Green’s Theorem

1. Let $C$ be the boundary of the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$, oriented counterclockwise, and let $\vec{F}$ be the vector field $\vec{F}(x,y) = (e^y + x, x^2 - y)$. Find $\int_C \vec{F} \cdot d\vec{r}$.

**Solution.** Let’s write $P(x,y) = e^y + x$ and $Q(x,y) = x^2 - y$, so that $\vec{F} = \langle P, Q \rangle$. Let $\mathcal{R}$ be the region $0 \leq x \leq 1, 0 \leq y \leq 1$. The boundary of $\mathcal{R}$, oriented “correctly” (so that a penguin walking along it keeps $\mathcal{R}$ on his left), is the given curve $C$. So, Green’s Theorem says that $\int_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{R}} (Q_x - P_y) \, dA = \iint_{\mathcal{R}} (2x - e^y) \, dA$. We compute this by converting it to an iterated integral:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{R}} (2x - e^y) \, dA$$

$$= \int_0^1 \int_0^1 (2x - e^y) \, dx \, dy$$

$$= \int_0^1 \left( x^2 - xe^y \bigg|_{x=0}^{x=1} \right) \, dy$$

$$= \int_0^1 (1 - e^y) \, dy$$

$$= y - e^y \bigg|_{y=0}^{y=1}$$

$$= 2 - e$$

2. Let $C$ be the oriented curve consisting of line segments from $(0,0)$ to $(2,3)$ to $(2,0)$ back to $(0,0)$, and let $\vec{F}(x,y) = (y^2, x^2)$. Find $\int_C \vec{F} \cdot d\vec{r}$.

**Solution.** Here is a picture of the curve $C$, along with the interior of the triangle, which we’ll call $\mathcal{R}$:

The boundary of $\mathcal{R}$, oriented “correctly” (so that a penguin walking along it keeps $\mathcal{R}$ on his left side), is $-C$ (that is, it’s $C$ with the opposite orientation). So, Green’s Theorem says that $\int_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{R}} (Q_x - P_y) \, dA$, where $\vec{F} = \langle P, Q \rangle$. We are looking for $\int_C \vec{F} \cdot d\vec{r}$, which we know is the negative of
\[ \int_{-C} F \cdot d\vec{r}. \] Therefore,

\[
\int_{C} F \cdot d\vec{r} = -\iint_{R} (Q_{x} - P_{y}) \, dA
\]

\[
= -\int_{R} (2x - 2y) \, dA
\]

\[
= -\int_{0}^{2} \int_{0}^{3x/2} (2x - 2y) \, dy \, dx
\]

\[
= -\int_{0}^{2} \left( 2xy - y^2 \bigg|_{y=0}^{y=3x/2} \right) \, dx
\]

\[
= -\int_{0}^{2} \frac{3x^2}{4} \, dx
\]

\[
= -\left( \frac{1}{4}x^3 \bigg|_{x=0}^{x=2} \right)
\]

\[
= -2
\]

3. Find the area of the region enclosed by the parameterized curve \( \vec{r}(t) = (t - t^2, t^3) \), \( 0 \leq t \leq 1 \).

Solution. Let \( R \) be the region in question. We know from #2(a) on the worksheet “Double Integrals” that the area of \( R \) is \( \iint_{R} 1 \, dA \). Normally, we would evaluate this by converting it to an iterated integral; in this case, that’s quite challenging, so we’ll instead use Green’s Theorem to evaluate this integral. If we can come up with a vector field \( \vec{F}(x,y) = (P(x,y), Q(x,y)) \) such that \( Q_{x} - P_{y} = 1 \), then Green’s Theorem will say that \( \iint_{R} 1 \, dA = \int_{C} \vec{F} \cdot d\vec{r} \), where \( C \) is the boundary of the region, traveled counterclockwise (so that a penguin walking along \( C \) keeps \( R \) on his left). One such vector field is \( \vec{F}(x,y) = (0, x) \).

