} \otimes f^{-1}\mathcal{O}_Y \cdot f^{-1}\mathcal{D}_Y.
\]

**Exercise 4.6.1.** Show that \( \text{DR}_{X/Y} \) takes filtered quasi-isomorphisms of complexes in \( \text{FM}_i(\mathcal{D}_X) \) to filtered quasi-isomorphisms of complexes in \( \text{FM}_i(f^{-1}\mathcal{D}_Y) \).

We next use the definitions and notation on derived categories discussed in Remark 4.5.9. Given the exercise above, we have an induced functor
\[
\text{DR}_{X/Y}: D(\text{FM}_i(\mathcal{D}_X)) \to D(\text{FM}_i(f^{-1}\mathcal{D}_Y)).
\]
In combination with Proposition 4.5.12 and its obvious analogue, we can in turn see this as a functor
\[
\text{DR}_{X/Y}: D(\text{FM}(\mathcal{D}_X)) \to D(\text{FM}(f^{-1}\mathcal{D}_Y)).
\]

We next define a direct image functor
\[
D(\text{FM}(f^{-1}\mathcal{D}_Y)) \to D(\text{FM}(\mathcal{D}_Y)),
\]
which composed with \( \text{DR}_{X/Y} \) will give rise to our desired filtered direct image functor
\[
f_+: D(\text{FM}(\mathcal{D}_X)) \to D(\text{FM}(\mathcal{D}_Y)).
\]

**Definition 4.6.2.** Let \( (\mathcal{M}, F) \) be a filtered \( f^{-1}\mathcal{D}_Y \)-module. Its \textit{topological direct image} is defined as \( f_*(\mathcal{M}, F) = (\mathcal{N}, F) \), where
\[
\mathcal{N} := \bigcup_{k \in \mathbb{Z}} f_*F_k\mathcal{M} \subseteq f_*\mathcal{M} \quad \text{and} \quad F_k\mathcal{N} := f_*F_k\mathcal{M}.
\]
Here on the right hand side we use the standard sheaf-theoretic direct image.

**Remark 4.6.3.** This definition can be made in great generality, and usually it is not necessarily the case that \( \mathcal{N} = f_*\mathcal{M} \). However this is always true in the case we are interested in, namely the case of algebraic varieties (since every open set is quasi-compact), and also in the case of complex analytic varieties if \( f \) is proper.

We therefore obtain a functor
\[
f_*: \text{FM}(f^{-1}\mathcal{D}_Y) \to \text{FM}(\mathcal{D}_Y).
\]

We would like to extend this functor to the derived category \( D(\text{FM}(f^{-1}\mathcal{D}_Y)) \), in order to finish our construction.
First recall that to every module $\mathcal{M}$ over a sheaf of rings on $X$, in particular over $f^{-1}\mathcal{D}_Y$, we can associate the flasque sheaf of discontinuous sections $\mathcal{I}^0(\mathcal{M})$ defined on every open set $U \subseteq X$ by

$$\Gamma(U, \mathcal{I}^0(\mathcal{M})) = \prod_{x \in U} \mathcal{M}_x,$$

and we have a functorial inclusion $\mathcal{M} \hookrightarrow \mathcal{I}^0(\mathcal{M})$.

Let’s now consider a filtered version. To $(\mathcal{M}, F) \in FM(f^{-1}\mathcal{D}_Y)$ we associate $\mathcal{I}^0(\mathcal{M}, F)$, namely the filtered $f^{-1}\mathcal{D}_Y$-module $(\mathcal{N}, F)$ given by

$$\mathcal{N} := \bigcup_{k \in \mathbb{Z}} \mathcal{I}^0(F_k \mathcal{M}) \subseteq \mathcal{I}^0(\mathcal{M}) \quad \text{and} \quad F_k \mathcal{N} := \mathcal{I}^0(F_k \mathcal{M}).$$

We have a filtered inclusion $(\mathcal{M}, F) \hookrightarrow \mathcal{I}^0(\mathcal{M}, F)$, and we define

$$\mathcal{I}^1(\mathcal{M}, F) := \text{Coker}(i).$$

Continuing in this fashion, we obtain a complex

$$0 \to \mathcal{I}^0(\mathcal{M}, F) \to \mathcal{I}^1(\mathcal{M}, F) \to \cdots$$

in $C^+(FM(f^{-1}\mathcal{D}_Y))$ which is filtered quasi-isomorphic to $(\mathcal{M}, F)$, and consequently a functor

$$\mathcal{I}^\bullet : FM(f^{-1}\mathcal{D}_Y) \to C^+(FM(f^{-1}\mathcal{D}_Y)).$$

Note now that in the context we are considering, basic properties of higher direct images tell us that there exists an integer $N > 0$ such that for every sheaf $\mathcal{F}$ of abelian groups on $X$ we have $R^i f_* \mathcal{F} = 0$ for $i > N$. (In the setting of algebraic varieties we can in fact take $N = \dim X$.) We modify our resolution by taking

$$\tilde\mathcal{I}^j(\mathcal{M}, F) = \mathcal{I}^j(\mathcal{M}, F) \quad \text{for} \quad j \leq N, \quad \tilde\mathcal{I}^j(\mathcal{M}, F) = 0 \quad \text{for} \quad j > N + 1,$$

and

$$\tilde\mathcal{I}^{N+1}(\mathcal{M}, F) = \text{Coker}(\mathcal{I}^{N-1}(\mathcal{M}, F) \to \mathcal{I}^N(\mathcal{M}, F)).$$

