# What makes a theorem beautiful?

We look at a few criteria for "beauty" in a theorem and illustrate with examples a few proposed taxonomies.

Oliver Knill
Harvard University
5/5/2026

# PLAN

1) INTRODUCTION

2) FOUR THEOREMS

3) DEFINITION ATTEMPTS

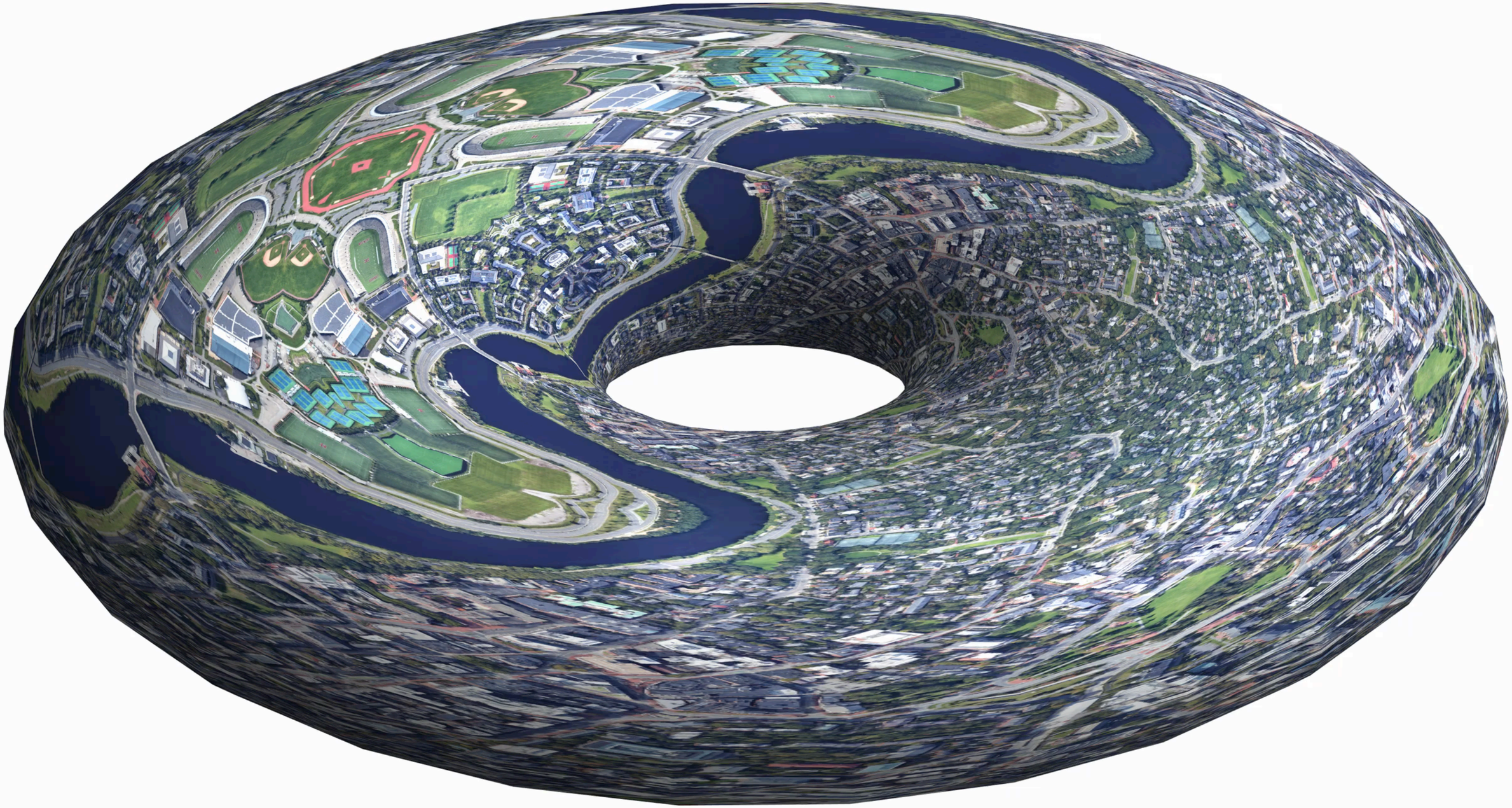
4) RANKING LISTS

1

# PART 1:

# INTRODUCTION

Science Center, yesterday 5/4/26
memorial hall
<--- where I live
me



Spieglein, Spieglein
an der Wand, was ist
der schönste Satz im
ganzen Land?

We look at a few criteria for "beauty" in a theorem and illustrate a few proposed taxonomies.

Otto Kubel, ca 1900, Grimm Märchen

"Theorems are the backbone of mathematics. A theory is rich, if it produces lots of nice theorems. "

But what is "nice?"

# BRAIN STORM

Symmetry

Simplicity

Clarity

Taste

Harmony

Surprise

History

Originality

Mix different categories

Is Deep

Uses simple definitions

Can be understood easily

Has many applications

Has prototype character

General and still concrete

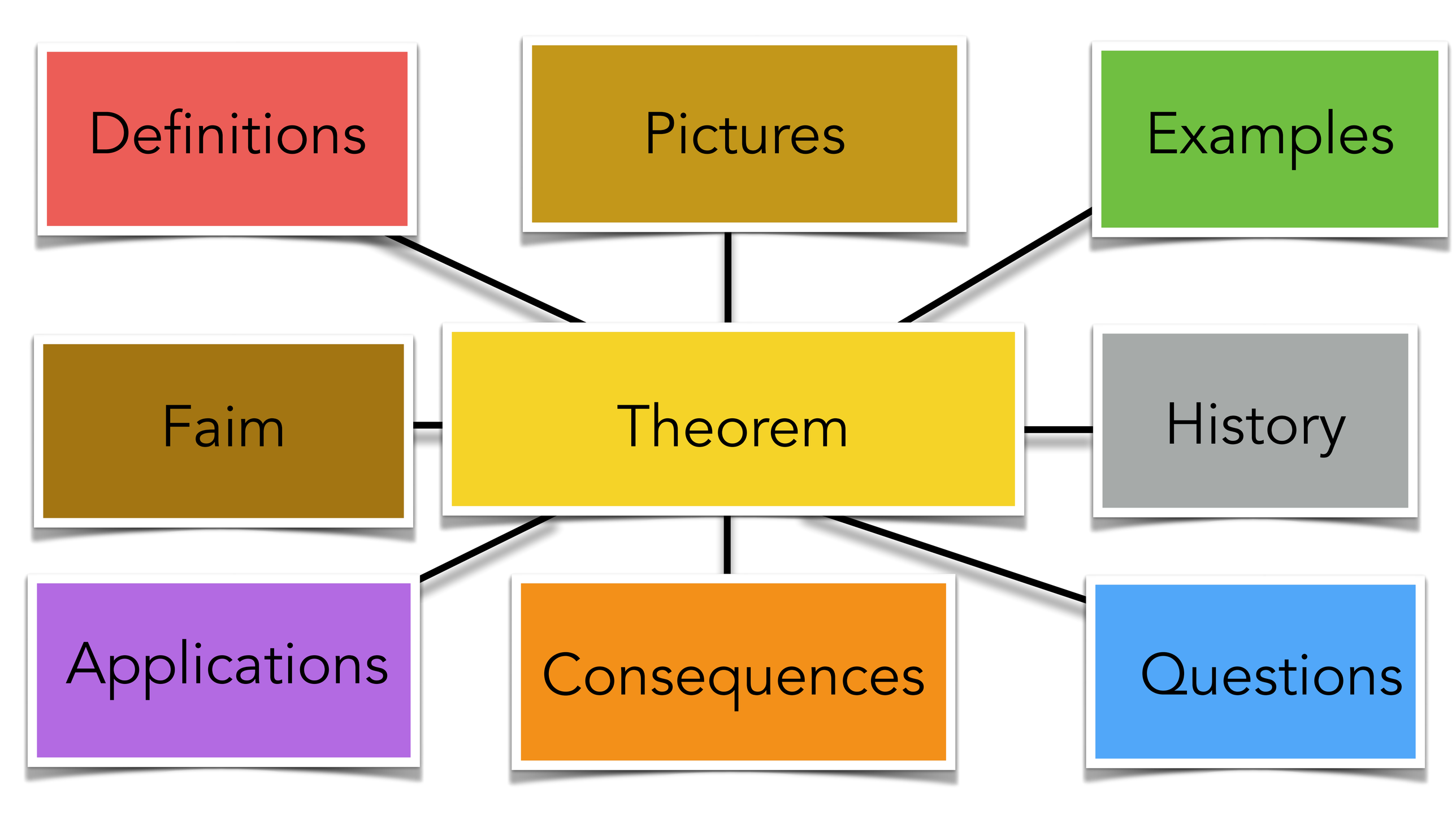
Simplicity and Complexity

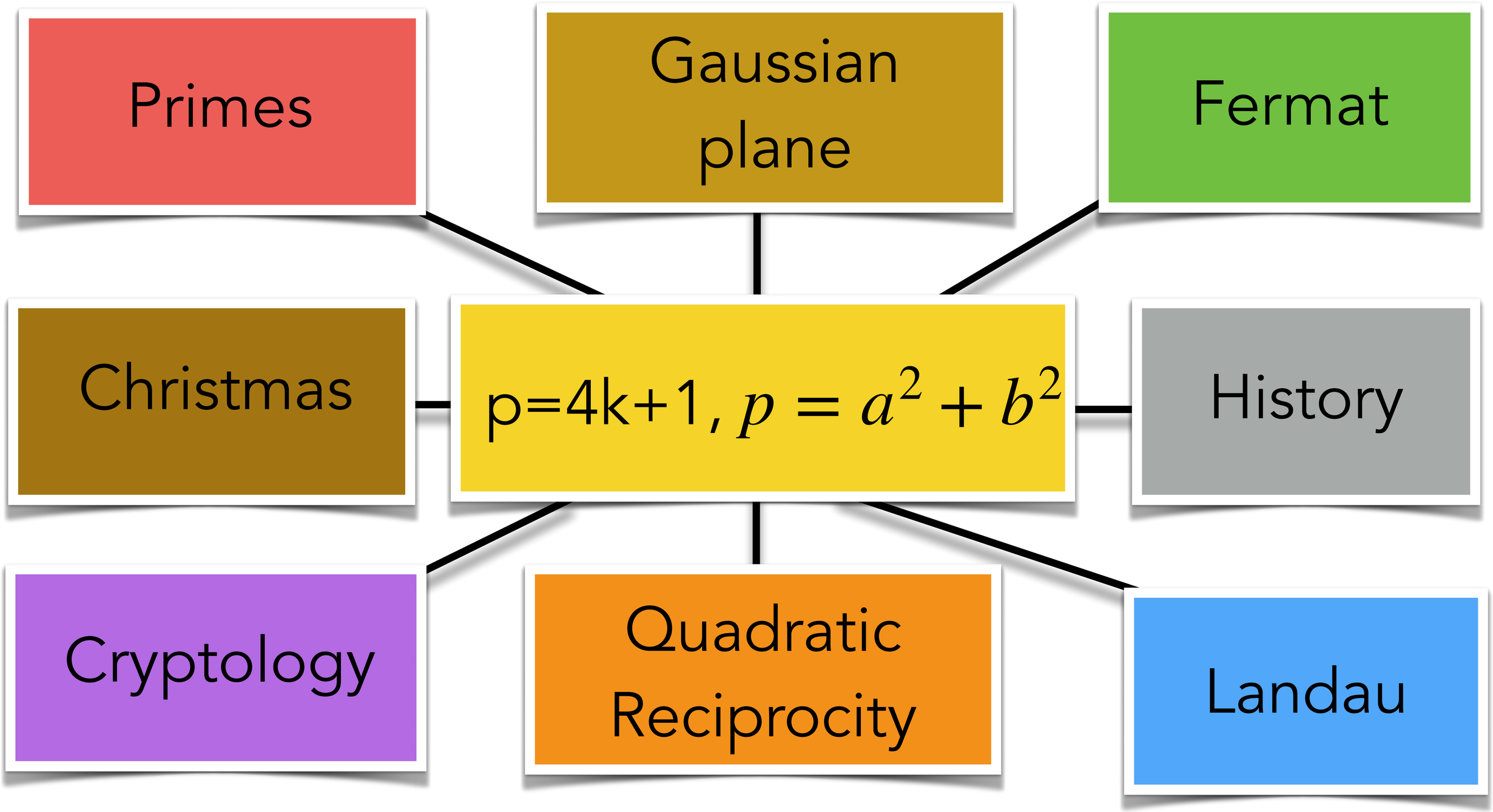
Novelty and Creativity

Familiarity and Surprise

Elegance and Richness

# THEOREMS ARE NOT ALONE


Definitions
Pictures
Examples
Faim
Theorem
History
Applications
Consequences
Questions


Primes
Gaussian plane
Fermat
Christmas
p=4k+1, $p = a^2 + b^2$
History
Cryptology
Quadratic Reciprocity
Landau

