Lecture 20: Double integrals

The integral \( \int \int_R f(x, y) \, dx \, dy \) is defined as the limit of Riemann sums

\[
\frac{1}{n^2} \sum_{(i/n, j/n) \in R} f(i/n, j/n)
\]

when \( n \to \infty \).

1. In order to integrate \( f(x, y) = xy \) over the unit square, we can sum up the Riemann sum for fixed \( y = j/n \) and get \( y/2 \). Integrating with respect to \( y \) from 0 to 1 gives 1/4. This example shows how we can reduce double integrals to single variable integrals.

2. If \( f(x, y) = 1 \), then the integral is the area of the region \( R \). The integral is the limit \( L(n)/n^2 \), where \( L(n) \) is the number of lattice points \((i/n, j/n)\) inside \( R \).

3. The integral \( \int \int_R f(x, y) \, dx \, dy \) as the signed volume of the solid below the graph of \( f \) and above the region \( R \) in the \( x-y \) plane. The volume below the \( xy \)-plane is counted negatively.

**Fubini’s theorem** allows to switch the order of integration over a rectangle, if the function \( f \) is continuous:

\[
\int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.
\]

Proof. For every \( n \), check the “quantum Fubini identity”

\[
\sum_{\frac{i}{n} \in [a,b]} \sum_{\frac{j}{n} \in [c,d]} f(i/n, j/n) = \sum_{\frac{j}{n} \in [c,d]} \sum_{\frac{i}{n} \in [a,b]} f(i/n, j/n)
\]

holds for all functions. Now divide both sides by \( n^2 \) and take the limit \( n \to \infty \).

A **dy dx region** is of the form

\[
R = \{(x, y) \mid a \leq x \leq b, \; c(x) \leq y \leq d(x) \}.
\]

An integral over such a region is called a **dy dx integral**

\[
\int \int_R f \, dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y) \, dy \, dx.
\]
A **dx dy region** is of the form

\[ R = \{(x, y) \mid c \leq y \leq d, a(y) \leq x \leq b(y) \} \, . \]

An integral over such a region is called a **dx dy integral**

\[ \int \int_{R} f \, dA = \int_{c}^{d} \int_{a(y)}^{b(y)} f(x, y) \, dx \, dy \, . \]

---

### 4

Integrate \( f(x, y) = x^2 \) over the region bounded above by \( \sin(x^3) \) and bounded below by the graph of \( -\sin(x^3) \) for \( 0 \leq x \leq \pi \). The value of this integral has a physical meaning. It is called **moment of inertia**.

\[
\int_{0}^{\pi/3} \int_{-\sin(x^3)}^{\sin(x^3)} x^2 \, dy \, dx = 2 \int_{0}^{\pi/3} \sin(x^3) x^2 \, dx
\]

This can be solved by substitution

\[
= -\frac{2}{3} \cos(x^3)|_{0}^{\pi/3} = \frac{4}{3} .
\]

### 5

Integrate \( f(x, y) = y^2 \) over the region bound by the \( x \)-axes, the lines \( y = x + 1 \) and \( y = 1 - x \). The problem is best solved as a dy dx integral. Because we would have to compute 2 different integrals as a dy dx integral. The \( y \) bounds are \( x = y - 1 \) and \( x = 1 - y \)

\[
\int_{0}^{1} \int_{y-1}^{1-y} y^3 \, dx \, dy = 2 \int_{0}^{1} y^3(1-y) \, dy = 2\left(\frac{1}{4} - \frac{1}{3}\right) = \frac{1}{10} .
\]

### 6

Let \( R \) be the triangle \( 1 \geq x \geq 0, 0 \leq y \leq x \). What is

\[ \int \int_{R} e^{-x^2} \, dxdy \, ? \]

The dx dy integral \( \int_{0}^{1} [\int_{y}^{1} e^{-x^2} \, dx] \, dy \) can not be solved because \( e^{-x^2} \) has no anti-derivative in terms of elementary functions. The dy dx integral \( \int_{0}^{1} [\int_{x}^{1} e^{-x^2} \, dy] \, dx \) however can be solved:

\[
= \int_{0}^{1} xe^{-x^2} \, dx = -\frac{e^{-x^2}}{2} \bigg|_{0}^{1} = \frac{(1-e^{-1})}{2} = 0.316... .
\]