We thus obtain a finite resolution

$$\tilde\mathcal{I}^\bullet(\mathcal{M}, F) : 0 \to \tilde\mathcal{I}^0(\mathcal{M}, F) \to \tilde\mathcal{I}^1(\mathcal{M}, F) \to \cdots \to \tilde\mathcal{I}^{N+1}(\mathcal{M}, F) \to 0$$

of $(\mathcal{M}, F)$ with filtered sheaves having the same properties as those in $\mathcal{I}^\bullet(\mathcal{M}, F)$. Moreover, all the entries $\tilde\mathcal{I}^j(\mathcal{M}, F)$ are filtered $f$-acyclic in the sense that

$$R^i f_*(F_k \tilde\mathcal{I}^j(\mathcal{M}, F)) = 0 \quad \text{for all} \quad k \in \mathbb{Z}, \quad i > 0.$$

This follows by construction and the assumption on $N$ (exercise!).

We can extend this construction to complexes. If $C^\bullet = (\mathcal{M}^\bullet, F^\bullet \mathcal{M}^\bullet)$ is an object in $C(FM(f^{-1}\mathcal{D}_Y))$, we can form the double complex $(\tilde\mathcal{I}^p(\mathcal{M}^q, F^p \mathcal{M}^q))_{p,q}$ and define $\tilde\mathcal{I}^\bullet(C^\bullet)$ to be the total complex of this double complex. We thus have a functor

$$\tilde\mathcal{I}^\bullet : C(FM(f^{-1}\mathcal{D}_Y)) \to C(FM(f^{-1}\mathcal{D}_Y))$$

such that $C^\bullet$ is filtered quasi-isomorphic to $\tilde\mathcal{I}^\bullet(C^\bullet)$, with $\cdot$, $+$ and bounded versions.
Finally, this allows us to define the exact functor of triangulated categories we are interested in, as

\[ Rf_* : \mathbf{D}(\text{FM}(f^{-1} \mathcal{D}Y)) \to \mathbf{D}(\text{FM}(\mathcal{D}Y)), \quad Rf_* C^\bullet := f_*(\tilde{\mathcal{F}}^\bullet(C^\bullet)). \]

This functor is well defined thanks to the following

**Exercise 4.6.4.** Show that if \( A^\bullet \to B^\bullet \) is a filtered quasi-isomorphism in \( \mathbf{C}(\text{FM}(f^{-1} \mathcal{D}Y)) \), then the induced \( f_*(\tilde{\mathcal{F}}^\bullet(A^\bullet)) \to f_*(\tilde{\mathcal{F}}^\bullet(B^\bullet)) \) is a filtered quasi-isomorphism as well.

The following property is a direct consequence of the definition and of filtered \( f \)-acyclicity:

**Corollary 4.6.5.** If \( C^\bullet \) represents an object in \( \mathbf{D}(\text{FM}(f^{-1} \mathcal{D}Y)) \), then

\[ H^iF_k(Rf_* C^\bullet) \simeq R^i f_*(F_k C^\bullet). \]

Note that this allows us to obtain the filtration on each \( R^i f_* C^\bullet \) as follows:

\[ F_k R^i f_* C^\bullet = \text{Im}[H^iF_k(Rf_* C^\bullet) \to H^i(Rf_* C^\bullet)] = \text{Im}[R^i f_*(F_k C^\bullet) \to R^i f_* C^\bullet]. \]

It is however not necessarily the case that this last map is injective, and therefore the filtration is in general not simply given by \( R^i f_*(F_k C^\bullet) \). That this is actually true for those filtered \( \mathcal{D} \)-modules that underlie Hodge modules is a deep property of Hodge-theoretic flavor that we will discuss in the next section.

Finally, as mentioned above, composing \( Rf_* \) with \( \text{DR}_{X/Y} \), we obtain the desired filtered direct image functor

\[ f_+: \mathbf{D}(\text{FM}(\mathcal{D}X)) \to \mathbf{D}(\text{FM}(\mathcal{D}Y)). \]

If \( f \) is proper, this induces a functor

\[ f_+: \mathbf{D}^b(\text{cof}(\text{FM}(\mathcal{D}X))) \to \mathbf{D}^b(\text{cof}(\text{FM}(\mathcal{D}Y))). \]

**Remark 4.6.6.** It is immediate from the definitions that if we forget the filtration, this functor coincides with the usual direct image functor \( f_+ \) on the derived category of \( \mathcal{D} \)-modules recalled at the beginning of this section.

### 4.7. Strictness

A special property that is crucial in the theory of filtered \( \mathcal{D} \)-modules underlying Hodge modules is the strictness of the filtration.