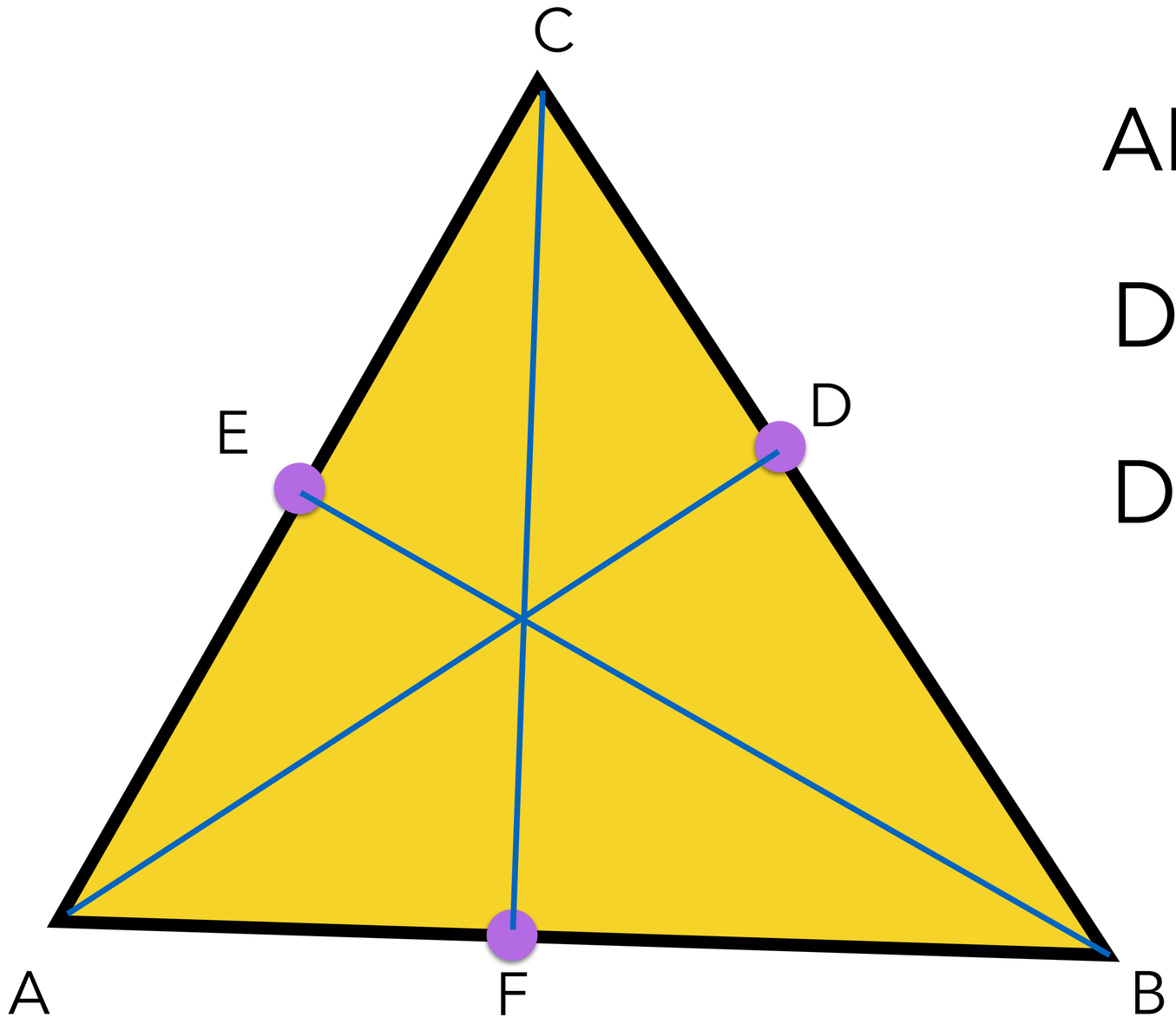
2

# PART II
# EXAMPLES

The choice of the following four theorems is pretty random. I just took the theorems related to projects, teaching, social media exposure and referee work. All from this spring.

1. theorem : referee work
2. theorem: research project
3. theorem: social media
4. theorem: teaching

