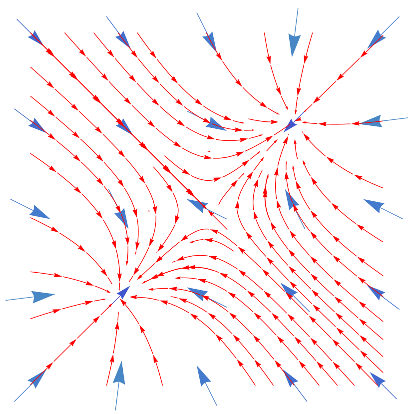
Lecture 19: Vectorfields

A **vector field** in the plane is a map, which assigns to each point (x, y) in the plane a vector $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$. A vector field in space is a map, which assigns to each point (x, y, z) in space a vector $\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$.

For example $\vec{F}(x, y) = \langle x-1, y\rangle/((x-1)^2+y^2)^{3/2} - \langle x+1, y\rangle/((x+1)^2+y^2)^{3/2}$ is the electric field of positive and negative point charge. It is called **dipole field**. It is shown in the picture below



If $f(x, y)$ is a function of two variables, then $\vec{F}(x, y) = \nabla f(x, y)$ is called a **gradient field**. Gradient fields in space are of the form $\vec{F}(x, y, z) = \nabla f(x, y, z)$.

When is a vector field a gradient field? $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = \nabla f(x, y)$ implies $Q_x(x, y) = P_y(x, y)$. If this does not hold at some point, F is no gradient field.

Clairot test: If $Q_x(x, y) - P_y(x, y)$ is not zero at some point, then $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is not a gradient field.

We will see next week that the condition $\text{curl}(F) = Q_x - P_y = 0$ is also necessary for F to be a gradient field. In class, we see more examples on how to construct the potential f from the gradient field F.

1 Is the vector field $\vec{F}(x, y) = \langle P, Q\rangle = \langle 3x^2y + y + 2, x^3 + x - 1\rangle$ a gradient field? **Solution**: the Clairot test shows $Q_x - P_y = 0$. We integrate the equation $f_x = P = 3x^2y + y + 2$ and get $f(x, y) = 2x + xy + x^3y + c(y)$. Now take the derivative of this with respect to y to get $x + x^2 + c'(y)$ and compare with $x^3 + x - 1$. We see $c'(y) = -1$ and so $c(y) = -y + c$. We see the solution $\boxed{x^3y + xy - y + 2x}$.

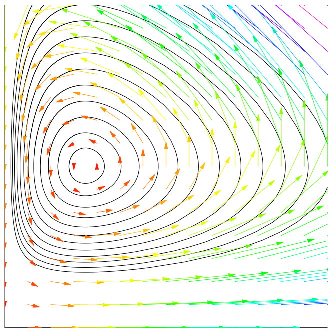
2 Is the vector field $\vec{F}(x, y) = \langle xy, 2xy^2\rangle$ a gradient field? **Solution:** No: $Q_x - P_y = 2y^2 - x$ is not zero.

Vector fields are important in differential equations. We look at some examples in population dynamics and mechanics. You can skip this part. This is more motivational.

3 Let $x(t)$ denote the population of a "prey species" like tuna fish and $y(t)$ is the population size of a "predator" like sharks. We have $x'(t) = ax(t) + bx(t)y(t)$ with positive a, b because both more predators and more prey species will lead to prey consumption. The rate of change of $y(t)$ is $-cy(t) + dxy$, where c, d are positive. We have a negative sign in the first part because predators would die out without food. The second term is explained because both more predators as well as more prey leads to a growth of predators through reproduction. A concrete example is the **Volterra-Lodka system**

$$\begin{aligned} \dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy\ . \end{aligned}$$

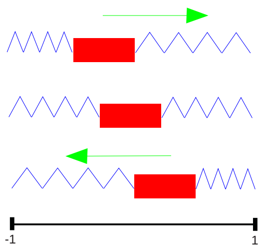
Volterra explained with such systems the oscillation of fish populations in the Mediterranean sea. At any specific point $\vec{r}(x, y) = \langle x(t), y(t)\rangle$, there is a curve $= \vec{r}(t) = \langle x(t), y(t)\rangle$ through that point for which the tangent $\vec{r}\,'(t) = (x'(t), y'(t)$ is the vector $\langle 0.4x - 0.4xy, -0.1y + 0.2xy\rangle$.

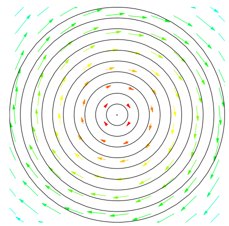


4 A class vector fields important in mechanics are **Hamiltonian fields**: If $H(x, y)$ is a function of two variables, then $\langle H_y(x, y), -H_x(x, y)\rangle$ is called a **Hamiltonian vector field**. An example is the harmonic oscillator $H(x, y) = x^2+y^2$. Its vector field $(H_y(x, y), -H_x(x, y)) = (y, -x)$. The flow lines of a Hamiltonian vector fields are located on the level curves of H (as you have shown in th homework with the chain rule).

5 Newton's law $m\vec{r}'' = F$ relates the acceleration $\vec{r}''$ of a body with the force F acting at the point. For example, if $x(t)$ is the position of a mass point in $[-1, 1]$ attached at two springs and the mass is $m = 2$, then the point experiences a force $(-x + (-x)) = -2x$ so that $mx'' = 2x$ or $x''(t) = -x(t)$. If we introduce $y(t) = x'(t)$ of t, then $x'(t) = y(t)$ and $y'(t) = -x(t)$. Of course y is the velocity of the mass point, so a pair (x, y), thought of as an initial condition, describes the system so that nature knows what the future evolution of the system has to be given that data.

