Math 261y: von Neumann Algebras (Lecture 34)

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In this lecture, we will study how Tomita-Takesaki theory plays out in some examples, mostly omitting proofs.

Let us first recall the basic setup. Let A be a von Neumann algebra and V a representation of A equipped with a cyclic and separating vector $v \in V$. Then the closure of the unbounded operator $xv \mapsto x^*v$ admits a polar decomposition $J\Delta^{1/2}$. The main theorem asserts that conjugation by J exchanges A with its commutant A'.

We may assume without loss of generality that $v \in V$ is a unit vector, so that v determines an ultraweakly continuous state $\phi : A \to \mathbf{C}$ by the formula $\phi(x) = (xv, v)$. Since $v \in V$ is cyclic, we can recover the pair (V, v) canonically from the state ϕ. Moreover, the condition that v is separating is equivalent to the requirement that ϕ be *faithful*: that is, if x is positive and $\phi(x) = 0$, then $x = 0$.

Example 1. Let (X, Σ, Σ_0) be a measurable space, and assume that $L^\infty(X)$ is a von Neumann algebra (we have seen that this is always the case if (X, Σ) admits a finite measure having Σ_0 as the collection of sets of measure zero). We can identify ultraweakly continuous states on $L^\infty(X)$ with probability measures on X having the property that $\mu(Y) = 0$ for $Y \in \Sigma_0$. Such a state is faithful if and only if the converse holds: that is, if $\mu(Y) \neq 0$ for $Y \notin \Sigma_0$. Given such a choice of measure μ, we can identify the corresponding representation with $L^2(X, \mu)$. Since μ is finite, we can regard $L^\infty(X)$ as a subspace of $L^2(X, \mu)$. The operator S in this case is actually bounded, and carries each function f to its complex conjugate. Since S is antiunitary we can identify S with J. Conjugation by J induces a map from A to itself, also given by $f \mapsto \overline{f}$.

If μ and μ' are two probability measures which both define faithful traces, then the Radon-Nikodym theorem implies that $\mu = \lambda\mu'$ for a unique measurable function $\lambda : X \to \mathbb{R}_{>0}$. Then multiplication by $\lambda^{1/2}$ gives an isometric isomorphism from $L^2(X, \mu)$ to $L^2(X, \mu')$. By means of these isomorphisms, we see that the representation $L^2(X, \mu)$ is canonically independent of the choice of μ. This Hilbert space is often referred to as the space of *half measures* on X.

Example 2. Let $A = B(W)$ for some Hilbert space W. We have seen that A admits a faithful ultraweakly continuous state if and only if W is separable. Let us now assume that W has countable dimension, and choose an orthonormal basis $e_1, e_2, \ldots, \in W$. Then an element $x \in A$ is determined by its matrix coefficients $x_{ij} = (xe_j, e_i)$.

Choose a sequence of nonnegative real numbers $\lambda_1, \lambda_2, \ldots$ with $\sum \lambda_i = 1$. Then the diagonal matrix with diagonal entries λ_i is a positive trace class operator y on W, so that the functional

$$x \mapsto \mathrm{tr}(xy) = \sum \lambda_i x_{i,i}$$

is a positive ultraweakly continuous state ϕ on A. Note that

$$\phi(xx^*) = \sum_{i,j} \lambda_i ||x_{i,j}||^2.$$

If each λ_i is positive, then ϕ is faithful. Let us henceforth assume this.

Let $V = \bigoplus_{i \geq 1} W$ be a direct sum of countably many copies of W. Then V has an orthonormal basis $e_{i,j}$, where $e_{i,j}$ denotes the image of e_i in the jth copy of W. Set $v = \sum \lambda_i^{1/2} e_{i,i}$. Then v is a vector of V satisfying satisfying $\phi(x) = (xv, v)$. It is not hard to see that v is a cyclic and separating vector for V.

If $x \in B(W)$, we have

$$xv = (\sum_i \lambda_1^{1/2} x_{i1} e_i, \sum_i \lambda^{1/2} 2 x_{i2} e_i, \ldots) = \sum_{i,j} \lambda_j^{1/2} x_{i,j} e_{i,j}.$$

Similarly,

$$x^* v = \sum_{i,j} \lambda_i^{1/2} \overline{x}_{i,j} e_{j,i}.$$

It follows that the unbounded operator S satisfies

$$S(\mu e_{i,j}) = \overline{\mu} \frac{\lambda_i^{1/2}}{\lambda_j^{1/2}} e_{j,i}$$

for each complex scalar μ. Writing the polar decomposition $S = J\Delta^{1/2} = \Delta^{-1/2} J$, we have

$$J(\sum \mu_{i,j} e_{i,j}) = \sum \overline{\mu}_{i,j} e_{j,i}$$

$$\Delta^{-1/2}(\sum \mu_{i,j} e_{i,j}) = \sum \frac{\lambda_i^{1/2}}{\lambda_j^{1/2}} \mu_{i,j} e_{i,j}.$$