# A FIRST BEAUTIFUL THEOREM



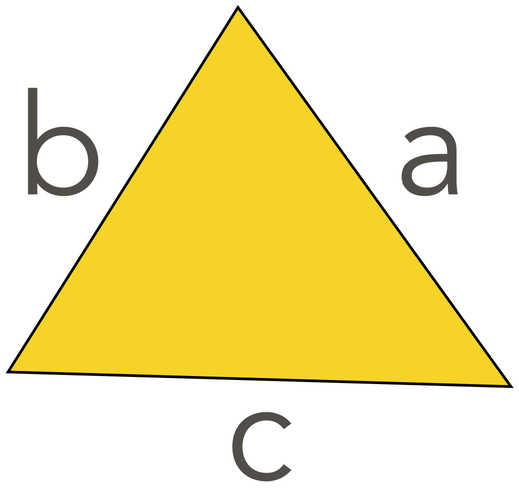

ABC arbitrary triangle

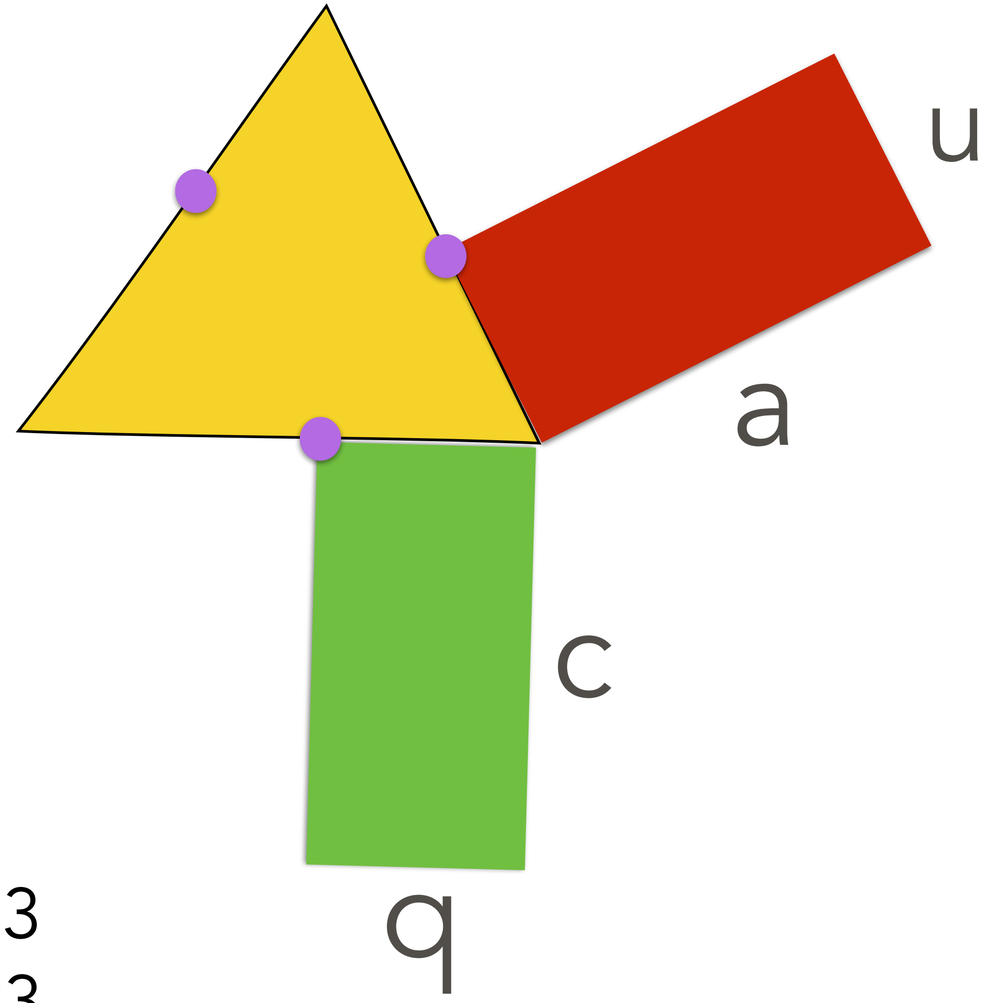
Draw the altitudes

DEF pedal triangle

# Theorem

$$q\ c = u\ a$$



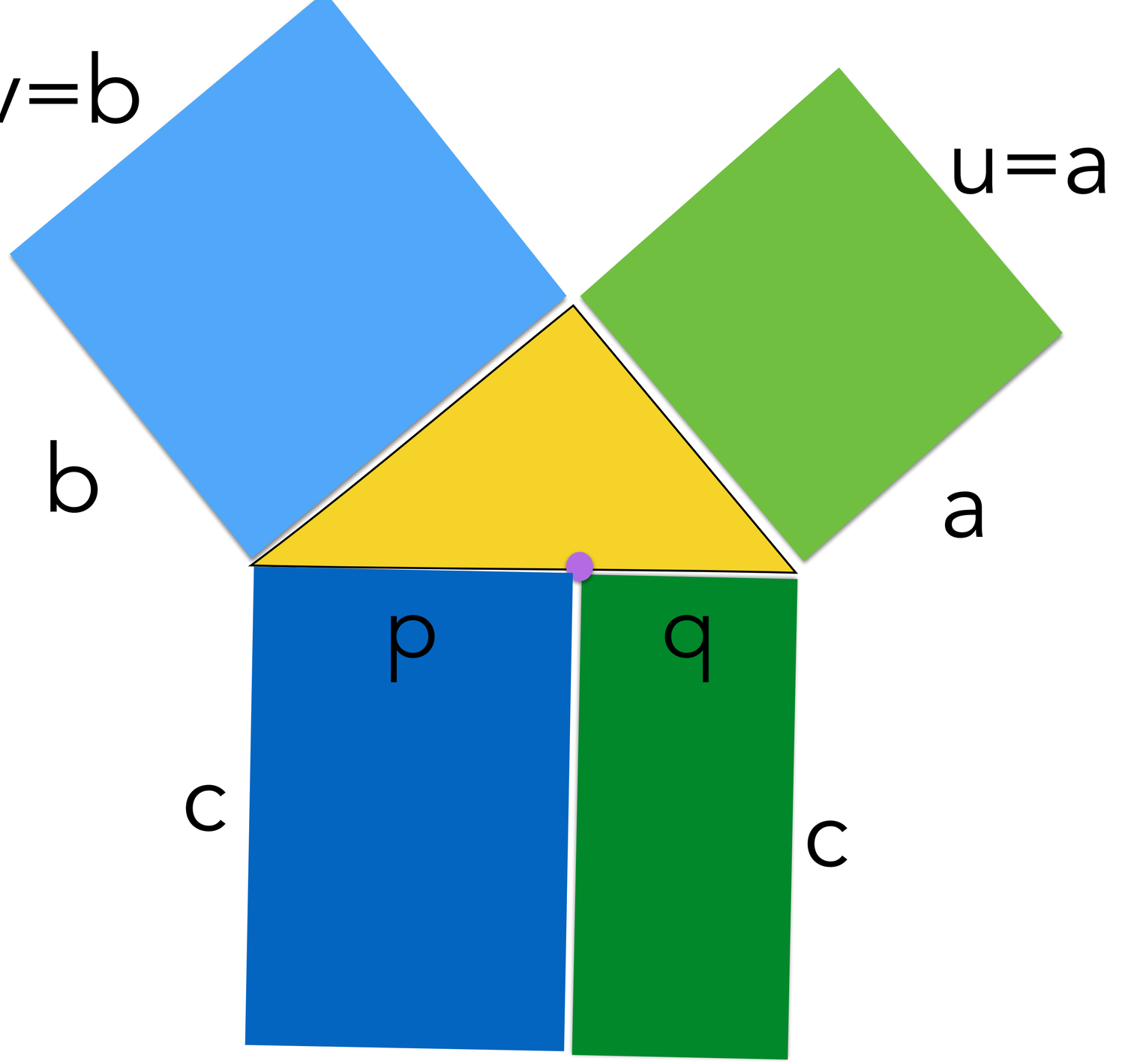




Quan Hung Tran: Math Magazine 96, 2023
Alexander Bogomolny: Cut the Knot, 2023

right angle
triangle case

q c=$a^2$

p c=$b^2$

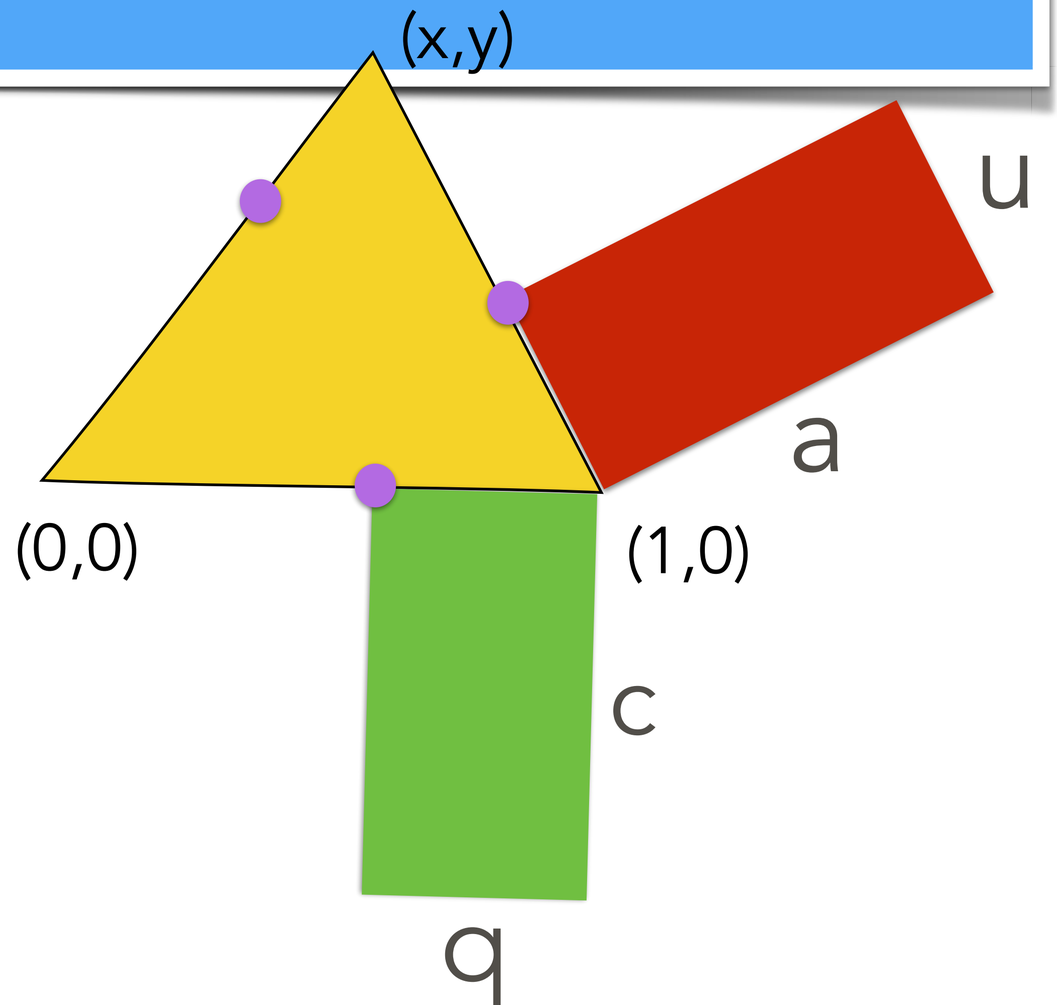


**Proof**. Rotate and scale to have A=(0,0),B=(1,0),C=(x,y). Then

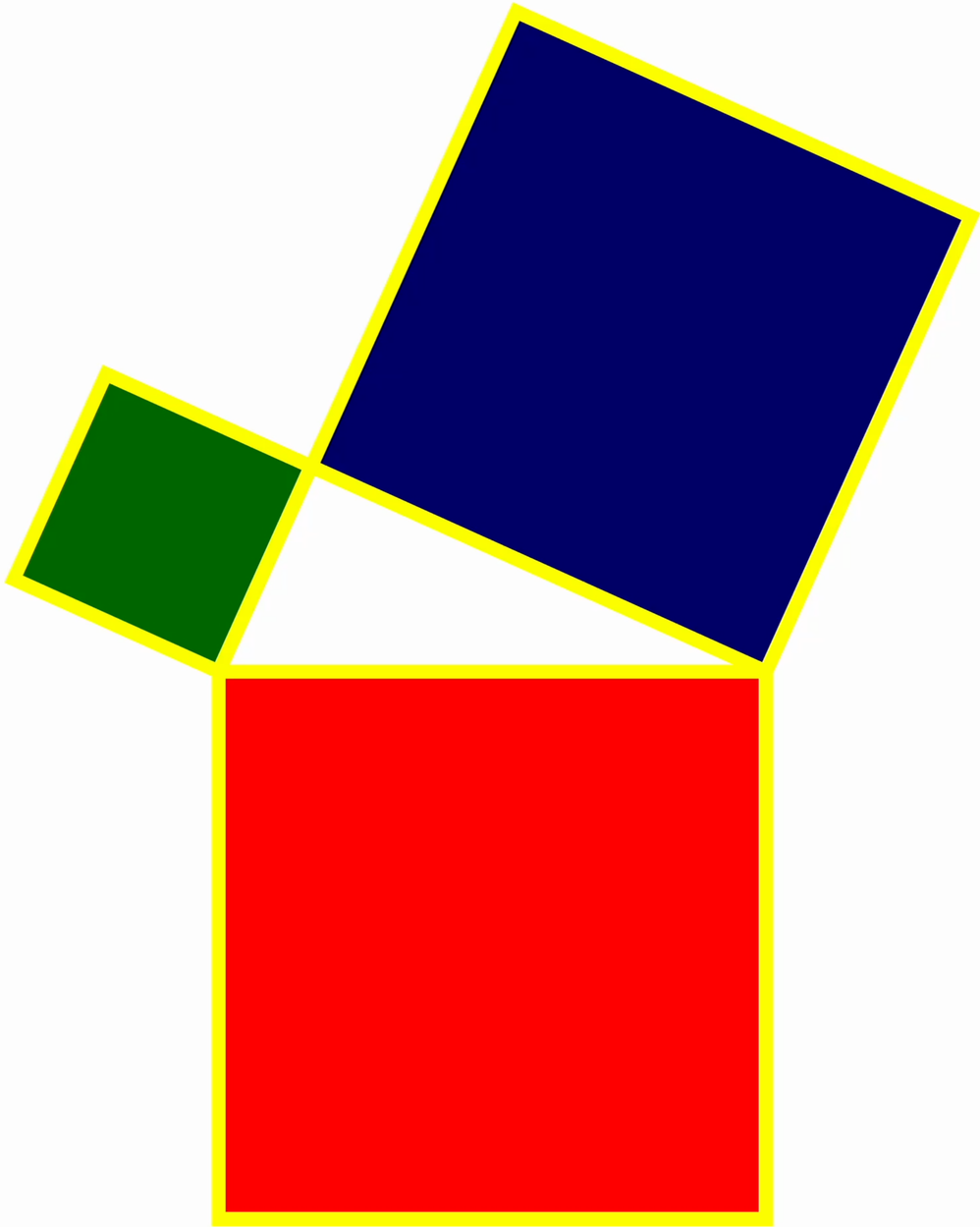
$$c = 1, b^2 = x^2 + y^2, a^2 = (1-x)^2 + y^2, q = 1 - x, u^2 = \frac{(1-x)^2}{(1-x)^2 + y^2}$$

Simplify q*c-u*a.  QED

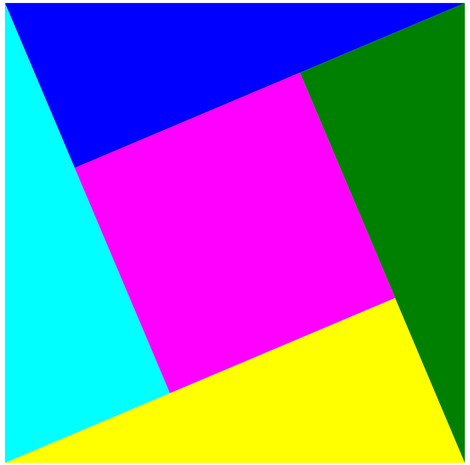
q c=u a



One of the possible first proofs by Pythagoras?