**Definition 4.7.1.** Let \( f : (\mathcal{M}, F) \to (\mathcal{N}, F) \) be a morphism of filtered \( \mathcal{D}X \)-modules. Then \( f \) is called **strict** if

\[ f(F_k \mathcal{M}) = F_k \mathcal{N} \cap f(\mathcal{M}) \quad \text{for all } k. \]

Similarly, a complex of filtered \( \mathcal{D}X \)-modules \( (\mathcal{M}^\bullet, F_* \mathcal{M}^\bullet) \) is called strict if all of its differentials are strict. Via a standard argument, the notion of strictness makes sense more generally for objects in the derived category \( \mathbf{D}(\text{FM}(\mathcal{D}X)) \) of filtered \( \mathcal{D}X \)-modules.
Note that it is only in the case of a strict complex that the cohomologies of $\mathcal{M}^i$ can also be seen as filtered $\mathcal{D}_X$-modules. An equivalent interpretation is given by the following:

**Exercise 4.7.2.** The complex $(\mathcal{M}^i, F^i \mathcal{M}^i)$ is strict if and only if, for every $i, k \in \mathbb{Z}$, the induced morphism
\[
\mathcal{H}^i F_k \mathcal{M}^i \to \mathcal{H}^i \mathcal{M}^i
\]
is injective.

As a preview, a crucial property of the filtered $\mathcal{D}$-modules underlying Hodge modules will be the following. If $f : X \to Y$ is a proper morphism of smooth varieties, and $(\mathcal{M}, F)$ is one such filtered $\mathcal{D}_X$-module, then $f_+(\mathcal{M}, F)$ is strict as an object in $\text{D}(\text{FM}(\mathcal{D}_Y))$; here $f_+$ is the filtered direct image functor discussed in the previous section. By the Exercise above, this means that
\[
\mathcal{H}^i F_k f_+(\mathcal{M}, F) \to \mathcal{H}^i f_+(\mathcal{M}, F)
\]
is injective for all integers $i$ and $k$. By Corollary 4.6.5 and the discussion right after, this is equivalent to the injectivity of the natural morphism
\[
(4.7.1) \quad R^i f_*(F_k (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_{X \to Y})) \to R^i f_*(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_{X \to Y}).
\]
Moreover, the image of this morphism, isomorphic to the term on the left hand side, is the term $F_k \mathcal{H}^i f_+(\mathcal{M}, F)$.

In conclusion, in the strict case the cohomologies of direct images of filtered $\mathcal{D}$-modules are themselves filtered $\mathcal{D}$-modules, and it is sometimes possible to have a reasonably good grasp of the filtration on such direct images.

**Example 4.7.3 (Absolute case).** The absolute case gives a good idea of the meaning of strictness, and how it is natural in Hodge theory. In this context it can be seen as a generalization of the degeneration at $E_1$ of the classical Hodge-to-de Rham spectral sequence. Concretely, let $X$ be a smooth variety, and $(\mathcal{M}, F)$ a filtered $\mathcal{D}$-module on $X$. The natural inclusion of complexes $F_k \text{DR}(\mathcal{M}) \to \text{DR}(\mathcal{M})$ induces, after passing to cohomology, a morphism
\[
\varphi_{k,i} : H^i(X, F_k \text{DR}(\mathcal{M})) \to H^i(X, \text{DR}(\mathcal{M})).
\]
Now for the constant map $f : X \to \text{pt}$, the definition of pushforward gives
\[
f_+ \mathcal{M} \simeq \mathcal{R} \Gamma(X, \text{DR}(\mathcal{M})),
\]
and by the discussion above, the image of $\varphi_{k,i}$ is $F_k H^i(X, \text{DR}(\mathcal{M}))$. The strictness of $f_+(\mathcal{M}, F)$ is captured by the injectivity of $\varphi_{k,i}$ for all $k$ and $i$, which is in turn equivalent to
\[
gr_{F_k} H^i(X, \text{DR}(\mathcal{M})) \simeq H^i(X, \text{gr}_{F_k} \text{DR}(\mathcal{M})).
\]
It is not hard to see that this is the same as the $E_1$-degeneration of the Hodge-to-de Rham spectral sequence
\[
E_1^{p,q} = H^{p+q}(X, \text{gr}_{F_q}^p \text{DR}(\mathcal{M})) \Rightarrow H^{p+q}(X, \text{DR}(\mathcal{M})).
\]
where on the left hand side we have the hypercohomology of a complex of coherent $\mathcal{O}_X$-modules, and on the right hand side the cohomology of the perverse sheaf $\text{DR}(\mathcal{M})$. 


For instance, when $X$ is projective and $M$ is given by the trivial VHS $C_X$, then this is the degeneration of the classical Hodge-to-de Rham spectral sequence, and same for any other VHS; see Examples 1.2.2, 1.2.3 and 4.3.3. This property extends to the filtered $\mathcal{D}$-modules associated to Hodge modules on $X$.

### 4.8. Localization and Hodge ideals

We now focus on one extended example, where many of the definitions and techniques in the previous sections can be seen in action.

Throughout this section $X$ is a smooth complex variety of dimension $n$ and $D$ is a reduced effective divisor on $X$. Recall that to this data we can associate the left $\mathcal{D}_X$-module of functions with poles along $D$,

$$\mathcal{O}_X(*D) = \bigcup_{k \geq 0} \mathcal{O}_X(kD),$$

i.e. the localization of $\mathcal{O}_X$ along $D$. If $f$ is a local defining equation for $D$, then this is $\mathcal{O}_X[\frac{1}{f}]$, with the obvious action of differential operators. The associated right $\mathcal{D}_X$-module is denoted $\omega_X(*D)$.

