THE AREA OF THE MANDELBROT SET

OLIVER KNILL

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DOUBLE INTEGRALS

The Riemann integral is a limit of a summation process. If \( f(x, y) \) is continuous on a region \( G \), the integral \( \iint_G f(x, y) \, dA \) is defined as the limit of the Riemann sums

\[
\frac{1}{n^2} \sum_{(i/n, j/n) \in G} f(i/n, j/n)
\]
as \( n \to \infty \). We write also \( \iint_G f(x, y) \, dA \), where \( dA \) is a notation standing for “area element”.

AN EXAMPLE

Here is an example, where it is not difficult to give an answer. Question: Using a double integral, compute the area of the region between the curve \( y = x^2 \) and \( y = x \), where \( 0 \leq x \leq 1 \). Answer: The result is \( \int_0^1 x - x^2 \, dx = 1/2 - 1/3 = 1/6 \). We can also compute it with a Monte Carlo computation.

\[
\begin{align*}
n = 10^6; & \quad S = \text{Sum}\{\{x, y\} = \{2 \, \text{Random[]} - 1, 2 \, \text{Random[]} - 1\}; \quad \text{If} [x^2 < y < x, 1, 0], \{n\}] / n \\
\end{align*}
\]

In some cases, it is not possible to find an analytic expression.

\( G = \{(x, y) \mid x^6 y + \sin(y)^2 x^2 \cos(x^2) < 0.02, x^2 + y^2 < 1\} \)

would be difficult to describe analytically as we can not find expressions for the boundary. What still can find random points \((x, y)\) in \([a, b] \times [c, d]\), then form

\[
\frac{A}{n} \sum_{k=1}^{n} f(x_k, y_k),
\]

where \( A = (d - c)(b - a) \) is the area in which we shoot randomly.

\[
\begin{align*}
n = 10^6; & \quad S = \text{Sum}\{\{x, y\} = \{2 \, \text{Random[]} - 1, 2 \, \text{Random[]} - 1\}; \quad (4/n) * \text{If} [x^6 + 2 \, \text{Random[]} > x^2 \, \text{Cos}[x^2] < 0.02 \&\& x^2 + y^2 < 1, 1, 0], \{n\}] \\
\end{align*}
\]

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The Mandelbrot set is
\[ M = \{ c = a + ib \in \mathbb{C} \in \mathbb{R}^2 \mid T^n_c(0) \text{ stays bounded} \}, \]
where \( T_c(z) = z^2 + c \). In real coordinates the map is \( T_c(x, y) = (x^2 - y^2 + a, 2xy + b) \). The notation \( T^n \) means applying \( T \) a number of times. For example \( T^3(x, y) = T(T(T(x, y))) \). One can draw it numerically or approximate curves one knows to be outside.

What is the area of \( M \)? We know it is contained in the rectangle \( x \in [-2, 1] \) and \( y \in [-3/2, 3/2] \) which has area 9. One way to get bounds on the area is to approximate the set with polynomials \( p_n \).

Here is the code to plot such polynomial contour curves and to compute its area (which is already challenging for a computer algebra system for \( n=2 \)).

```math
\begin{align*}
p[0, z] &:= z; 
p[n, z] := p[n-1, z]^2 + z; 
ContourPlot[\text{Abs}[p[8, x+I y]] == 2, \{x, -2, 1\}, \{y, -3/2, 3/2\}]
f &= p[2, x+I y] \ast p[2, x-I y] \quad /\ Expand
\end{align*}
```

The polynomial \( p_2(x, y) \) is

\[
x^8 + 4x^7 + 4x^6y^2 + 6x^5 + 12x^3 + y^4 + 14x^3y^2 + 5x^4 + 12x^3 + y^4 + 4x^3y^2 + 2x^2y^6 + 4x^2y^2 + x^2y^4 - 2xy^4 + 2xy^2 + y^8 + 2y^6 - 3y^4 + y^2
\]

But computing the area of such regions bounded by polynomial curves is already difficult for computer algebra systems.

```math
\text{Area[ImplicitRegion[f<4,\{x,y\}]]}
```

We can also just randomly shoot into this rectangle and see whether we are in the Mandelbrot set or not after 1000 iterations:

```math
M := \text{Compile[}\{x, y\}, \text{Module[}\{z=x+I y, k=0\}, 
\text{While[} \text{Abs}[z]<2 \&\& k<999, z=\text{N}[-2+\text{Random}[\{2\}]+I \text{Random}[\{2\}]], ++k]\}; \text{Floor}[k/999]]; 
n = 10^6; 9*\text{Sum}[\text{M}[-2+3*\text{Random}[\{2\}], \{-1.5+3*\text{Random}[\{2\}]\}, \{n\}]/n
```

Here is a value computed with this code:

1.51116


\text{Math 21A}