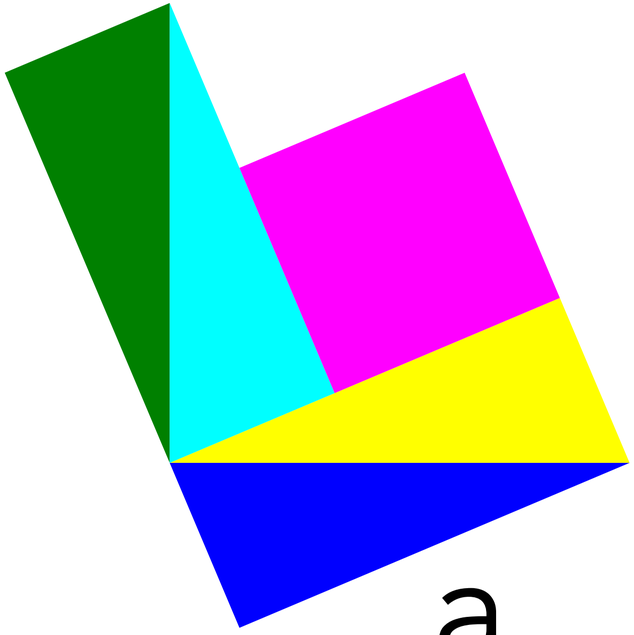


# An other possible first proof.



c

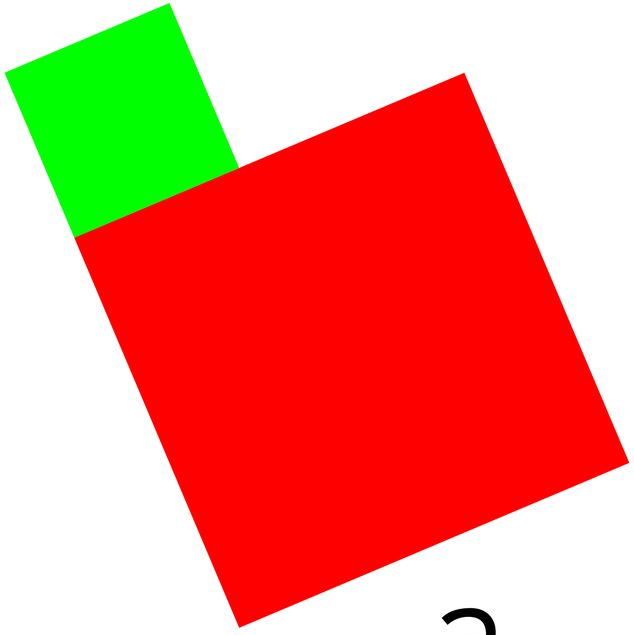
Cut into
5 pieces



a

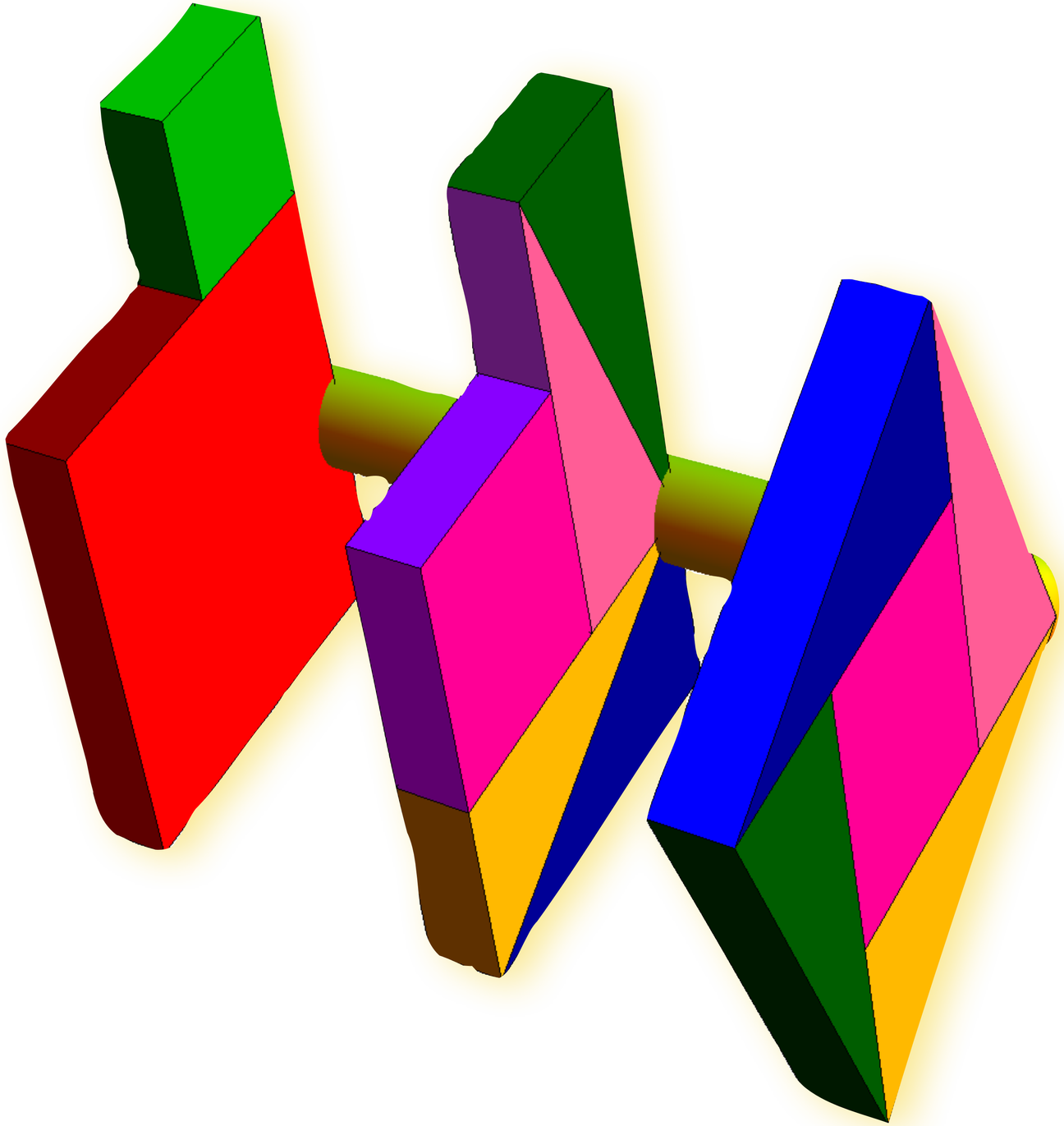
Rearrange
by translation

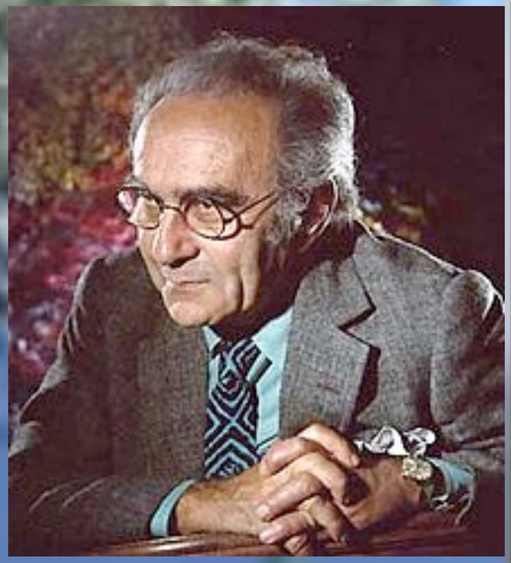
b



a

Join and
cut into two





*The Ascent of man, The Music of the Spheres,*

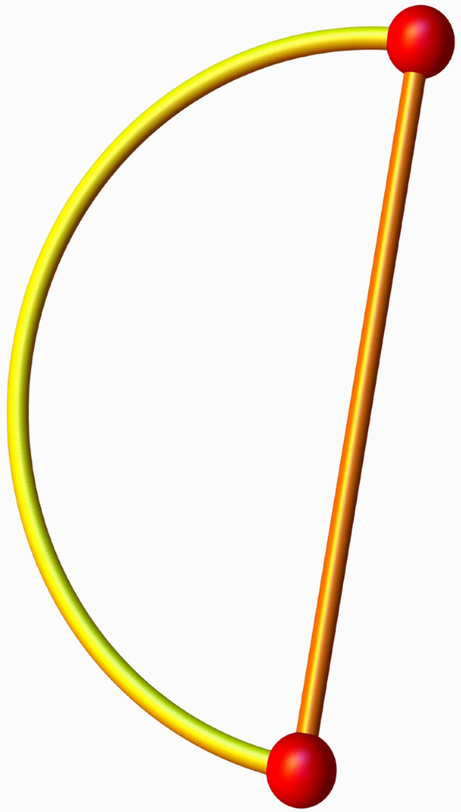
*BBC 1973, Jacob Bronowski 1908-1974*

500 BC
Pythagoreans

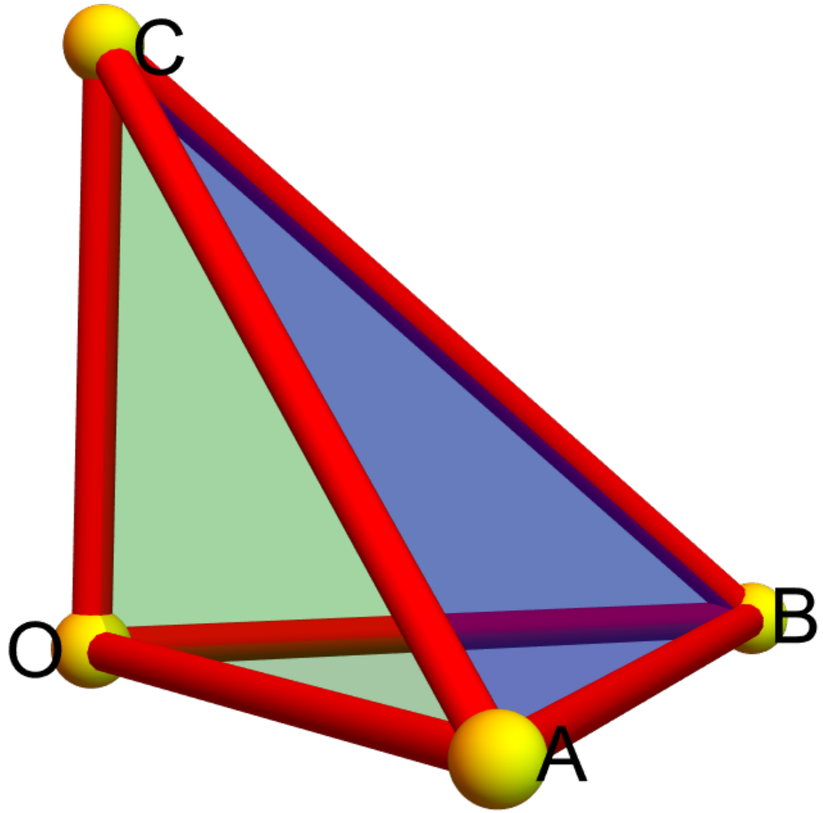
# The time of Pythagoras

# Two nice consequences of Pythagoras

The Lunes of Hippocrates



3D Pythagoras



# Why is this theorem beautiful?

New: 2023

Simple

Historical context

Surprise

Geometry Gem

# Caveats

Applicability?

We gave a proof using Pythagoras

# A SECOND BEAUTIFUL THEOREM

# Polarization Identity

Given n variables $x_1, \ldots, x_n$ in commutative ring R with 1.

Define $S = \{-1,1\}^n$ the set of all 0-1-strings of length n.

$$n! \prod_{i=1}^{n} x_i = |S|^{-1} \sum_{s \in S} \left( \prod_j s_j \right) \left( \sum_i s_i x_i \right)^n$$

Take n=2, variables x,y. Symmetry set S={ {1,1},{-1,1},{1,-1},{-1,-1}}

$$2xy = \frac{1}{4}[(x+y)^2 - (x-y)^2 - (-x+y)^2 + (-x-y)^2]$$

- Inner products are determined from lengths
- Length measurements determine angles
- Covariance can be determined from Variances
- Ricci(v,v) determines the Ricci tensor

# Why is this theorem beautiful?

Simple proof

Not well known

Application context

Algebra Gem

# Caveats

Looks a bit messy at first

Not so well known

# A THIRD BEAUTIFUL THEOREM

All binary operations xy, x/y, x+y, x-y, $x^y$ can be reduced to one single binary operation

# eml(x,y)= exp(x)-log(y)

Proof:

eml(1,x) = e-log(x)

eml(x,1) = exp(x)

eml(1,eml(eml(1,x),1)) = e-(e-log(x)) = log(x)

eml(log(x),exp(y))= x-y

eml(log(x),exp(-y)) = x+y

eml(log(x) + log(y),1) = x y

eml(log(x) - log(y),1) = x/y

Andrzej Odrzywołek 2026

# All elementary functions from a single operator

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E-mail: andrzej.odrzywolek@uj.edu.pl

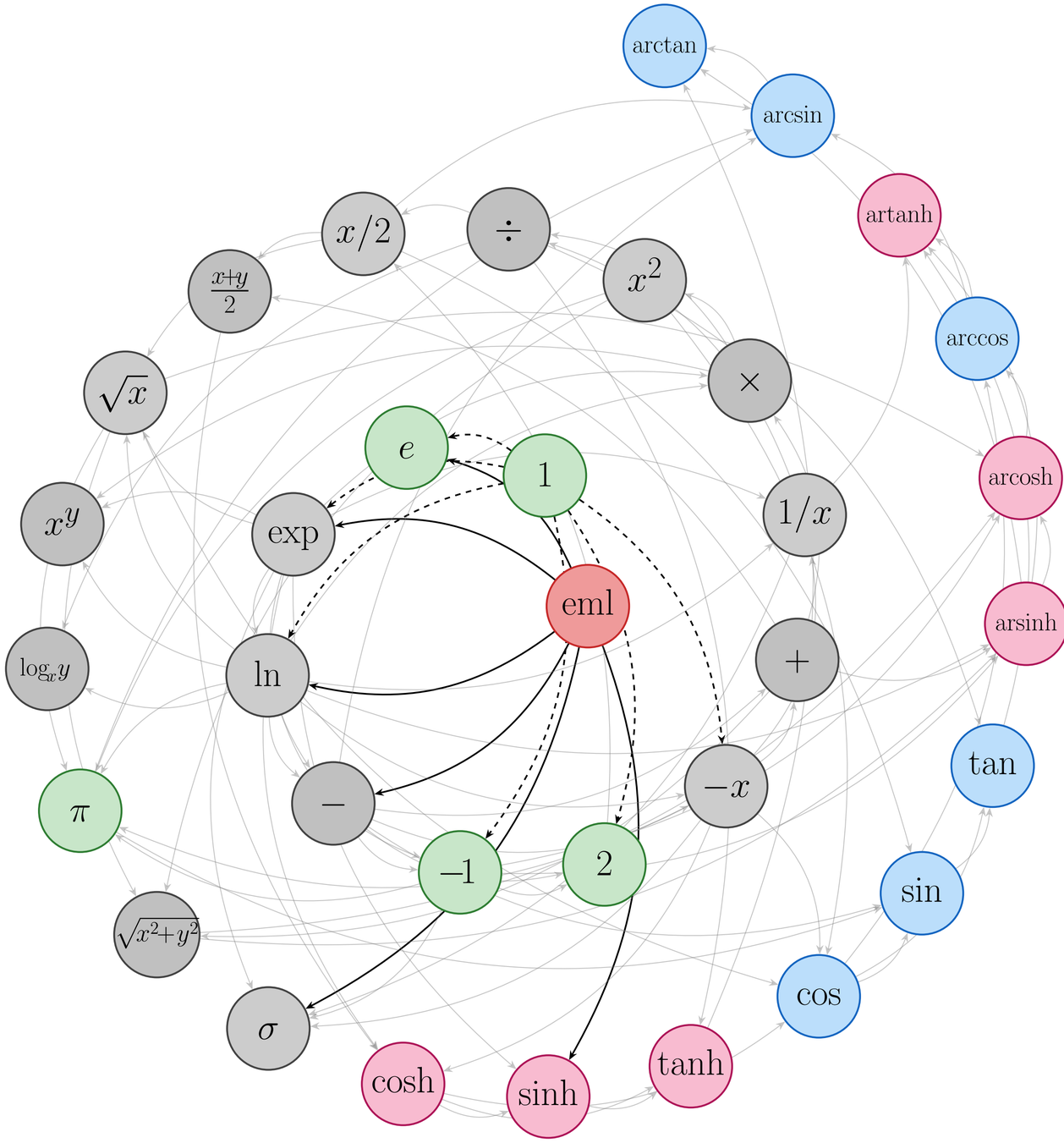
April 7, 2026

**Abstract**

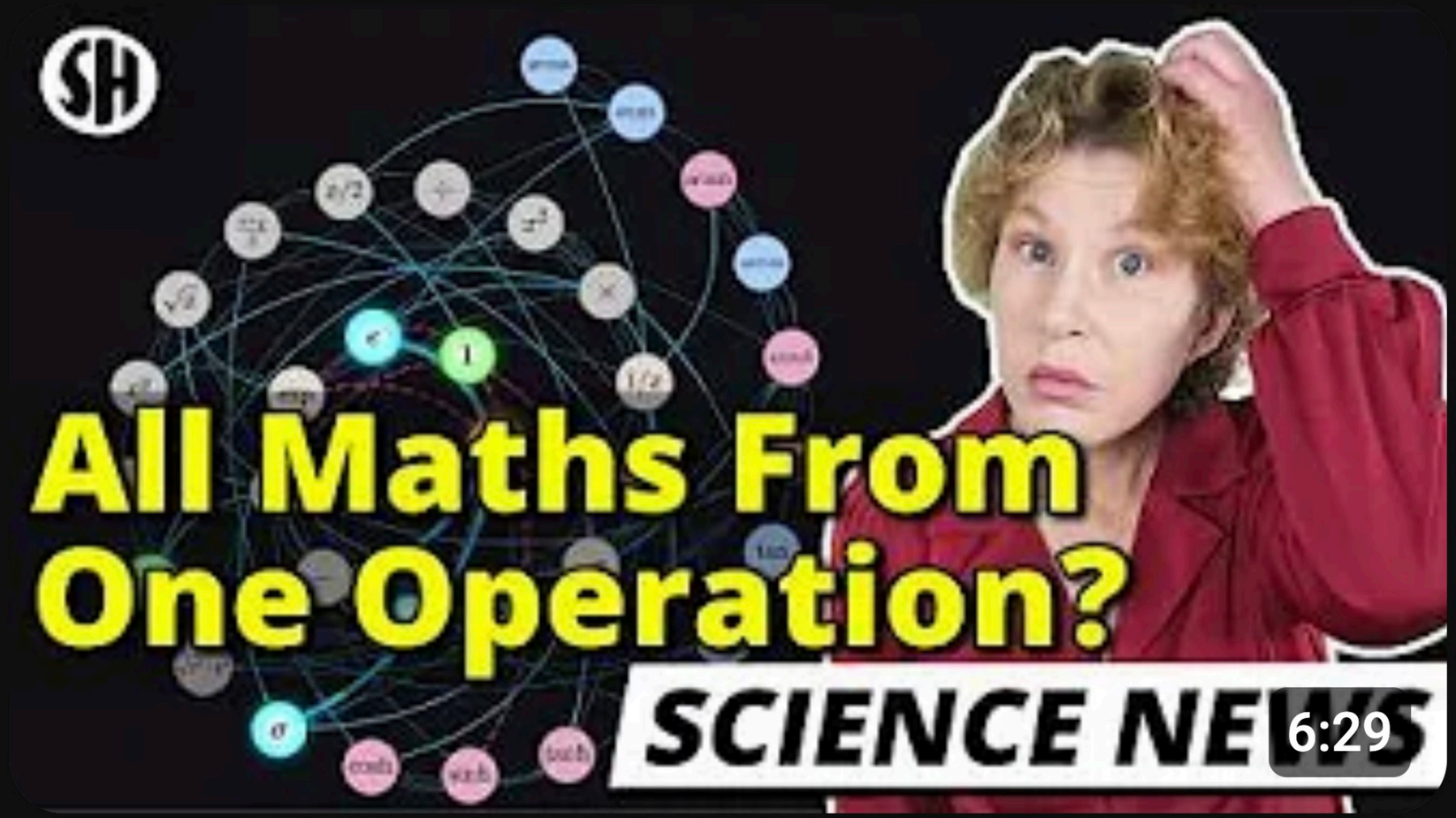
A single two-input gate suffices for all of Boolean logic in digital hardware. No comparable primitive has been known for continuous mathematics: computing elementary functions such as sin, cos, $\sqrt{\ }$, and log has always required multiple distinct operations. Here we show that a single binary operator,

$$\mathrm{eml}(x, y) = \exp(x) - \ln(y),$$

together with the constant 1, generates the standard repertoire of a scientific calculator. This includes constants such as $e$, $\pi$, and $i$; arithmetic operations including $+$, $-$, $\times$, $/$, and exponentiation as well as the usual transcendental and algebraic functions. For example, $e^x = \mathrm{eml}(x, 1)$, $\ln x = \mathrm{eml}(1, \mathrm{eml}(\mathrm{eml}(1, x), 1))$, and likewise for all other operations. That such an operator exists was not anticipated; I found it by systematic exhaustive search and established constructively that it suffices for the concrete scientific-calculator basis. In EML (Exp-Minus-Log) form, every such expression becomes a binary tree of identical nodes, yielding a grammar as simple as $S \to 1 \mid \mathrm{eml}(S, S)$. This uniform structure also enables gradient-based symbolic regression: using EML trees as trainable circuits with standard optimizers (Adam), I demonstrate the feasibility of exact recovery of closed-form elementary functions from numerical data at shallow tree depths up to 4. The same architecture can fit arbitrary data, but when the generating law is elementary, it may recover the exact formula.