This $\mathcal{D}$-module underlies the mixed Hodge module $j_* \mathbb{Q}^H_U \mathbb{Q}[n]$, where $U = X \setminus D$ and $j : U \hookrightarrow X$ is the inclusion map. It therefore comes with an attached Hodge filtration $F_k \mathcal{O}_X(*D)$, where $k \geq 0$. When $D$ is an SNC divisor we have already seen this in Example 4.4.9, while in general one uses resolution of singularities and the push-forward functor; see Lemma 4.8.7 below, and the discussion thereafter.

Note that there is also a more obvious filtration on $\mathcal{O}_X(*D)$, namely the pole order filtration, whose nonzero terms are taken by convention to be

$$P_k \mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D) \quad \text{for} \quad k \geq 0.$$

This filtration is in general too coarse to be of much use; it is usually not even a good filtration. We will see however that it is very useful to measure the Hodge filtration against it, and indeed we have:

**Lemma 4.8.1.** For every $k \geq 0$, we have an inclusion

$$F_k \mathcal{O}_X(*D) \subseteq P_k \mathcal{O}_X(*D).$$

It is possible to give a proof of this Lemma using the interaction between the $F$-filtration and the $V$-filtration. We will however give a proof using resolution of singularities below.

**Definition 4.8.2 (Hodge ideals).** Given the inclusion in Lemma 4.8.1, for each $k \geq 0$ we define the ideal sheaf $I_k(D)$ on $X$ by the formula

$$F_k \mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D) \otimes I_k(D).$$

We call $I_k(D)$ the $k$-th Hodge ideal of $D$. 

In what follows we will analyze these ideals (and in the process prove Lemma 4.8.1 on which their definition relies) by first looking at the SNC case and then making use of log resolutions and filtered push-forward for Hodge modules.

**Simple normal crossings case.** When \( D \) is a simple normal crossing divisor, we saw in Example 4.4.9 that \( F_k \mathcal{O}_X(*D) = F_k \mathcal{D}_X \cdot \mathcal{O}_X(D) \).

**Exercise 4.8.3.** Verify directly in this setting that indeed \( F_k \mathcal{D}_X \cdot \mathcal{O}_X(D) \subseteq \mathcal{O}_X((k+1)D) \) for all \( k \geq 0 \).

We conclude that in this case the ideals \( I_k(D) \) are given by the following expression:

\[
I_k(D) = (F_k \mathcal{D}_X \cdot \mathcal{O}_X(D)) \otimes \mathcal{O}_X(-(k+1)D) \quad \text{for all } k \geq 0, \tag{4.8.1}
\]

Note that in particular \( I_0(D) = \mathcal{O}_X \). We can in fact completely describe a set of generators for all of these ideals in local coordinates.

**Proposition 4.8.4.** Suppose that around a point \( p \in X \) we have coordinates \( x_1, \ldots, x_n \) such that \( D \) is defined by \( (x_1 \cdots x_r = 0) \). Then, for every \( k \geq 0 \), the ideal \( I_k(D) \) is generated around \( p \) by

\[
\{ x_1^{a_1} \cdots x_r^{a_r} \mid 0 \leq a_i \leq k, \sum a_i = k(r-1) \}.
\]

In particular, if \( r = 1 \) (that is, when \( D \) is smooth), we have \( I_k(D) = \mathcal{O}_X \) and if \( r = 2 \), then \( I_k(D) = (x_1, x_2)^k \).

**Proof.** It is clear that \( F_k \mathcal{D}_X \cdot \mathcal{O}_X(D) \) is generated as an \( \mathcal{O}_X \)-module by

\[
\{ x_1^{-b_1} \cdots x_r^{-b_r} \mid b_i \geq 1, \sum b_i = r+k \}.
\]

According to (4.8.1), the expression for \( I_k(D) \) now follows by multiplying these generators by \( (x_1 \cdots x_r)^{k+1} \). The assertions in the special cases \( r = 1 \) and \( r = 2 \) are clear. \( \square \)

For convenience, let’s record separately the case \( r = 1 \) in the Proposition above.

**Corollary 4.8.5.** If \( D \) is a smooth divisor, then \( I_k(D) = \mathcal{O}_X \) for all \( k \geq 0 \).

**Remark 4.8.6.** It turns out that the converse of this statement holds as well, but this requires some serious work; see [MP1, Theorem A].

The general case. When \( D \) is arbitrary, we consider a log resolution \( f: Y \to X \) of the pair \( (X, D) \) which is an isomorphism over \( U = X \setminus D \), and let \( E = (f^*D)_{\text{red}} \), an SNC divisor.

**Lemma 4.8.7.** There is a natural isomorphism

\[
f_+ \omega_Y(*E) \simeq H^0 f_+ \omega_Y(*E) \simeq \omega_X(*D).
\]

\[9\text{We will see below that one can also define Hodge ideals directly using log resolutions, without appealing to the theory of Hodge modules, in which case we would need to take this as the definition.}\]
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Proof. Let $V = Y \setminus E = f^{-1}(U)$, and denote by $j_U$ and $j_V$ the inclusions of $U$ and $V$ into $X$ and $Y$ respectively. Note that $j_{U+}\omega_U \simeq \omega_X(*D)$ and $j_{V+}\omega_V \simeq \omega_Y(*E)$. The result then follows from the fact that $j_U = f \circ j_V$. 

