Math 155 (Lecture 12)

September 28, 2011

In this lecture we will work out some more examples of cycle indices and applications of Polya's theorem.

Question 1. Up to rotational symmetry, how many ways can we color the faces of a cube using a set of colors $T = \{y_1, y_2, \ldots, y_t\}$?

Let G be the group of rotational symmetries of the cube, and let G act on the set X of faces of the cube. We first determine the cycle index Z_G. Note that G has exactly 24 elements: it permutes the six faces of the cube transitively, and the subgroup that fixes a face is cyclic of order 4. In fact, the group G is isomorphic to the symmetric group Σ_4. The isomorphism can be given by looking at the action of G on the set of diagonals of the cube.

Let us now study the different kinds of elements $g \in G$. Recall that every non-identity rotation of 3-dimensional space is given by a rotation around some axis. There are several axes of symmetry to consider.

(a) The element $g \in G$ could be the identity. This has 6 fixed points on X, so the cycle monomial is given by $Z_g = s_1^6$.

(b) The element $g \in G$ could be a rotation around the axis passing through a vertex of the cube. There are four such axes, and there are two non-identity rotational symmetries around each axis, for a total of eight elements in all. In terms of the isomorphism $G \simeq \Sigma_4$, these elements correspond to permutations with cycle structure (123)(4). On the set of faces of the cube, they have two orbits, each of size 3: the cycle monomial is s_3^2.

(c) The element $g \in G$ could be a rotation around the axis passing through the center of an edge. There are six such axes (one for each edge, but opposite edges give the same axis). Under the isomorphism $G \simeq \Sigma_4$, these correspond to permutations with cycle structure (12)(3)(4). On the set of faces, these have three orbits, each of size 2. The cycle monomial is therefore given by s_2^3.

(d) The element $g \in G$ could be a rotation of 180° through around the center of some face. There are three such rotations (half the number of faces), and under the isomorphism $G \simeq \Sigma_4$ these correspond to permutations with cycle structure (12)(34). On the set of faces, such a rotation has two fixed points (the faces meeting the axis of rotation) and two other orbits of size 2, so the cycle monomial is $s_1^2 s_2^2$.

(e) The element $g \in G$ could be a rotation of 90° (in either direction) through the center of some face. There are six such rotations (three choices for the axis of rotation, and two choices for the direction of the rotation). Under the isomorphism with Σ_4, these correspond to the cyclic permutations. On the set of faces, there are two fixed points (the faces meeting the axis of rotation) and one orbit of size 4. The cycle monomial is therefore $s_1^2 s_4$.

This is an exhaustive list, since $1 + 8 + 6 + 3 + 6 = 24$. We can therefore write the cycle index as

$$Z_G(s_1, s_2, \ldots) = \frac{1}{24}(s_1^6 + 8s_3^2 + 6s_2^3 + 3s_1^2 s_2^2 + 6s_1^2 s_4) = \frac{s_1^6}{24} + \frac{s_3^2}{3} + \frac{s_2^3}{4} + \frac{s_1^2 s_2^2}{8} + \frac{s_1^2 s_4}{4}.$$

We can now answer Question 1. Using the simple version of Polya's theorem, we see that the total number of colorings is given by

$$Z_G(t,t,t,\ldots) = \frac{1}{24}(t^6 + 3t^4 + 12t^3 + 8t^2).$$

For example, if we allow two colors, the number of colorings is $\frac{1}{24}(64 + 48 + 96 + 32) = 10$.

Using the version of Polya's theorem we proved in the last lecture, we can obtain more refined information. For example, suppose there are six colors, and we ask in how many ways we can color the cube using each color exactly once. To answer this, we evaluate the cycle index $Z_G(s_1, s_2, \ldots)$ on the power sums

$$s_i = Y_1^i + Y_2^i + \cdots + Y_6^i,$$

and extract the coefficient of $Y_1 Y_2 \cdots Y_6$. Most of terms in the cycle index do not contribute: we have a contribution only from $\frac{s_1^6}{24}$. This contribution is $\frac{6!}{24} = 30$, for a total of 30 such colorings. Of course, this number is easy to extract directly. The group G acts freely on the collection of all colorings where each color is used exactly once, and the number of such colorings is the number of ways to order the set of faces: namely, 6!. It follows that the number of colorings up to symmetry is given by $\frac{6!}{|G|} = 30$.