Home Videos Shorts Playlists Posts



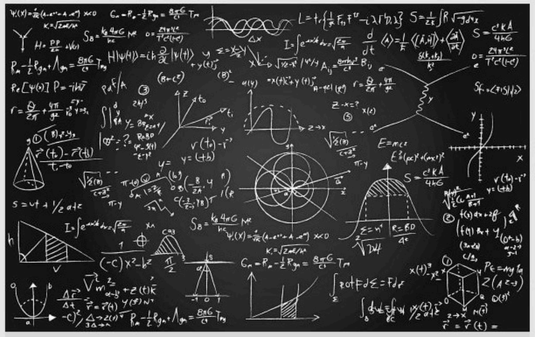

Mathematician Collapses All Functions to One

ÜBERRASCHEND WENIG

# Die Mathematik lässt sich auf eine einzige Grundrechnungsart reduzieren

Eine einzelne Funktion und die Zahl Eins genügen, um daraus die wichtigsten Konstanten und Funktionen auf jedem Taschenrechner zu gewinnen

Mathematik ist, in all ihrer Komplexität, eigentlich erstaunlich simpel. Aus einer Reihe von Zahlen, die sich an Fingern abzählen lassen, und wenigen Rechenoperationen, die Kinder bereits in ihren ersten Lebensjahren in der Schule lernen, lässt sich Schritt für Schritt die Mathematik aufbauen – inklusive komplexer Zahlen, unendlichdimensionaler Funktionenräume und allem, was man sonst so an Handwerkszeug braucht, um etwa mittels Physik die Welt bis ins kleinste Detail zu beschreiben. Was nicht heißen soll, dass sich die Möglichkeiten der Mathematik nur darauf beschränken.



In der Schule lernen wir Grundrechenarten und darauf aufbauend immer komplexere Methoden. Eine einzelne Grundrechenart wäre dabei ausreichend, wenn auch nicht einfacher.

Getty Images


## Willkommen bei DER STANDARD

**Sie entscheiden darüber, wie Sie unsere Inhalte nutzen wollen. Ihr Gerät erlaubt uns derzeit leider nicht, die entsprechenden Optionen anzuzeigen.**

**Bitte deaktivieren** Sie sämtliche Hard- und Software-Komponenten, die in der Lage sind Teile unserer Website zu blockieren. Z.B. Browser-AddOns wie Adblocker oder auch netzwerktechnische Filter.

Sie haben ein PUR-Abo? Hier anmelden.

# Why is this theorem beautiful?

Elegant

Not well known

Fresh: 2026

Nice Exercise

# Caveats

Use high level stuff to get basic stuff.

Over-hyped by media

# A FOURTH BEAUTIFUL THEOREM

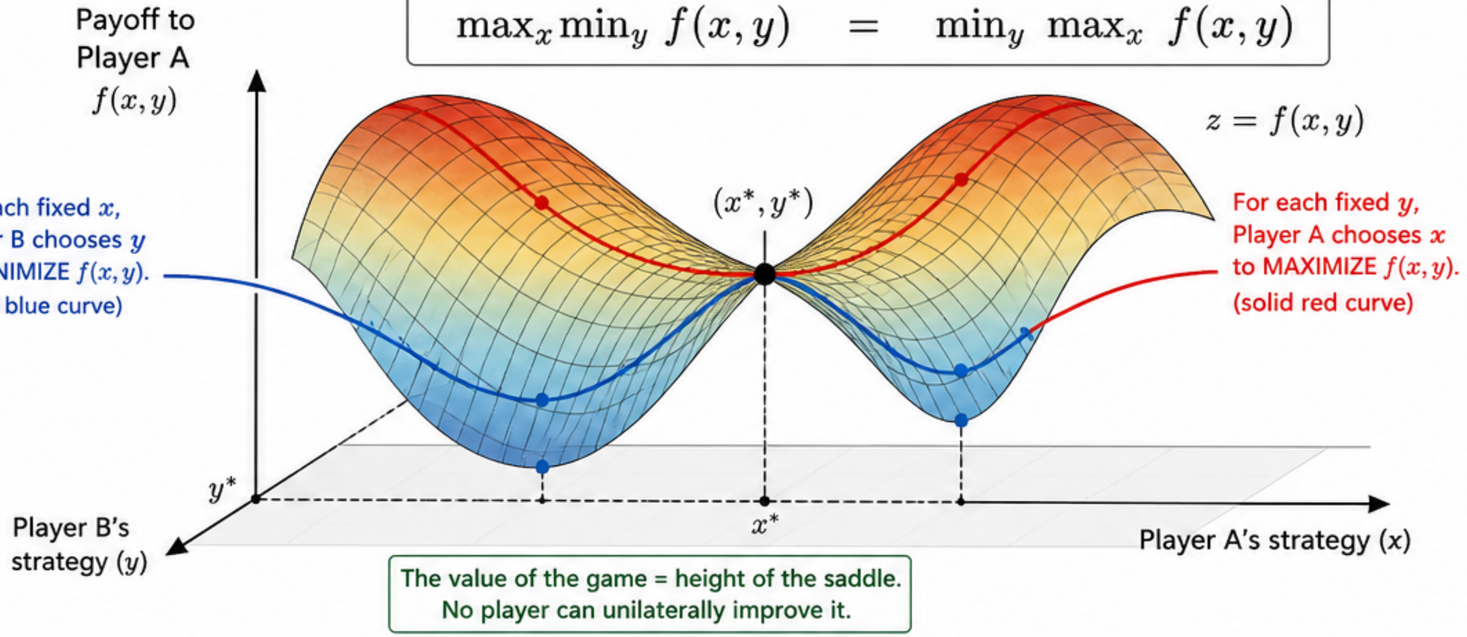
# Von Neumann's Mini-Max Theorem

$P^n$ set of probability vectors in dimension n. A any $m \times n$ matrix

$$\max_{x \in P^m} \min_{y \in P^n} x^T A y = \min_{y \in P^n} \max_{x \in P^m} x^T A y$$

# von Neumann Minimax Theorem

$$\max_x \min_y f(x,y) \quad = \quad \min_y \max_x f(x,y)$$




The value of the game = height of the saddle.
No player can unilaterally improve it.

At the saddle point $(x^*, y^*)$:

- in the $x$-direction: minimum (no player B can do better)
- in the $y$-direction: maximum (no player A can do better)
- both values are equal:

$$\max_x \min_y f(x,y) = \min_y \max_x f(x,y)$$

## 1. FINITE GAME (Payoff Matrix)

Payoff to Player A

Player B (columns)

| Player A (rows) | $B_1$ | $B_2$ | $\cdots$ | $B_n$ | row mins |
|---|---|---|---|---|---|
| $A_1$ | $a_{11}$ | $a_{12}$ | $\cdots$ | $a_{1n}$ | $r_1$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | $\cdots$ | $a_{2n}$ | $r_2$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\vdots$ |
| $A_m$ | $a_{m1}$ | $a_{m2}$ | $\cdots$ | $a_{mn}$ | $r_m$ |
| | | | | | mins |

For each row (A's pure strategy): take the MIN in that row

maximin $= \max_i r_i$

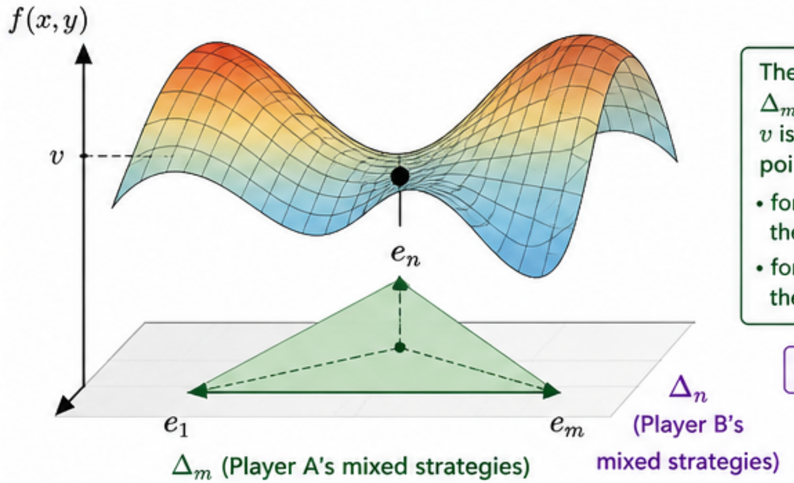
For each column (B's pure strategy): take the MAX in that column

$c_1$ $c_2$ $\cdots$ $c_n$

minimax $= \min_j c_j$

**von Neumann theorem: maximin = minimax**

## 2. MIXED STRATEGIES (Geometric View)




The payoff is bilinear on $\Delta_m \times \Delta_n$ and the value $v$ is attained at a saddle point $(x^*, y^*)$:

- for fixed $x^*$, $y^*$ minimizes the payoff;
- for fixed $y^*$, $x^*$ maximizes the payoff.

Value of the game = $v$

**Interpretation:** Player A wants to lift the payoff up; Player B wants to push it down. At the saddle $(x^*, y^*)$, both are optimally balanced.

GPT 5.4

$$\max_{x \in P^m} \min_{y \in P^n} x^T A y = \min_{y \in P^n} \max_{x \in P^m} x^T A y$$

$$\exists x^* \quad \exists y^* \; \exists c$$

$$\min_{y \in P^n} (x^*)^T A y \geq c \geq \max_{x \in P^m} x^T A y^*$$

$$\min_{y \in P^n} x^T A y \leq \max_{x \in P^m} x^T A y$$

# Beautiful Proof by Beautiful mind:

$X \times Y = P^n \times P^m$ is compact, contractible.

$(x, y) \in X \times Y$ gives pay-off $f = x^T A y$

$\alpha_i = \max\{0, (Ay)_i - f\}, \quad i = 1, \ldots, m$ positive if row strategy >than payoff

$\beta_j = \max\{0, (x^T A)_j - f\}, \quad j = 1, \ldots, n$ positive if column strategy >than payoff

$$u_i(x, y) = \frac{x_i + \alpha_i}{1 + \sum_{k=1}^{m} \alpha_k}, \quad i = 1, \ldots, m,$$

$$v_j(x, y) = \frac{y_j + \beta_j}{1 + \sum_{\ell=1}^{n} \beta_\ell}, \quad j = 1, \ldots, n$$

defines continuous map T(x,y) =(u,v)

Brouwer fixed point theorem: exists (x,y) = T(x,y) . This implies $\alpha_i = 0, \beta_j = 0$

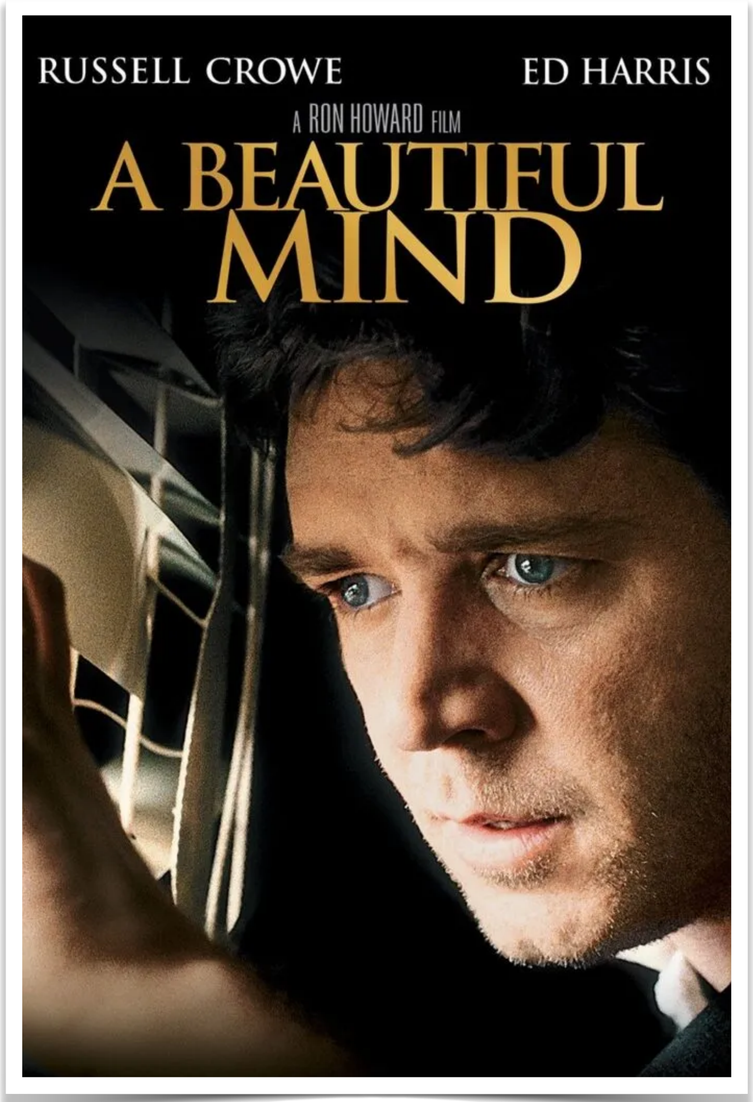
# Why is this theorem beautiful?

Important

First of its kind

Elegant

Nice proof by beautiful mind



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# PART 3: HOW TO WE DEFINE BEAUTY?