We therefore need to understand the isomorphism of filtered $\mathcal{D}$-modules

$$f_+ (\omega_Y(*E), F) \simeq (\omega_X(*D), F),$$

where in both cases we consider the Hodge filtration underlying the mixed Hodge module structure, and the filtered push-forward is the construction described in Section 4.6. Since $f_+ \omega_Y(*E) \simeq Rf_* (\omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \to X})$, the first step is to consider a convenient (filtered) representative for $\omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \to X}$. It turns out that this object is supported only in degree zero; even though not entirely necessarily, let’s go one step further and provide a precise description.

First, recall that since $\mathcal{D}_{Y \to X}$ is a left $\mathcal{D}_Y$-module, we have a canonical morphism of $(\mathcal{D}_Y, f^{-1}\mathcal{O}_X)$ bimodules

$$\varphi: \mathcal{D}_Y \to f^* \mathcal{D}_X$$

that maps 1 to 1, and is clearly an isomorphism over $V = Y \setminus E$. Since $\mathcal{D}_Y$ is torsion-free, we conclude that $\varphi$ is injective, with cokernel supported on $E$. For each $k$, we have induced inclusions $F_k \mathcal{D}_Y \hookrightarrow f^* F_k \mathcal{D}_X$ of $(\mathcal{O}_Y, f^{-1}\mathcal{O}_X)$ bimodules.

**Proposition 4.8.8.** The canonical morphism induced by $\varphi$ in the derived category of right $f^{-1}\mathcal{O}_X$-modules

$$\omega_Y(*E) \to \omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \to X}$$

is an isomorphism.

The proof we give below is inspired in part by arguments in [HTT, §5.2].

**Lemma 4.8.9.** The induced morphism

$$\text{Id} \otimes \varphi: \mathcal{O}_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_Y \to \mathcal{O}_Y(*E) \otimes_{\mathcal{D}_Y} f^* \mathcal{D}_X$$

is an isomorphism.

**Proof.** It suffices to show that the induced mappings

$$\mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} F_k \mathcal{D}_Y \to \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} f^* F_k \mathcal{D}_X$$

are all isomorphisms for $k \geq 0$. But this follows immediately from Lemma 4.8.10 below (note that since $\mathcal{O}_Y(*E)$ is flat over $\mathcal{O}_Y$, these maps are injective). 

In the proof above we used the following well-known observation; see [HTT, Lemma 5.2.7].

**Lemma 4.8.10.** If $\mathcal{F}$ is a coherent $\mathcal{O}_X(*D)$-module supported on $D$, then $\mathcal{F} = 0$. 

Let us now use the notation
\[ \mathcal{D}_Y(*E) := \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y. \]
This is a sheaf of rings, and one can identify it with the subalgebra of \( \text{End}_C(\mathcal{O}_Y(*E)) \) generated by \( \mathcal{D}_Y \) and \( \mathcal{O}_Y(*E) \). Note that since \( \mathcal{O}_Y(*E) \) is a flat \( \mathcal{O}_Y \)-module, we have \( \mathcal{D}_Y(*E) \cong \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \). A basic fact is the following:

**Lemma 4.8.11.** The canonical morphism
\[ \mathcal{D}_Y(*E) \rightarrow \mathcal{D}_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_Y \rightarrow \mathcal{X} \]
induced by \( \varphi \) is an isomorphism.

**Proof.** Via the isomorphism \( \mathcal{D}_Y(*E) \cong \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \), the morphism in the statement gets identified to the morphism
\begin{equation}
\mathcal{D}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow \mathcal{D}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{D}_Y} \mathcal{D}_Y \rightarrow \mathcal{D}_Y(*E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X
\end{equation}
induced by \( \varphi \). Moreover, since \( \mathcal{O}_Y(*E) \) is a flat \( \mathcal{O}_Y \)-module, the morphism (4.8.2) gets identified with the isomorphism in Lemma 4.8.9. \( \square \)

**Proof of Proposition 4.8.8.** Via the right \( \mathcal{D} \)-module structure on \( \omega_Y \), we have that \( \omega_Y(*E) \) has a natural right \( \mathcal{D}_Y(*E) \)-module structure. The morphism in the proposition gets identified with the morphism
\[ \omega_Y(*E) \otimes_{\mathcal{D}_Y(*E)} \mathcal{D}_Y(*E) \rightarrow \omega_Y(*E) \otimes_{\mathcal{D}_Y(*E)} \mathcal{D}_Y(*E) \otimes_{\mathcal{D}_Y(*E)} \mathcal{D}_Y \rightarrow \mathcal{X} \]
induced by \( \varphi \). In turn, this is obtained by applying \( \omega_Y(*E) \otimes_{\mathcal{D}_Y(*E)} \mathcal{D}_Y \) – to the isomorphism in Lemma 4.8.11, hence it is an isomorphism. \( \square \)

Next we address the filtration. On the tensor product \( \omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_Y \rightarrow \mathcal{X} \) we consider the tensor product filtration, that is,
\[ F_k(\omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_Y \rightarrow \mathcal{X}) := \text{Im} \left[ \bigoplus_{i \geq -n} F_i \omega_Y(*E) \otimes_{\mathcal{O}_Y} f^* F_{k-i} \mathcal{D}_X \rightarrow \omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_Y \rightarrow \mathcal{X} \right], \]
where the map in the parenthesis is the natural map between the tensor product over \( \mathcal{O}_Y \) and that over \( \mathcal{D}_Y \).

