Unit 6: Arc length

In this worksheet, we find the arc length of the cycloid

$$\vec{r}(t) = \begin{bmatrix} t - \sin(t) \\ \cos(t) \end{bmatrix}$$

from 0 to $2\pi$. The curve is the solution to the famous Brachistochrone problem, the curve along which a ball descents fastest.

1. Compute the velocity $\vec{r}'(t)$.
2. Verify that $|\vec{r}'(t)| = \sqrt{2 - 2\cos t}$.
3. Use the double angle formula identity

$$2 - 2\cos(t) = 4\sin^2\left(\frac{t}{2}\right).$$

to find the arc length

$$\int_0^{2\pi} |\vec{r}'(t)| \, dt.$$

Johann Bernoulli asked the Brachistochrone problem in 1696. The problem marks the start of a mathematical area called the calculus of variations in which one extremizes functions on infinite dimensional spaces.

Cycloids are curves traced by your feet, when you bike. It is a natural curve because it combines linear and circular motion.
About integration techniques

When looking at arc length integrals, basic integration techniques come back. Can you solve the following problems?

1. \[ \int_{0}^{2\pi} x \sin(x) \, dx \]

   2. \[ \int_{0}^{\sqrt{\pi}} 2x \sin(x^2) \, dx. \]

   3. \[ \int_{0}^{1} \frac{1}{1+x^2} \, dx. \]

   4. \[ \int_{0}^{2\pi} \cos^2(x) \, dx \]

1) \(-2\pi\), 2) 2, 3) \(\pi/4\). 4) \(\pi\) (use double angle formula)