Ioannis Keppleri

# HARMONICES MVNDI

## LIBRI V. QVORVM

Primus GEOMETRICVS, De Figurarum Regularium, quæ Proportiones Harmonicas constituunt, ortu & demonstrationibus.

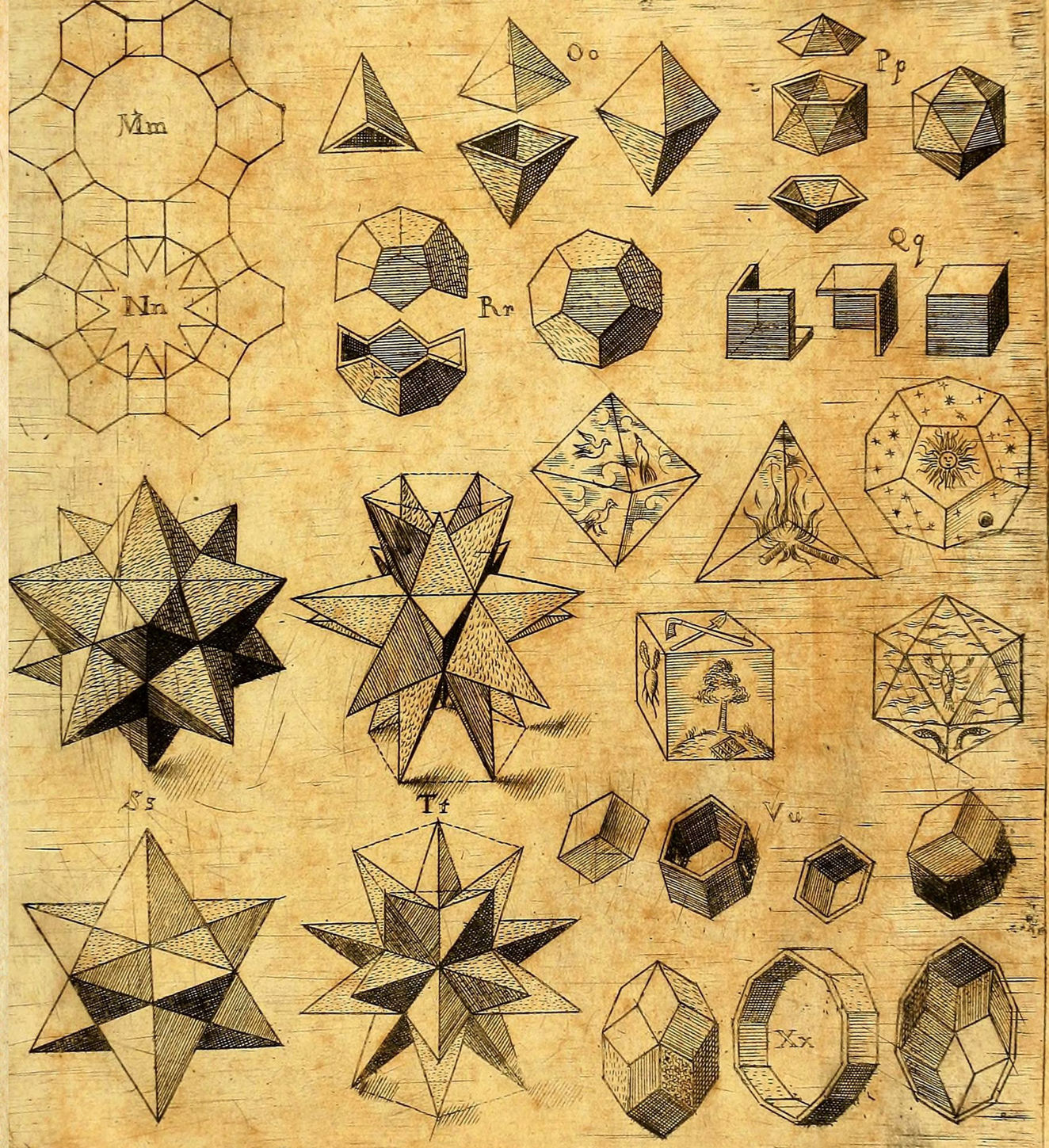
Secundus ARCHITECTONICVS, seu ex GEOMETRIA FIGVRATA, De Figurarum Regularium Congruentia in plano vel solido:

Tertius propriè HARMONICVS, De Proportionum Harmonicarum ortu ex Figuris; deque Naturâ & Differentiis rerum ad cantum pertinentium, contra Veteres:

Quartus METAPHYSICVS, PSYCHOLOGICVS & ASTROLOGICVS, De Harmoniarum mentali Essentiâ earumque generibus in Mundo; præsertim de Harmonia radiorum, ex corporibus cœlestibus in Terram descendentibus, eiusque effectu in Natura seu Anima sublunari & Humana:

Quintus ASTRONOMICVS & METAPHYSICVS, De Harmoniis absolutissimis motuum cœlestium, ortuque Eccentricitatum ex proportionibus Harmonicis.

Appendix habet comparationem huius Operis cum Harmonices Cl. Ptolemæi libro III. cumque Roberti de Fluctibus, dicti Flud. Medici Oxonienſis ſpeculationibus Harmonicis, operi de Macrocosmo & Microcosmo insertis.

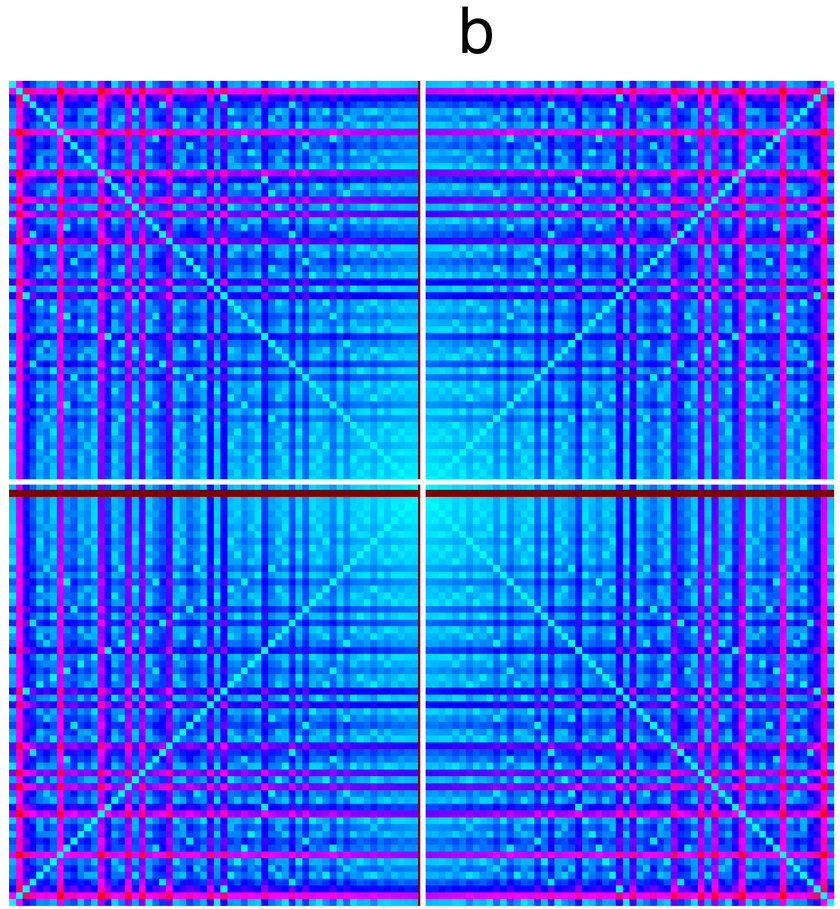


# Euler's Gradus Suavitatis

$$G(a/b) = 1+\sum_{p \text{ in } a*b} (p-1)$$

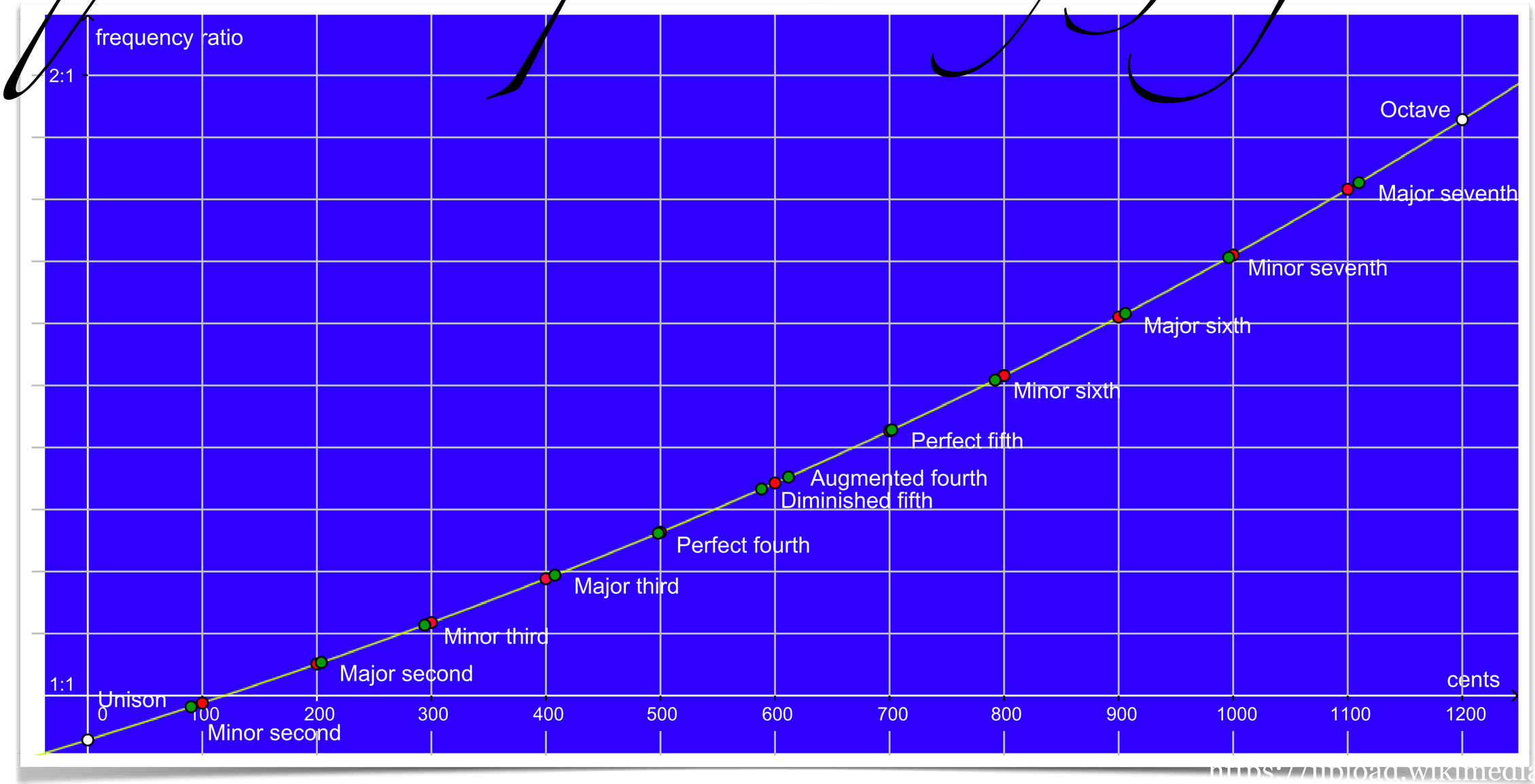
$$G(5/12) = 1+2(2-1)+1\,(3-1)+1\,(5-1)=9$$

n(p) is multiplicity with which prime p appears in a/b after factoring out common divisors




```
G[a_,b_]:=Module[{s=FactorInteger[LCM[a,b]/GCD[a,b]]},
1+Sum[s[[k,2]]*(s[[k,1]]-1),{k,Length[s]}]]
```