**Lemma 4.8.12.** The definition above simplifies to
\[ F_k(\omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_Y \rightarrow \mathcal{X}) = \text{Im} \left[ \omega_Y(E) \otimes_{\mathcal{O}_Y} f^* F_{k+n} \mathcal{D}_X \rightarrow \omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_Y \rightarrow \mathcal{X} \right]. \]

**Proof.** Fix \( i \geq -n \) and recall that \( F_i \omega_Y(*E) = \omega_Y(E) \cdot F_{i+n} \mathcal{D}_Y \). The factor \( F_{i+n} \mathcal{D}_Y \) can be moved over the tensor product once we pass to the image in the tensor product over \( \mathcal{D}_Y \), and moreover we have an inclusion
\[ F_{i+n} \mathcal{D}_Y \cdot f^* F_{k-i} \mathcal{D}_X \subseteq f^* F_{k+n} \mathcal{D}_X. \]
Therefore inside $\omega_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \rightarrow X}$, the image of $F_i \omega_Y(*E) \otimes_{\mathcal{O}_Y} f^*F_{k-n} \mathcal{D}_X$ is contained in the image of $\omega_Y(E) \otimes_{\mathcal{O}_Y} f^*F_{k+n} \mathcal{D}_X$. □

Propositions 4.5.14 and 4.8.8 have the following immediate consequence:

**Corollary 4.8.13.** On $Y$ there is a filtered complex of right $f^{-1} \mathcal{D}_X$-modules

$$0 \to f^* \mathcal{D}_X \to \Omega_Y^1(\log E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \to \cdots \to \Omega_Y^{n-1}(\log E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \to \omega_Y(E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \to 0$$

which is exact (though not necessarily filtered exact).

**Proof.** It follows from Proposition 4.5.14 that the complex

$$0 \to f^* \mathcal{D}_X \to \Omega_Y^1(\log E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \to \cdots \to \omega_Y(E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \to 0$$

represents the object $\omega_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \rightarrow X}$ in the derived category, hence Proposition 4.8.8 implies the exactness of the entire complex in the statement. □

With these preparations in place, let’s go back to understanding Hodge ideals. We denote by $A^\bullet$ the complex

$$0 \to f^* \mathcal{D}_X \to \Omega_Y^1(\log E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \to \cdots \to \omega_Y(E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \to 0$$

placed in degrees $-n, \ldots, 0$. It is filtered by the subcomplexes $C^\bullet_{k-n} = F_{k-n}A^\bullet$ of $A^\bullet$, for $k \geq 0$, given by

$$0 \to f^*F_{k-n} \mathcal{D}_X \to \Omega_Y^1(\log E) \otimes_{\mathcal{O}_Y} f^*F_{k-n+1} \mathcal{D}_X \to \cdots \to \Omega_Y^{n-1}(\log E) \otimes_{\mathcal{O}_Y} f^*F_{k-1} \mathcal{D}_X \to \omega_Y(E) \otimes_{\mathcal{O}_Y} f^*F_k \mathcal{D}_X \to 0.$$

With this filtration, it represents the object

$$\omega_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \rightarrow X} \simeq \omega_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \rightarrow X}$$

in the derived category of filtered right $f^{-1} \mathcal{D}_X$-modules. In other words, in the language of Section 4.6, we have

$$\text{DR}_{Y/X}(\omega_Y(*E), F) \simeq (A^\bullet, F).$$

**Remark 4.8.14.** Moreover, we have seen in Corollary 4.8.13 that the natural mapping

$$A^\bullet \longrightarrow H^0A^\bullet \simeq \omega_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \rightarrow X}$$

is a quasi-isomorphism.; again though, this is not necessarily a filtered quasi-isomorphism, meaning $C^\bullet_{k-n}$ are not necessarily exact away from the right-most term.

Lemma 4.8.7 combined with Corollary 4.6.5 and the subsequent discussion then gives

$$F_{k-n} \omega_X(*D) = F_{k-n} \mathcal{H}^0 f_+ \omega_Y(*E) \simeq \text{Im} \left[ R^0 f_* C^\bullet_{k-n} \rightarrow R^0 f_* A^\bullet \right],$$

where the map of $\mathcal{O}_X$-modules in the parenthesis is induced by the inclusion $C^\bullet_{k-n} \hookrightarrow A^\bullet$. With this description, we are now finally able to prove the crucial Lemma 4.8.1.
Proof of Lemma 4.8.1. Passing to right \( \mathcal{D} \)-modules, equivalently we need to show

\[ F_{k-n} \omega_X(*D) \subseteq \omega_X((k+1)D) \]

for all \( k \). Since \( D \) is reduced, we can find an open subset \( U \subseteq X \) with the property that \( \text{codim}(X \setminus U, U) \geq 2 \), the induced morphism \( f^{-1}(U) \to U \) is an isomorphism, and \( D|_U \) is a smooth (possibly disconnected) divisor. Let \( j: U \to X \) be the inclusion. By assumption, on \( f^{-1}(U) \) we have \( \mathcal{D}_{Y \to X} = \mathcal{D}_Y \). From Proposition 4.5.14, on \( U \) we obtain

\[ F_{k-n} \omega_X(*D) = \omega_X((k+1)D) \]