There is a map from V to $B(W)$, which carries $\sum \mu_{i,j} e_{i,j}$ to the operator x with matrix coefficients $\mu_{i,j}$. This map is injective, and identifies W with the two-sided $*$-ideal of $B(W)$ consisting of Hilbert-Schmidt operators. Let us denote this ideal by I. We note that the action of A on V corresponds to the action of A on I by left multiplication, and that the operator J on V is given by the map $x \mapsto x^*$ on I. Conjugation by J therefore exchanges the left action of A on I with the right action of A on I. We also note that the modular operator $\Delta^{1/2}$ determines a 1-parameter group of unitary operators Δ^{it}, given by

$$\Delta^{it}(\sum \mu_{j,k} e^{j,k}) = \sum e^{it(\log \lambda_k - \log \lambda_j)} \mu_{j,k} e_{j,k}.$$

Translating to the ideal I, we see that Δ^{it} is given by conjugation by the unitary operator $e_j \mapsto e^{it \log \lambda_j} = \lambda_j^{it} e_j$. From this description, it is clear that conjugation by Δ^{it} preserves A (as a space of operators on I).

The analysis of Example 2 suggests that things would be particularly simple if we could take all of the scalars λ_i to be equal to 1: that is, if we could take ϕ to be given by

$$\phi(x) = \mathrm{tr}(x) = \sum (xe_i, e_i).$$

Unforunately, this operator is not bounded (unless W is finite-dimensional). It is therefore convenient to allow a more general notion of "state" which permits this sort of unbounded behavior.

Definition 3. Let A be a von Neumann algebra and let A_+ be the set of positive elements of A. A *weight* on A is a function $\phi : A_+ \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying the following conditions:

(*a*) $\phi(0) = 0$

(*b*) $\phi(x + y) = \phi(x) + \phi(y)$

(*c*) $\phi(\lambda x) = \lambda \phi(x)$ for $\lambda \in \mathbb{R}_{\geq 0}$ (with the conventiion that $0\infty = 0$).

A weight ϕ said to be *normal* if it is lower semi-continuous with respect to the ultraweak topology on A_+. We say that ϕ is *faithful* if $\phi(x) = 0$ implies that $x = 0$, and *semi-finite* if, whenever $\phi(x) = \infty$ and $t < \infty$, we can find $y \leq x$ with $t \leq \phi(y) < \infty$.

Example 4. If ϕ is a state on A, then the restriction $\phi|A_+$ is a weight on A. This weight is normal if and only if ϕ is ultraweakly continuous.

Construction 5. If ϕ is a weight on a von Neumann algebra A, we define $\mathfrak{n}_\phi$ to be the subset of A consisting of those elements x such that $\phi(x^*x) < \infty$. Note that this is a left ideal of A: if $x \in \mathfrak{n}_\phi$ and $y \in A$, we have

$$\phi(x^*y^*yx) \leq \phi(x^*||y||^2x) = ||y||^2\phi(x^*x) < \infty$$

so that $yx \in \mathfrak{n}_\phi$.

We can equip $\mathfrak{n}_\phi$ with an inner product with associated quadratic form given by $(x, x) = \phi(x^*x)$. We denote the Hilbert space completion of $\mathfrak{n}_\phi$ by V_ϕ. The left action of A on $\mathfrak{n}_\phi$ extends to a representation of A on V_ϕ. This action is ultraweakly continuous if ϕ is normal.

Representations of A having the form V_ϕ are called *semicyclic representations*. Suppose that ϕ is normal, faithful, and semi-finite. Let j denote the canonical map from $\mathfrak{n}_\phi$ to V_ϕ (since ϕ is faithful, this map is injective). We can then define an unbounded operator S_0 on V_ϕ by the formula

$$S_0(j(x)) = j(x^*)$$

for $x \in \mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*$. The main results of Tomita-Takesaki theory extend to this setting: S_0 is closable, its closure S has a polar decomposition $S = J\Delta^{1/2}$, conjugation by J exchanges A with its commutant, and so forth.

Remark 6. One can show that every von Neumann algebra A admits a faithful semifinite normal weight ϕ. The associated semicyclic representation is denoted by $L^2(A)$. Our earlier arguments can be generalized to show that $L^2(A)$ is canonically independent of the choice of ϕ.

Example 7. Let $A = L^\infty(X)$ be an abelian von Neumann algebra, for some measurable space (X, Σ, Σ_0). There is a one-to-one correspondence between semifinite normal weights on A and semi-finite measures μ on (X, Σ) with $\mu(Y) = 0$ for $Y \in \Sigma_0$. The corresponding weight is faithful if and only if $\mu(Y) \neq 0$ for $Y \notin \Sigma_0$.

Example 8. Let W be an arbitrary Hilbert space, let $B^{\mathrm{tc}}(W)$ be the collection of trace-class operators on W. Then there is a faithful semi-finite normal weight $\phi : B(W)_+ \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, given by $\phi(x) = \begin{cases} \mathrm{tr}(x) & \text{if } x \in B^{\mathrm{tc}}(W) \\ \infty & \text{otherwise.} \end{cases}$ The associated semicyclic representation of A can be identified with the space of Hilbert-Schmidt operators on W. In this case, Δ is the identity, and J is given by the construction $x \mapsto x^*$.