6 We don't yet know yet the curve $t \mapsto \vec{r}(t) = \langle x(t), y(t) \rangle$, but we know the tangents $\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle = \langle y(t), -x(t) \rangle$. In other words, we know a direction at each point. The equation $(x' = y, y' = -x)$ is called a system of ordinary differential equations (ODE's) More generally, the problem when studying ODE's is to find solutions $x(t), y(t)$ of equations $x'(t) = f(x(t), y(t)), y'(t) = g(x(t), y(t))$. Here we look for curves $x(t), y(t)$ so that at any given point (x, y), the tangent vector $(x'(t), y'(t))$ is $(y, -x)$. You can check by differentiation that the circles $(x(t), y(t)) = (r\sin(t), r\cos(t))$ are solutions. They form a family of curves.





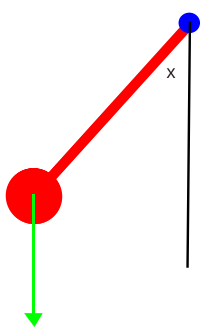
7 If $x(t)$ is the angle of a pendulum, then the gravity acting on it produces a force $G(x) = -gm\sin(x)$, where m is the mass of the pendulum and where g is a constant. For example, if $x = 0$ (pendulum at bottom) or $x = \pi$ (pendulum at the top), then the force is zero. The Newton equation "mass times acceleration = force" gives

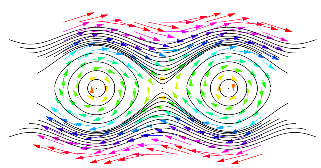
$$\ddot{x}(t) = -g\sin(x(t)) \; .$$

8 The equation of motion for the pendulum $\ddot{x}(t) = -g\sin(x(t))$ can be written with $y = \dot{x}$ also as

$$\frac{d}{dt}(x(t), y(t)) = (y(t), -g\sin(x(t)) \; .$$

Each possible motion of the pendulum $x(t)$ is described by a curve $\vec{r}(t) = (x(t), y(t))$. Writing down explicit formulas for $(x(t), y(t))$ is in this case not possible with known functions like $\sin, \cos, \exp, \log$ etc. However, one still can understand the curves.





Curves on the top of the picture represent situations where the velocity y is large. They describe the pendulum spinning around fast in the clockwise direction. Curves starting near the point $(0, 0)$, where the pendulum is at a stable rest, describe small oscillations of the pendulum.

Vector fields in weather forecast On weather maps, one can see **isoterms**, curves of constant temperature or **isobars**, curves $p(x, y) = c$ of constant pressure. These are level curves. The wind velocity $\vec{F}(x, y)$ is close but not always exactly perpendicular to the **isobars**, the lines of equal pressure p. In reality, the scalar pressure field p and the velocity field $\vec{F}$ also depend on time. The equations which describe the weather dynamics are called the **Navier Stokes equations**

$$\frac{d}{dt}\vec{F} + \vec{F} \cdot \nabla\vec{F} = \nu\Delta\vec{F} - \nabla p + f, \mathrm{div}\vec{F} = 0$$

(where Δ and div are defined later. This is an other example of a **partial differential equation**. It is one of the millenium problems to prove that these equations have smooth solutions in space.

Homework

1 The vector field $\vec{F}(x, y) = \langle x/r^3, y/r^3 \rangle$ appears in electrostatics, where $r = \sqrt{x^2 + y^2}$ is the distance to the charge. Find a function $f(x, y)$ such that $\vec{F} = \nabla f$. Hint. Write out the vector field $F = \langle P, Q \rangle$ where P, Q are functions of x, y. Then integrate P with respect to x.

2 a) Draw the gradient vector field of the function $f(x, y) = \sin(x + y)$.
b) Draw the gradient vector field of the function $f(x, y) = (x - 1)^2 + (y - 2)^2$.
Hint: In both cases, draw first a contour map of f and use a property of gradients to draw the vector field $F(x, y) = \nabla f$.

3 a) Is the vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle xy, x^2 \rangle$ a gradient field?
b) Is the vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle \sin(x) + y, \cos(y) + x \rangle$ a gradient field?
In both cases, give the potential $f(x, y)$ satisfying $\nabla f(x, y) = \vec{F}(x, y)$ if it exists and if there is no gradient, give a reason, why it is not a gradient field.

4 Which of the following vector fields $\vec{F} = \langle P, Q \rangle$ can be written as $\vec{F} = \langle P, Q \rangle = \langle f_x, f_y \rangle$? Make use of Clairot's identity which implies that $Q_x = P_y$, if a function f exists. If f exists, find the potential f.
a) $\vec{F}(x, y) = \langle x^5, y^7 \rangle$.
b) $\vec{F}(x, y) = \langle y^5, x^7 \rangle$.
c) $\vec{F}(x, y) = \langle y, x \rangle$.
d) $\vec{F}(x, y) = \langle y^2 + x^2, y^2 + x^2 \rangle$.
e) $\vec{F}(x, y) = \langle 5 - y^2 + 4x^3y^3, -2xy + 3x^4y^2 \rangle$.

5 a) The vector field

$$\vec{F}(x, y, z) = \langle 5x^4y + z^4 + y\cos(x * y), x^5 + x\cos(xy), 4xz^3 \rangle$$

is a gradient field. Find the potential function f.
b) Can you find conditions for a vector field $\vec{F} = \langle P, Q, R \rangle$ so that $\vec{F} = \nabla f$?