Now \( F_{k-n} \omega_X(*D) \) is torsion-free, being a subsheaf of \( \omega_X(*D) \), and so the following canonical map is injective:

\[ F_{k-n} \omega_X(*D) \to j_*(F_{k-n} \omega_X(*D)|_U) = j_*(\omega_X((k+1)D)|_U) = \omega_X((k+1)D) \]

This completes the proof. \( \square \)

Recall that this means that we can define a sequence of coherent sheaves of ideals \( I_k(D) \) associated to \( D \), by the formula

\[ F_{k-n} \omega_X(*D) = \omega_X((k+1)D) \otimes I_k(D). \]

Remark 4.8.15. The formula above, together with (4.8.3), can be alternatively taken as a definition of the Hodge ideals \( I_k(D) \) using log resolutions, without appealing to the theory of mixed Hodge modules. In this approach however, one has to prove that the definition is independent of the choice of log resolution, which can be done using a Nakano-type local vanishing theorem; see [MP1, Theorem 11.1].

Computations for \( k = 0, 1 \). The first in the sequence of Hodge ideals can be identified with a multiplier ideal; for the general theory of multiplier ideals see [La, Ch.9] or §2.6 above.

Proposition 4.8.16. We have

\[ I_0(D) = \mathcal{J}(X, (1-\varepsilon)D), \]

the multiplier ideal associated to the \( \mathbb{Q} \)-divisor \( (1-\varepsilon)D \) on \( X \), for any \( 0 < \varepsilon \ll 1 \).

Proof. Recall that \( C_{\bullet,n} = \omega_Y(E) \), hence

\[ F_0 \omega_X(*D) = \text{Im}(f_*(\omega_Y(E) \to \omega_X(D))) = f_* \mathcal{O}_Y(K_{Y/X} + E - f^*D) \otimes \omega_X(D). \]

Therefore the statement to be proved is that

\[ f_* \mathcal{O}_Y(K_{Y/X} + E - f^*D) = \mathcal{J}(X, (1-\varepsilon)D). \]

On the other hand, the right-hand side is by definition

\[ f_* \mathcal{O}_Y(K_{Y/X} - [(1-\varepsilon)f^*D]) = f_* \mathcal{O}_Y(K_{Y/X} + (f^*D)_{\text{red}} - f^*D), \]

which implies the desired equality. \( \square \)

The following is a direct consequence of the definition; see Exercise 2.6.13.

Corollary 4.8.17. We have \( I_0(D) = \mathcal{O}_X \) if and only if the pair \( (X, D) \) is log-canonical.
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We now move to a description of the case $k = 1$. We begin by noting the following fact about the complex $C^{\bullet}_{1-n}$.

**Lemma 4.8.18.** The morphism

$$\Omega_{Y}^{n-1}(\log E) \longrightarrow \omega_{Y}(E) \otimes_{\Theta_{Y}} f^{*}F_{1}\mathcal{D}_{X}$$

is injective.

**Proof.** Note that we have a commutative diagram

$$\begin{array}{ccc}
\Omega_{Y}^{n-1}(\log E) & \xrightarrow{\beta} & \omega_{Y}(E) \otimes_{\Theta_{Y}} F_{1}\mathcal{D}_{Y} \\
\downarrow{\text{id}} & & \downarrow{\gamma} \\
\Omega_{Y}^{n-1}(\log E) & \xrightarrow{\alpha} & \omega_{Y}(E) \otimes_{\Theta_{Y}} f^{*}F_{1}\mathcal{D}_{X},
\end{array}$$

in which $\gamma$ is the canonical inclusion. Since $\beta$ is injective by Proposition 4.5.14, it follows that $\alpha$ is injective, too. $\square$

Denoting by $\mathcal{F}_{1}$ the cokernel of the map in the Lemma above, we therefore have a short exact sequence

(4.8.4) $0 \rightarrow \Omega_{Y}^{n-1}(\log E) \rightarrow \omega_{Y}(E) \otimes f^{*}F_{1}\mathcal{D}_{X} \rightarrow \mathcal{F}_{1} \rightarrow 0,$

hence in particular $\mathcal{F}_{1}$ is quasi-isomorphic to the complex $C^{\bullet}_{1-n}$. We have:

**Corollary 4.8.19.** We have an isomorphism

$$\omega_{X}(2D) \otimes I_{1}(D) \simeq f_{*}\mathcal{F}_{1}.$$ 

**Proof.** It is clear by definition and the exact sequence above that

$$R^{0}f_{*}C^{\bullet}_{1-n} \simeq f_{*}\mathcal{F}_{1}.$$ 

By definition we therefore have that $\omega_{X}(2D) \otimes I_{1}(D)$ is isomorphic to the image of the canonical morphism $f_{*}\mathcal{F}_{1} \rightarrow \omega_{X}(\ast D)$. However it is not hard to see that this morphism is injective; in this case this can be seen directly (exercise!), but note that it is in fact a special case of the general Corollary 4.8.22(1) below. $\square$