We are given a parameterization \( \vec{r}(t) \) of the curve, and this parameterization does in fact travel the
\[ \int \int_R 1 \, dA = \int_C \vec{F} \cdot d\vec{r} \]
\[ = \int_{t=0}^{1} \langle 0, t - t^2 \rangle \cdot \langle 1 - 2t, 1 - 3t^2 \rangle \, dt \]
\[ = \int_{t=0}^{1} (t - t^2)(1 - 3t^2) \, dt \]
\[ = \int_{t=0}^{1} (t - t^2 - 3t^3 + 3t^4) \, dt \]
\[ = \frac{1}{2} t^2 - \frac{1}{3} t^3 - \frac{3}{4} t^4 + \frac{3}{5} t^5 \bigg|_{t=0}^{t=1} \]
\[ = \frac{1}{60} \]

4. Let \( \vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle \) be any vector field defined on the region \( \mathcal{R} \) (in \( \mathbb{R}^2 \)) shown in the picture, and let \( C_1 \) and \( C_2 \) be the oriented curves shown in the picture. What does Green’s Theorem say about \( \int_{C_1} \vec{F} \cdot d\vec{r} \), \( \int_{C_2} \vec{F} \cdot d\vec{r} \), and \( \int \int_{\mathcal{R}} (Q_x - P_y) \, dA \)?

\[ \text{Solution.} \] The boundary of \( \mathcal{R} \) consists of two curves, \( C_1 \) and \( C_2 \). A penguin walking along \( C_1 \) in the indicated direction would indeed keep \( \mathcal{R} \) on his left, but a penguin walking along \( C_2 \) in the indicated direction would have \( \mathcal{R} \) on his right. So, the boundary of \( \mathcal{R} \) is really \( C_1 \) together with \( -C_2 \), which means
\[ \int \int_{\mathcal{R}} (Q_x - P_y) \, dA = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} . \]

5. Let \( \vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) \). You can check that \( P_y = Q_x \).

(a) **What is wrong with the following reasoning?** “\( P_y = Q_x \), so \( \vec{F} \) is conservative.”

**Solution.** \( \vec{F} \) is not defined at the origin, so its domain is \( \mathbb{R}^2 \) except the point \((0, 0)\). This domain is not simply connected, so we cannot conclude anything from the fact that \( P_y = Q_x \).

(b) **Let \( C \) be any simple closed curve in \( \mathbb{R}^2 \) that does not enclose the origin, oriented counterclockwise.**

\( ^{(1)} \) This is not completely obvious, but there’s an easy way to tell at the end whether the parameterization went the right way --- we are looking for an area, so our final answer must be positive.
(A simple curve is a curve that does not cross itself.) Use Green’s Theorem to explain why
\[ \int_C \vec{F} \cdot d\vec{r} = 0. \]

**Solution.** Since \( C \) does not go around the origin, \( \vec{F} \) is defined on the interior \( \mathcal{R} \) of \( C \). (The only point where \( \vec{F} \) is not defined is the origin, but that’s not in \( \mathcal{R} \).) Therefore, we can use Green’s Theorem, which says
\[ \int_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{R}} (Q_x - P_y) \, dA. \]
Since \( Q_x - P_y = 0 \), this says that \( \int_C \vec{F} \cdot d\vec{r} = 0 \).

(c) Let \( a \) be a positive constant, and let \( C \) be the circle \( x^2 + y^2 = a^2 \), oriented counterclockwise. Parameterize \( C \) (check your parameterization by plugging it into the equation \( x^2 + y^2 = a^2 \)), and use the definition of the line integral to show that \( \int_C \vec{F} \cdot d\vec{r} = 0 \). (Why doesn’t the reasoning from (b) work in this case?)

**Solution.** One possible parameterization of \( C \) is \( \vec{r}(t) = (a \cos t, a \sin t) \), \( 0 \leq t \leq 2\pi \). Then,
\[ \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \]
\[ = \int_0^{2\pi} \left( \frac{a \cos t}{\sqrt{(a \cos t)^2 + (a \sin t)^2}} \cdot \frac{a \sin t}{\sqrt{(a \cos t)^2 + (a \sin t)^2}} \right) \cdot (-a \sin t, a \cos t) \, dt \]
\[ = \int_0^{2\pi} 0 \, dt \]
\[ = 0, \]
as we wanted.
We cannot use the reasoning from (b) since \( \vec{F} \) is not defined in the whole interior of \( C \) (in particular, it’s not defined at the origin, which is inside \( C \)).