For one more example, let us suppose that we are given exactly two colors. Then the relevant expression is given by

$$\frac{(Y_1+Y_2)^6}{24} + \frac{(Y_1^3+Y_2^3)^2}{3} + \frac{(Y_1^2+Y_2^2)^3}{4} + \frac{(Y_1+Y_2)^2(Y_1^2+Y_2^2)^2}{8} + \frac{(Y_1+Y_2)^2(Y_1^4+Y_2^4)}{4}$$

or

$$Y_1^6 + Y_1^5 Y_2 + 2Y_1^4 Y_2^2 + 2Y_1^3 Y_2^3 + 2Y_1^2 Y_2^4 + Y_1 Y_2^5 + Y_2^6$$

which gives a refinement of our earlier count of 10: for example, it tells us that there are exactly two colorings (up to symmetry) which use each of the colors three times.

Question 2. Up to symmetry, how many ways are there to color the faces of a regular octahedron using the set of colors $T = \{y_1, \ldots, y_t\}$?

The symmetry group for Question 2 is identical to the symmetry group for Question **??**, since the octahedron and the cube are dual platonic solids. However, the relevant group action is different: now we must study the action of G on collection of eight faces of an octahedron, or equivalently the 8 vertices of the cube. Once again, there are five types of element $g \in G$ to consider:

(a) The identity, with cycle monomial s_1^8.

(b) Rotation around a vertex of the cube. This has two fixed points and two orbits of order 3, so the cycle monomial is $s_1^2 s_3^2$.

(c) Rotation through the center of an edge. This has four orbits of size 2, so cycle monomial s_2^4.

(d) Rotation of 180° around the center of a face of the cube. This again has four orbits of size 2, so cycle monomial s_2^4.

(e) Rotation of 90° around the the face of a cube. These rotations have two orbits of size four, so cycle monomial s_4^2.

The cycle index is therefore given by

$$Z_G(s_1, s_2, \ldots) = \frac{s_1^8 + 8s_1^2 s_3^2 + 9s_2^4 + 6s_4^2}{24}.$$

Invoking Polya's theorem, we see that the answer to Question 2 is

$$\frac{t^8 + 17t^4 + 6t^2}{24}.$$

For example, if there are exactly two colors, then the number of colorings is

$$\frac{256 + 17 \times 16 + 6 \times 4}{24} = 23.$$

We can also answer more refined questions. What we have eight colors and want to use each one exactly once? This we can answer without Polya's theorem. The group G acts freely on such colorings, so the answer is $\frac{8!}{24} = 1120$. What if we have two colors and would like to use each one four times? In this case, we need to extract the coefficient of $Y_1^4 Y_2^4$ in the polynomial

$$\frac{(Y_1 + Y_2)^8 + 8(Y_1 + Y_2)^2(Y_1^3 + Y_2^3)^2 + 9(Y_1^2 + Y_2^2)^4 + 6(Y_1^4 + Y_2^4)^2}{24}$$

which is given by

$$\frac{70 + 32 + 54 + 12}{24} = \frac{168}{24} = 7.$$

Question 3. Suppose we are given a circular wheel with n spokes. How many ways can the wheel be colored with t colors (up to rotational symmetries)?

In the situation of Question 3, the relevant symmetry group is the cyclic group $G = \mathbf{Z}/n\mathbf{Z}$ of order n, acting on itself by translation. Let's first consider the case where $n = p$ is a prime number. In this case, every non-identity element of G is a generator, and has cycle monomial s_p. We conclude that the cycle index Z_G is given by

$$Z_G(s_1, s_2, \ldots) = \frac{s_1^p + (p-1)s_p}{p} = s_p + \frac{s_1^p - s_p}{p}.$$

Applying Polya's theorem, we deduce that the total number of colorings is

$$t + \frac{t^p - t}{p}.$$

Since this must be an integer, we deduce that $t^p - t$ is divisible by p. This reproduces the proof of Fermat's little theorem given in the first class.

Let's consider a slightly more complicated version. Let $n = p^2$ be the square of a prime number. In this, case, the group $G = \mathbf{Z}/p^2\mathbf{Z}$ has three types of elements:

- (i) The identity, which has cycle monomial $s_1^{p^2}$.
- (ii) Elements of order p. There are $(p-1)$ of these, each of which has p orbits of size p and therefore cycle monomial s_p^p.
- (iii) Generators of G: there are $p(p-1)$ of these, each of which has a single orbit and therefore cycle monomial s_{p^2}.

We conclude that the cycle index is given by

$$Z_G(s_1, \ldots) = \frac{s_1^{p^2} + (p-1)s_p^p + (p^2 - p)s_{p^2}}{p^2}.$$

Applying Polya's formula, we deduce that the answer to Question 3 is

$$\frac{t^{p^2} + (p-1)t^p + (p^2 - p)t}{p^2} = t + \frac{t^p - t}{p} + \frac{t^{p^2} - t^p}{p^2}.$$

Since the first two expressions are integers, the third must be an integer as well. This proves the following variant of Fermat's little theorem:

Proposition 4. *Let p be a prime number and let t be a nonnegative integer. Then t^{p^2} is congruent to t^p modulo p^2.*