**Corollary 4.8.20.** There is a four-term exact sequence

$$0 \rightarrow f_{*}\Omega_{Y}^{1}(\log E) \rightarrow \omega_{X}(D) \otimes I_{0}(D) \otimes F_{1}\mathcal{D}_{X} \xrightarrow{\psi} \omega_{X}(2D) \otimes I_{1}(D) \rightarrow R^{1}f_{*}\Omega_{Y}(\log E) \rightarrow 0,$$

where the image of the map $\psi$ corresponds to the image of the natural filtered $\mathcal{D}_{X}$-module map

$$F_{-n}\omega_{X}(\ast D) : F_{1}\mathcal{D}_{X} \subseteq F_{1-n}\omega_{X}(\ast D).$$

**Proof.** The exact sequence is obtained by pushing forward the short exact sequence (4.8.4). The zero on the right comes from applying the projection formula, and using the fact that

$$R^{i}f_{*}\omega_{Y}(E) = 0 \quad \text{for all} \quad i > 0.$$ 

This is the well-known Local Vanishing theorem for multiplier ideals, but see also Corollary 4.8.22(2) below and the Remark thereafter. $\square$
Remark 4.8.21. The Corollary above shows the main obstruction to having an easy calculation of $I_1(D)$ once we have good understanding of the multiplier ideal $I_0(D)$, namely the appearance of the higher direct image $R^if_*\Omega_Y^{n-1}(\log E)$. In general this sheaf does not vanish, and is not straightforward to compute. Similar phenomena occur for the higher Hodge ideals $I_k(D)$.

Applications of strictness. Recall the definition of the Hodge filtration on $\omega_X(*)$ as in (4.8.3). This is a filtration on the push-forward of a mixed Hodge module, and is therefore strict. This has useful consequences:

Corollary 4.8.22. The following hold for every $k \geq 0$:

1. The map
   $$R^0 f_* C_{k-n}^* \to R^0 f_* A^* = \omega_X(*)$$
   is injective, and therefore
   $$\omega_X((k+1)) \otimes I_k(D) \simeq R^0 f_* C_{k-n}^*.$$

2. (Local vanishing for $I_k(D)$.) We have
   $$R^if_* C_{k-n}^* = 0 \quad \text{for} \quad i \neq 0.$$

Proof. We have $R^if_* A^* = 0$ for all $i \neq 0$ by Lemma 4.8.7 and Proposition 4.8.8. On the other hand, the strictness property (4.7.1) of the Hodge filtration implies that $R^if_* C_{k-n}^*$ injects in $R^if_* A^*$.

Remark 4.8.23. The case $k = 0$ in part (2) of the Corollary is the well-known Local Vanishing for multiplier ideals (see [La, Theorem 9.4.1]), a consequence of the Kawamata-Viehweg vanishing theorem; in this case $C_{-n}^* = \omega_Y(E)$.

Chain of inclusions. From the $\mathcal{D}$-module filtration property
$$F_1 \mathcal{D}_X \cdot F_{k-1} \mathcal{O}_X(*) \subseteq F_k \mathcal{O}_X(*)$$
it follows immediately that
$$I_{k-1}(D) \cdot \mathcal{O}_X(-D) \subseteq I_k(D)$$
for each $k \geq 1$. However, the Hodge ideals also satisfy another natural but more subtle sequence of inclusions.

Proposition 4.8.24. For every reduced effective divisor $D$ on the smooth variety $X$, and for every $k \geq 1$, we have
$$I_k(D) \subseteq I_{k-1}(D).$$

Proof. Consider the canonical inclusion
$$\iota: \mathcal{O}_X \hookrightarrow \mathcal{O}_X(*)$$

of filtered left $\mathcal{D}_X$-modules that underlie mixed Hodge modules. Since the category $\text{MHM}(X)$ of mixed Hodge modules on $X$ is abelian, the cokernel $\mathcal{M}$ of $\iota$ underlies a mixed Hodge module on $X$ too, and it is clear that $\mathcal{M}$ has support $D$. Since morphisms
between Hodge $\mathcal{D}$-modules preserve the filtrations and are strict, for each $k \geq 0$ we have a short exact sequence

$$0 \rightarrow F_k \mathcal{O}_X \rightarrow F_k \mathcal{O}_X(*D) \rightarrow F_k \mathcal{M} \rightarrow 0.$$ 

Recall now that $F_k \mathcal{O}_X = \mathcal{O}_X$ for all $k \geq 0$. On the other hand, if $f$ is a local equation of $D$, then by Lemma 4.1.7 we have

$$f \cdot F_k \mathcal{M} \subseteq F_{k-1} \mathcal{M}.$$ 

It then follows that $f \cdot F_k \mathcal{O}_X(*D) \subseteq F_{k-1} \mathcal{O}_X(*D)$ as well, which implies the assertion in the theorem by definition of Hodge ideals. \qed
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