(d) Let \( C \) be any simple closed curve in \( \mathbb{R}^2 \) that does enclose the origin, oriented counterclockwise. Explain why \( \int_C \vec{F} \cdot d\vec{r} = 0 \). (Hint: Use (c) and #4.)

**Solution.** No matter what \( C \) looks like, we can draw a giant circle \( x^2 + y^2 = a^2 \) around the origin that encloses all of \( C \). Let’s orient this giant circle counterclockwise and call it \( C' \), and let’s have \( \mathcal{R} \) be the region between \( C \) and \( C' \): \( \mathcal{R} \)

Notice that \( \vec{F} \) is defined on all of \( \mathcal{R} \) (because it is defined everywhere except the origin, and \( \mathcal{R} \)
doesn’t include the origin). So, #4 tells us that
\[
\int \int_{\mathcal{R}} (Q_x - P_y) \, dA = \int_{C'} \vec{F} \cdot d\vec{r} - \int_{C} \vec{F} \cdot d\vec{r}.
\]
We showed in (c) that \(\int_{C'} \vec{F} \cdot d\vec{r} = 0\), so this simplifies to
\[
\int \int_{\mathcal{R}} (Q_x - P_y) \, dA = -\int_{C} \vec{F} \cdot d\vec{r}.
\]
Since \(Q_x = P_y\) inside of \(\mathcal{R}\), the double integral is really a double integral of 0, so it’s equal to 0. Therefore, we conclude that \(\int_{C} \vec{F} \cdot d\vec{r} = 0\) as well.

(c) Is it valid to conclude from the above reasoning that, if \(\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle\) is a vector field defined everywhere except the origin and \(P_y = Q_x\), then \(\vec{F}\) is conservative?

Solution. No! The calculation in (c) only applied to this particular vector field \(\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle x/\sqrt{x^2+y^2}, y/\sqrt{x^2+y^2} \rangle\).

There are vector fields that are defined everywhere except the origin and satisfy \(P_y = Q_x\) but are still not conservative; the vector field in #4(b) of the worksheet “The Fundamental Theorem for Line Integrals; Gradient Vector Fields” is an example.

6. In this problem, you’ll prove Green’s Theorem in the case where the region is a rectangle. Let \(\vec{F}(x,y) = (P(x,y), Q(x,y))\) be a vector field on the rectangle \(\mathcal{R} = [a,b] \times [c,d]\) in \(\mathbb{R}^2\).

(a) Show that \(\int \int_{\mathcal{R}} \left[ Q_x(x,y) - P_y(x,y) \right] \, dA = \int_{c}^{d} \left[ Q(b,y) - Q(a,y) \right] \, dy - \int_{a}^{b} \left[ P(x,d) - P(x,c) \right] \, dx\).

Solution. Let’s first break the given double integral into a difference of two double integrals:
\[
\int \int_{\mathcal{R}} \left[ Q_x(x,y) - P_y(x,y) \right] \, dA = \int \int_{\mathcal{R}} Q_x(x,y) \, dA - \int \int_{\mathcal{R}} P_y(x,y) \, dA.
\]
Now, we’ll convert the double integrals on the right side to iterated integrals. This is easy, since the region \(\mathcal{R}\) is just a rectangle. However, we’re going to do the two iterated integrals in different orders: it makes sense to first integrate \(Q_x\) with respect to \(x\) (since it’s a derivative with respect to \(x\)) and to first integrate \(P_y\) with respect to \(y\):
\[
\int \int_{\mathcal{R}} \left[ Q_x(x,y) - P_y(x,y) \right] \, dA = \int_{c}^{d} \int_{a}^{b} Q_x(x,y) \, dx \, dy - \int_{c}^{b} \int_{a}^{d} P_y(x,y) \, dy \, dx.
\]
When we integrate \(Q_x\) with respect to \(x\), we just get \(Q\); similarly, when we integrate \(P_y\) with respect to \(y\), we just get \(P\):
\[
\int \int_{\mathcal{R}} \left[ Q_x(x,y) - P_y(x,y) \right] \, dA = \int_{c}^{d} \left( Q(x,y) \bigg|_{x=b}^{x=a} \right) \, dx - \int_{a}^{b} \left( P(x,y) \bigg|_{y=d}^{y=c} \right) \, dx
\]
\[
= \int_{c}^{d} \left( Q(b,y) - Q(a,y) \right) \, dy - \int_{a}^{b} \left[ P(x,d) - P(x,c) \right] \, dx,
\]
which is exactly what we were asked to show.
(b) Let $C$ be the boundary of $\mathcal{R}$, traversed counterclockwise. Show that $\int_C \vec{F} \cdot d\vec{r}$ is also equal to

