Lecture 21: Polar integration

1. The area of a disc of radius $R$ is

$$\int_{-R}^{R} \int_{\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1 \, dy \, dx = \int_{-R}^{R} 2\sqrt{R^2-x^2} \, dx .$$

This integral can be solved with the substitution $x = R \sin(u), \, dx = R \cos(u)$

$$\int_{-\pi/2}^{\pi/2} 2\sqrt{R^2 - R^2 \sin^2(u)} R \cos(u) \, du = \int_{-\pi/2}^{\pi/2} 2R^2 \cos^2(u) \, du .$$

Using a double angle formula we get

$$R^2 \int_{-\pi/2}^{\pi/2} 2\frac{1+\cos(2u)}{2} \, du = R^2 \pi .$$

We will now see how to do that better in polar coordinates.

A polar region is a region bound by a simple closed curve given in polar coordinates as the curve $(r(t), \theta(t))$.

In Cartesian coordinates the parametrization of the boundary curve is $\vec{r}(t) = [r(t) \cos(\theta(t)), r(t) \sin(\theta(t))]$. We are especially interested in regions which are bound by polar graphs, where $\theta(t) = t$.

2. The polar region defined by $r \leq |\cos(3\theta)|$ belongs to the class of roses $r(t) = |\cos(nt)|$ they are also called rhododenea. These names reflect that polar regions model flowers well.

3. The polar curve $r(\theta) = 1 + \sin(\theta)$ is called a cardioid. It looks like a heart. It is a special case of a limacon a polar curve of the form $r(\theta) = 1 + b \sin(\theta)$.

4. The polar curve $r(\theta) = |\sqrt{\cos(2t)}|$ is called a lemniscate. It looks like an infinity sign. It encloses a flower with two petals.

To integrate in polar coordinates, we evaluate the integral

$$\int \int_{R} f(x, y) \, dxdy = \int \int_{R} f(r \cos(\theta), r \sin(\theta)) r \, dr \, d\theta$$
Integrate \( f(x, y) = x^2 + x^2 + xy \), over the unit disc. We have \( f(x, y) = f(r \cos(\theta), r \sin(\theta)) = r^2 + r^2 \cos(\theta) \sin(\theta) \) so that \( \iint f(x, y) \, dx \, dy = \int_0^1 \int_0^{2\pi} (r^2 + r^2 \cos(\theta) \sin(\theta)) r \, d\theta \, dr = 2\pi/4. \)

We have earlier computed area of the disc \( \{ x^2 + y^2 \leq R^2 \} \) using substitution. It is more elegant to do this integral in polar coordinates: \( \frac{2\pi^2}{2} \int_0^R r \, dr = \frac{\pi R^2}{2}. \)

Why do we have to include the factor \( r \), when we move to polar coordinates? The reason is that a small rectangle \( R \) with dimensions \( d\theta \, dr \) in the \((r, \theta)\) plane is mapped by \( T : (r, \theta) \mapsto (r \cos(\theta), r \sin(\theta)) \) to a sector segment \( S \) in the \((x, y)\) plane. It has the area \( r \, d\theta \, dr \).

Integrate the function \( f(x, y) = 1 \{ (\theta, r(\theta)) \mid r(\theta) \leq | \cos(3\theta)| \} \).

\[
\iint_R 1 \, dx \, dy = \int_0^{2\pi} \int_0^{\cos(3\theta)} r \, dr \, d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2} \, d\theta = \frac{\pi}{2}.
\]

Integrate \( f(x, y) = y \sqrt{x^2 + y^2} \) over the region \( R = \{ (x, y) \mid 1 < x^2 + y^2 < 4, y > 0 \} \).

\[
\int_1^4 \int_0^\pi r \sin(\theta) r \, d\theta \, dr = \int_1^4 r^3 \int_0^\pi \sin(\theta) \, d\theta \, dr = \frac{(2^4 - 1^4)}{4} \int_0^\pi \sin(\theta) \, d\theta = \frac{15}{2}.
\]

For integration problems, where the region is part of an annular region, or if you see function with terms \( x^2 + y^2 \) try to use polar coordinates \( x = r \cos(\theta), y = r \sin(\theta) \).

The Belgian Biologist Johan Gielis defined in 1997 with the family of curves given in polar coordinates as

\[
r(\phi) = \left( \frac{|\cos\left(\frac{m\phi}{4}\right)|^{n_1}}{a} + \frac{|\sin\left(\frac{m\phi}{4}\right)|^{n_2}}{b} \right)^{-1/n_3}
\]

This super-curve can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce ”super-shapes”. The super-curve generalizes the super-ellipse which had been discussed in 1818 by Lamé and helps to describe forms in biology. \(^1\)

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