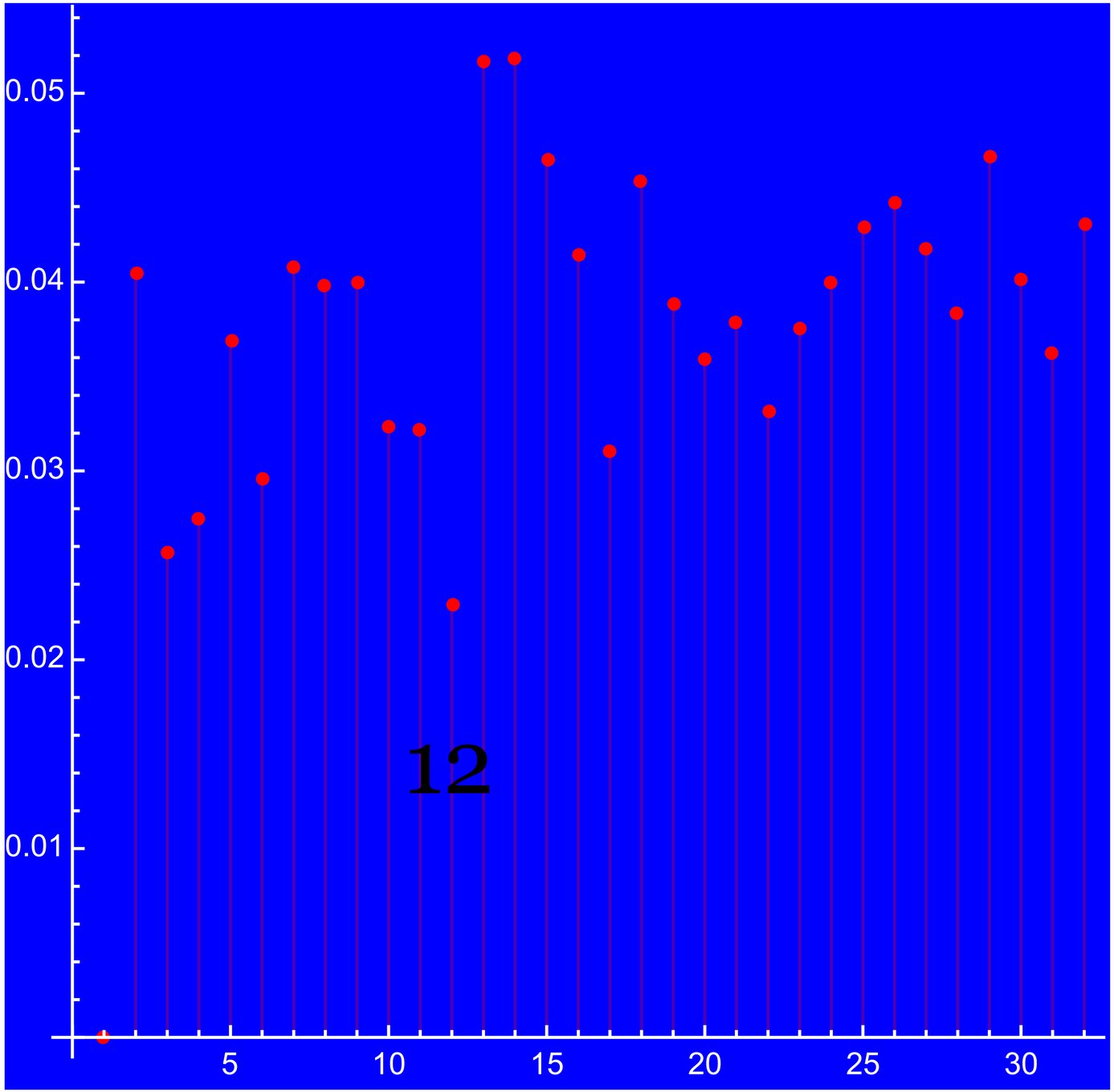
# Equal Tempered-Pythagorean




https://upload.wikimedia

# Why 12 ?

The distance to small rational numbers is measured using Euler Gradus Suavitats

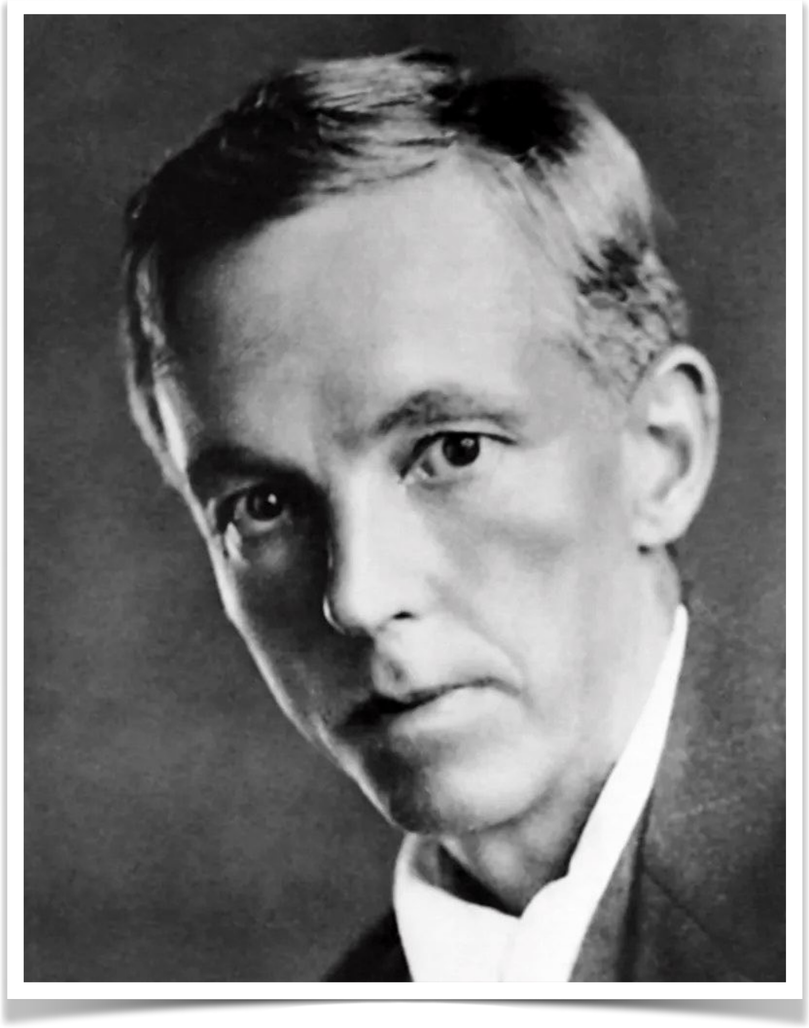



http://www.mathematik.com/Piano/

# Gian-Carlo Rota (1932-1999)

"Beauty is a rather objective property which depends on historic-social contexts."

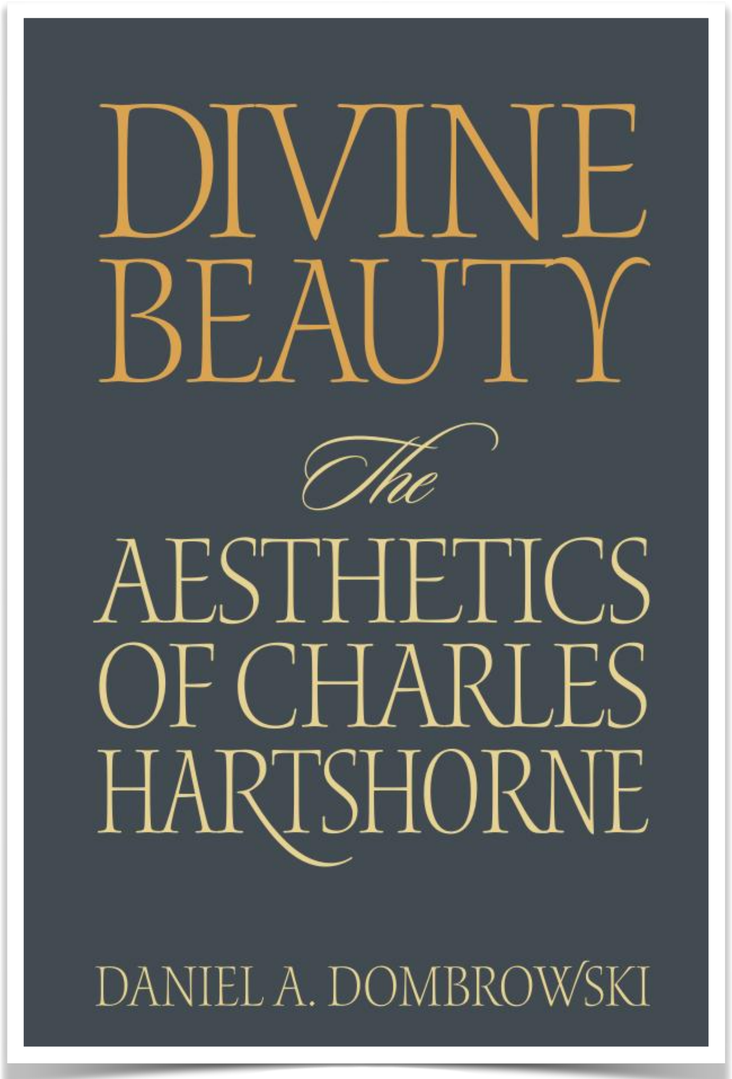
# Godfrey Harold Hardy (1877-1947)



"Beauty is the first test. There is no permanent place in the world for ugly mathematics"

# Charles Hartshorne

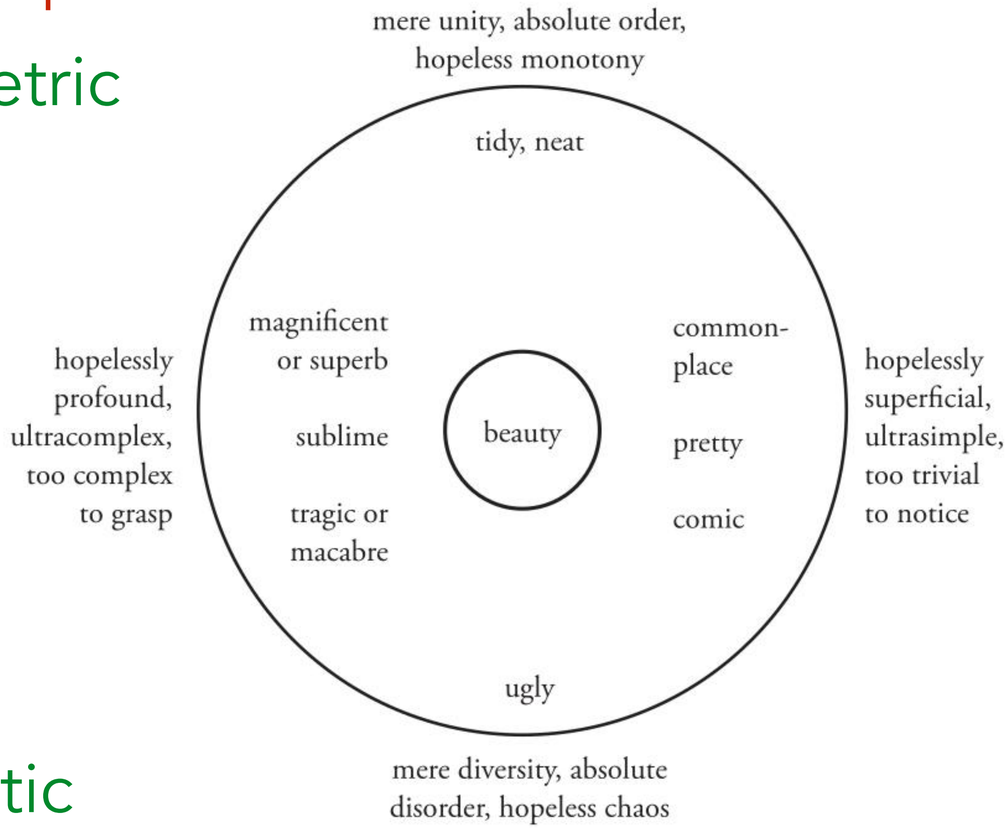
Charles Hartshorne was born in the nineteenth century and lived to philosophize in the twenty-first. Perhaps the most neglected aspect of his extensive and highly nuanced thought is his aesthetics, a discipline within philosophy to which he contributed as early as the 1920s in his Harvard doctoral dissertation (he minored in English literature at Harvard). His efforts in aesthetics quite incredibly lasted into the 1990s, over half a century after he cofounded the American Society for Aesthetics (and its associated journal, *The Journal of Aesthetics and Art Criticism*) in 1942 (HL, 49).