$$\int_c^d [Q(b, y) - Q(a, y)] \, dy - \int_a^b [P(x, d) - P(x, c)] \, dx.$$ 

Solution. Here is a picture of $C$:

As we can see, it’s composed of 4 pieces, and we’ll parameterize each separately. The bottom piece has $y = c$, so only $x$ varies, and we can parameterize it using $\vec{r}_1(t) = \langle t, c \rangle$ with $a \leq t \leq b$. The right piece has $x = b$, so only $y$ varies, and we can parameterize it using $\vec{r}_2(t) = \langle b, t \rangle$, $c \leq t \leq d$.

The top piece has $y = d$, so only $x$ varies, and we’d like to parameterize it using $\vec{r}_3(t) = \langle t, d \rangle$. The slight problem with this is that it goes the wrong direction: as $t$ increases, $\langle t, d \rangle$ goes to the right. This is actually not a problem, as long as we account for it later. So, we’ll go ahead and use $\vec{r}_3(t) = \langle t, d \rangle$ with $a \leq t \leq b$. Similarly, for the left piece, we’ll use $\vec{r}_4(t) = \langle a, t \rangle$, $c \leq t \leq d$.

Here’s a diagram showing the various things we’ve parameterized:

As we can see from the two diagrams,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\vec{r}_1(t)} \vec{F} \cdot d\vec{r} + \int_{\vec{r}_2(t)} \vec{F} \cdot d\vec{r} - \int_{\vec{r}_3(t)} \vec{F} \cdot d\vec{r} - \int_{\vec{r}_4(t)} \vec{F} \cdot d\vec{r}.$$ 

Plugging the four parameterizations into this, $\int_C \vec{F} \cdot d\vec{r}$ is equal to

$$\int_a^b \vec{F}(t, c) \cdot \langle 1, 0 \rangle \, dt + \int_c^d \vec{F}(b, t) \cdot \langle 0, 1 \rangle \, dt - \int_a^b \vec{F}(t, d) \cdot \langle 1, 0 \rangle \, dt - \int_c^d \vec{F}(a, t) \cdot \langle 0, 1 \rangle \, dt.$$ 

Writing $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$, we can simplify this to

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b P(t, c) \, dt + \int_c^d Q(b, t) \, dt - \int_a^b P(t, d) \, dt - \int_c^d Q(a, t) \, dt.$$ 

This is exactly what we were supposed to show, which is more obvious if we rename $t$ to be $x$ in the first and third integrals, rename $t$ to be $y$ in the second and fourth integrals, and rearrange
the terms:

$$\int_{C} \vec{F} \cdot d\vec{r} = \int_{a}^{b} P(x, c) \, dx + \int_{c}^{d} Q(b, y) \, dy - \int_{a}^{b} P(x, d) \, dx - \int_{c}^{d} Q(a, y) \, dy$$

$$= \int_{c}^{d} Q(b, y) \, dy - \int_{c}^{d} Q(a, y) \, dy - \int_{a}^{b} P(x, d) \, dx + \int_{a}^{b} P(x, c) \, dx$$

$$= \int_{c}^{d} [Q(b, y) - Q(a, y)] \, dy - \int_{a}^{b} [P(x, d) - P(x, c)] \, dx$$