### 3: Lines and Planes

A point \( P = (x_0, y_0, z_0) \) and a vector \( \vec{v} = [a, b, c] \) define the line

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
x_0 \\
y_0 \\
z_0
\end{bmatrix} + t \begin{bmatrix}
a \\
b \\
c
\end{bmatrix}.
\]

It is called the \textbf{parameterization} of the line.

Every vector contained inside the line is parallel to \( \vec{v} \). We think about the parameter \( t \) as "time" and about \( \vec{v} \) as the \textbf{velocity}. For \( t = 0 \), we are at \( P \) identified with \( \vec{OP} \).

Given two points like \((2, 3, 4)\) with \((3, 3, 5)\), the line

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
2 \\
3 \\
4
\end{bmatrix} + t \begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix}
\]

connects the points \((2, 3, 4)\) with \((3, 3, 5)\). If \( t \) is restricted to \([0, 1]\) we get the \textbf{line segment} defined by the two points.

**1 Problem.** Parametrize the line through \( P = (1, 1, 2) \) and \( Q = (2, 4, 6) \).

**Solution.** with \( \vec{v} = \vec{PQ} = [1, 3, 4] \) we get get the line

\[
[x, y, z] = [1, 1, 2] + t[1, 3, 4]
\]

which is \( \vec{r}(t) = [1 + t, 1 + 3t, 2 + 4t] \). Since \( [x, y, z] = [1, 1, 2] + t[1, 3, 4] \) consists of three equations \( x = 1 + 2t, y = 1 + 3t, z = 2 + 4t \) we can solve each for \( t \) to get \( t = (x - 1)/2 = (y - 1)/3 = (z - 2)/4 \).

The line \( \vec{r} = \vec{OP} + t\vec{v} \) defined by \( P = (p, q, r) \) and vector \( \vec{v} = [a, b, c] \) with nonzero \( a, b, c \) satisfies the \textbf{symmetric equations}

\[
\frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c}.
\]

**Proof.** Each of these expressions is equal to \( t \). These symmetric equations have to be modified a bit if one or two of the numbers \( a, b, c \) are zero. If \( a = 0 \), replace the first equation with \( x = p \), if \( b = 0 \) replace the second equation with \( y = q \) and if \( c = 0 \) replace third equation with \( z = r \).
2 Find the symmetric equations for the line through the two points \( P = (0, 1, 1) \) and \( Q = (2, 3, 4) \) Solution. first first form the parametric equations \([x, y, z] = [0, 1, 1] + t[2, 2, 3]\) or \( x = 2t, y = 1 + 2t, z = 1 + 3t\) and solve for \( t\) to get \( x/2 = (y - 1)/2 = (z - 1)/3\).

3 Problem: Find the symmetric equation for the z axes. Answer: This is a situation where \( a = b = 0\) and \( c = 1\). The symmetric equations are simply \( x = 0, y = 0\). If two of the numbers \( a, b, c\) are zero, we have a coordinate plane. If one of the numbers are zero, then the line is contained in a coordinate plane.

A point \( P\) and two vectors \( \vec{v}, \vec{w}\) define a plane \( \vec{r}(t) = \vec{O}P + t\vec{v} + s\vec{w}\), where \( t, s\) are real numbers.

This is called the parametric description of a plane.

A point \( P = (x_0, y_0, z_0)\) and two vectors \( \vec{v}, \vec{w}\) defines the plane

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
x_0 \\
y_0 \\
z_0
\end{bmatrix} + t \begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} + s \begin{bmatrix}
w_1 \\
w_2 \\
w_3
\end{bmatrix}.
\]

It is called the parameterization of the plane.

If a plane contains the two vectors \( \vec{v}\) and \( \vec{w}\), then the vector

\( \vec{n} = \vec{v} \times \vec{w} \)

is orthogonal to both \( \vec{v}\) and \( \vec{w}\). Because also the vector \( \vec{P}Q = \vec{O}Q - \vec{OP}\) is perpendicular to \( \vec{n}\), we have \((Q - P) \cdot \vec{n} = 0\). With \( Q = (x_0, y_0, z_0)\), \( P = (x, y, z)\), and \( \vec{n} = [a, b, c]\), this means \( ax + by + cz = ax_0by_0 + cz_0 = d\). The plane is therefore described by a single equation \( ax + by + cz = d\), where \( d\) is a constant obtained by plugging in a point. We have just shown

The equation of the plane \( \vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}\)

\[ax + by + cz = d,\]

where \([a, b, c] = \vec{v} \times \vec{w}\) and \(d\) is obtained by plugging in \(x_0\).

3 Problem: Find the equation of a plane which contains the three points \( P = (-1, -1, 1)\), \( Q = (0, 1, 1)\), \( R = (1, 1, 3)\). Answer: The plane contains the two vectors \( \vec{v} = [1, 2, 0]\) and \( \vec{w} = [2, 2, 2]\). We have \( \vec{n} = [4, -2, -2]\) and the equation is \(4x - 2y - 2z = d\). The constant \(d\) is obtained by plugging in the coordinates of a point to the left. In our case, it is \(4x - 2y - 2z = -4\).

4 Problem: Find the angle between the planes \( x + y = -1\) and \( x + y + z = 2\). Answer: find the angle between \( \vec{n} = [1, 1, 0]\) and \( \vec{m} = [1, 1, 1]\). It is \( \arccos(2/\sqrt{6})\).