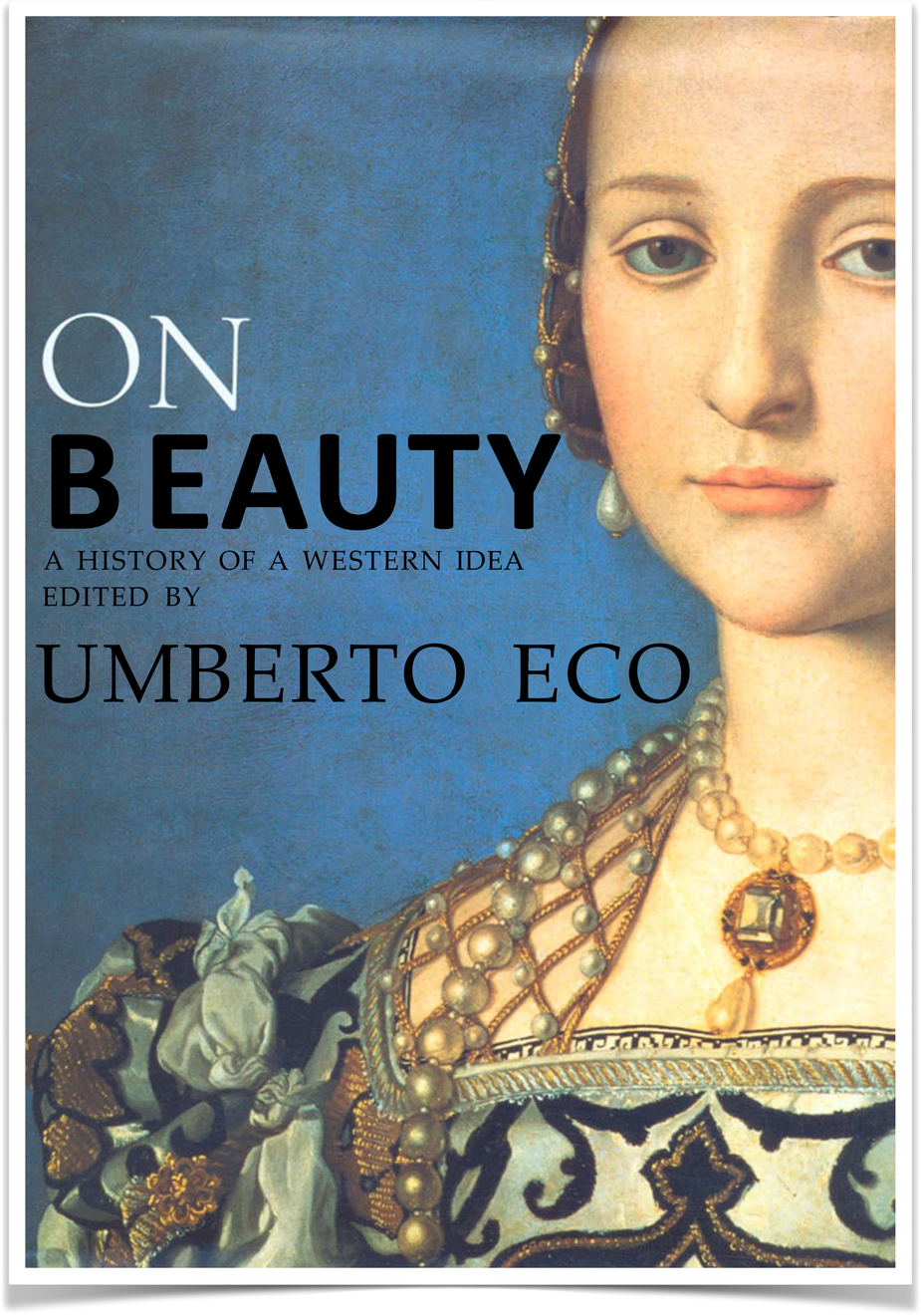



complex ← → trivial

symmetric ↑ ↓ chaotic

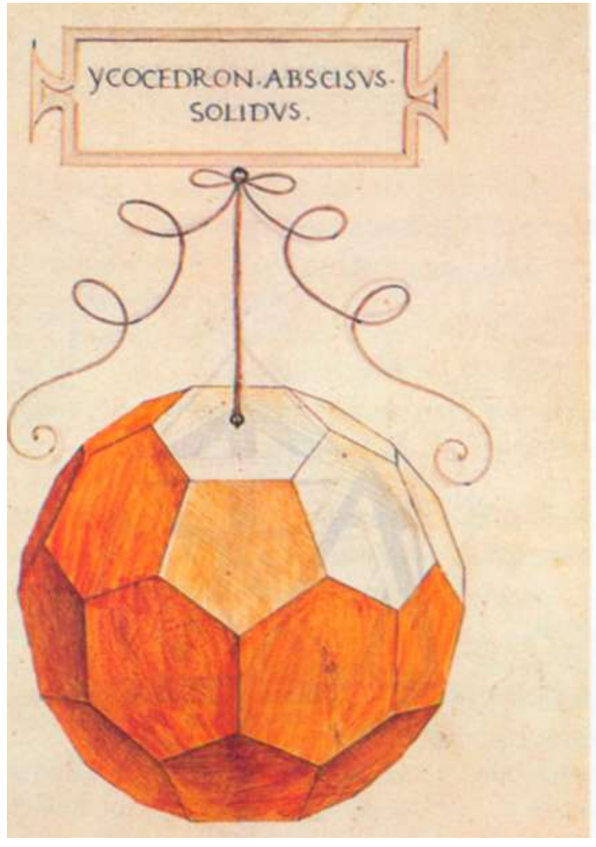



It should also be noted that there is an imaginary horizontal axis. A musical chord or a little flower is a superficial harmony, in contrast to a Mahler symphony or an old-growth forest. Deviations from beauty, however, when seen in a Hartshornian way as the central aesthetic value and as a double mean between vertical and horizontal extremes,




# Umberto Eco:

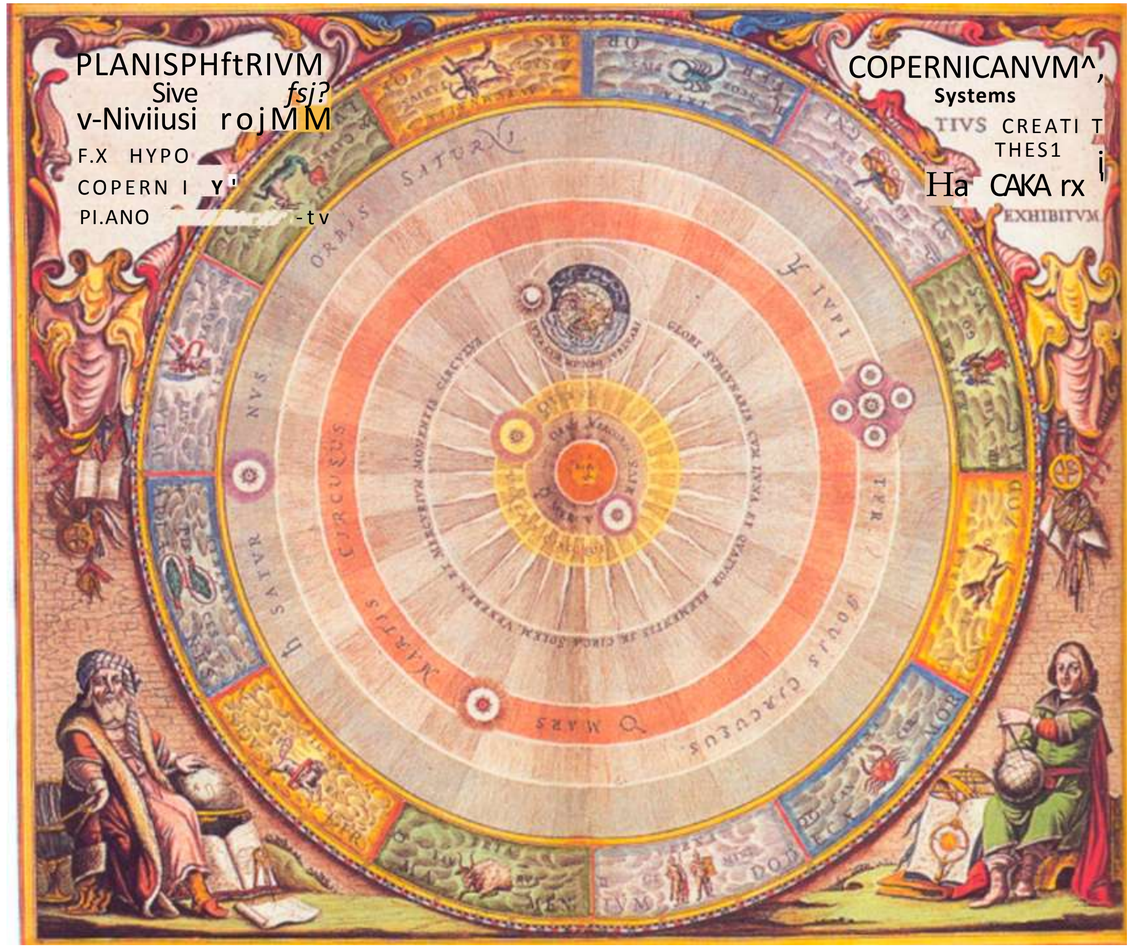
# Notions of Beauty shift with time



# From Eco's book

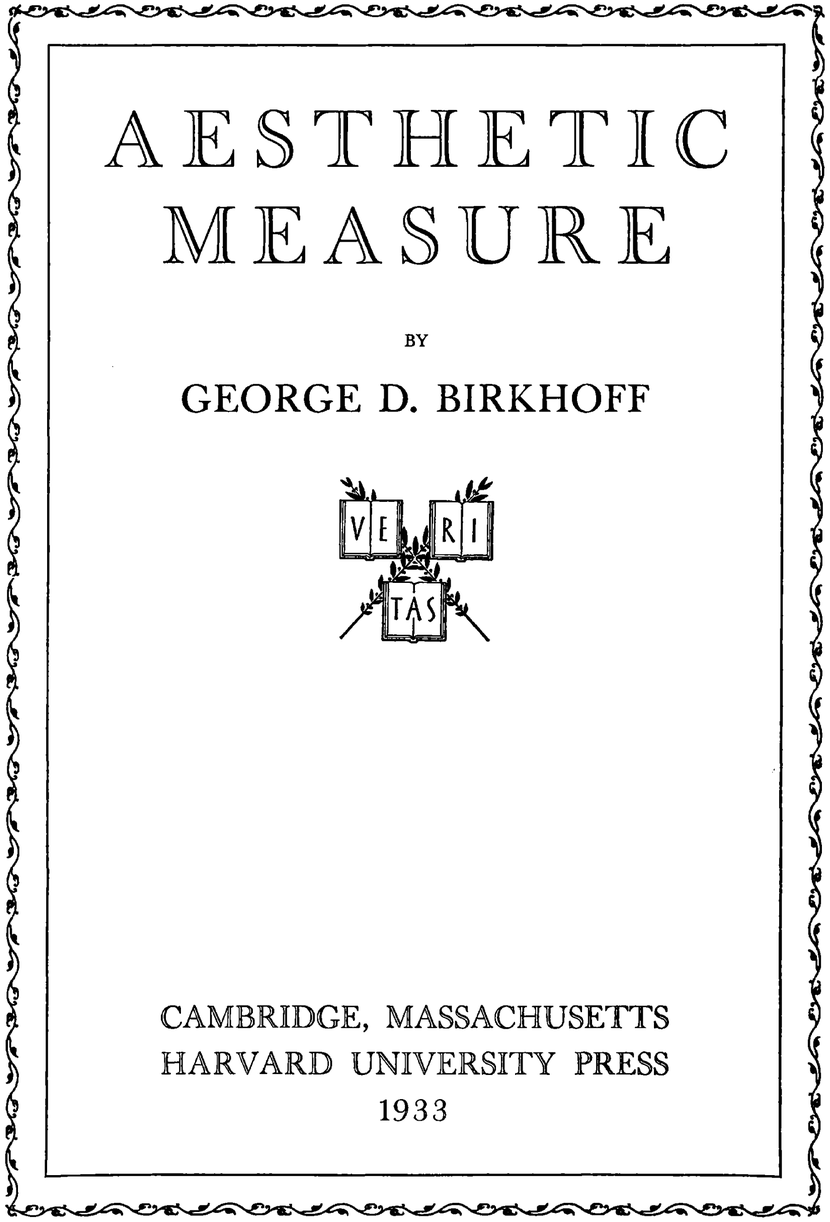
# Proportion and Harmony

III. BEAUTY AS PROPORTION AND HARMONY




Andreas Cellarius,
*Harmonia Macrocosmica,*
Amsterdam,
1660

# Birkhoff, 1933


AESTHETIC
MEASURE

BY

GEORGE D. BIRKHOFF

VE RI TAS

CAMBRIDGE, MASSACHUSETTS
HARVARD UNIVERSITY PRESS
1933

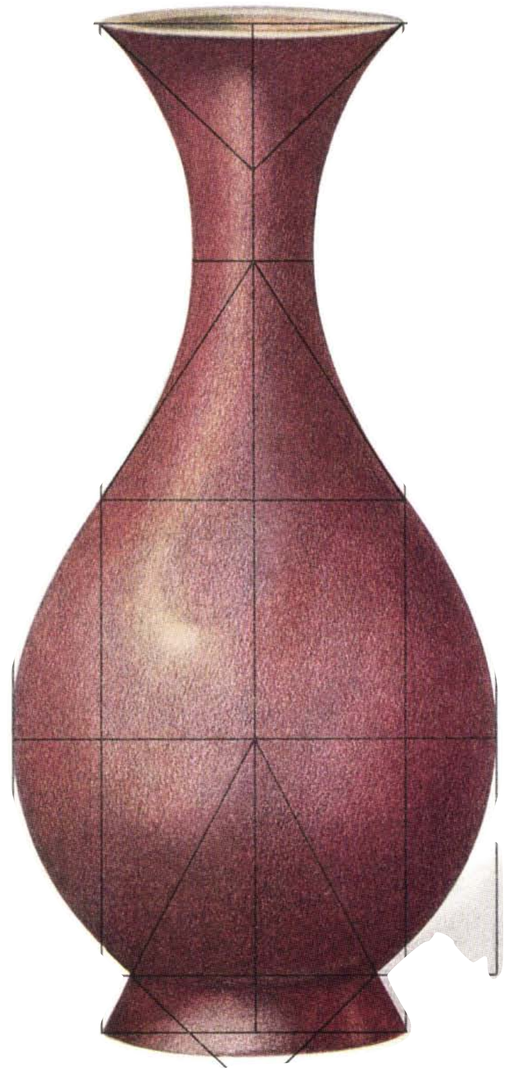
$$M = \frac{O}{C}$$

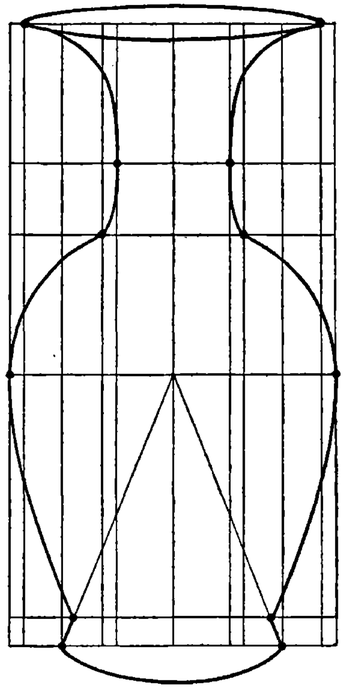
Beauty = Order / Complexity

George Birkhoff, 1881-1944

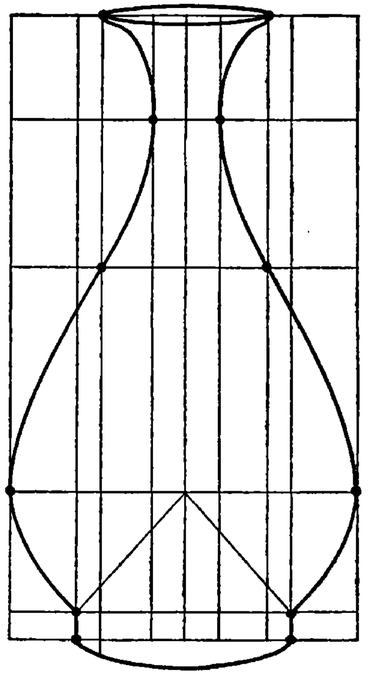
BA Harvard1905 MA. in 1906.

1920-1928 occupied by George Birkhoff

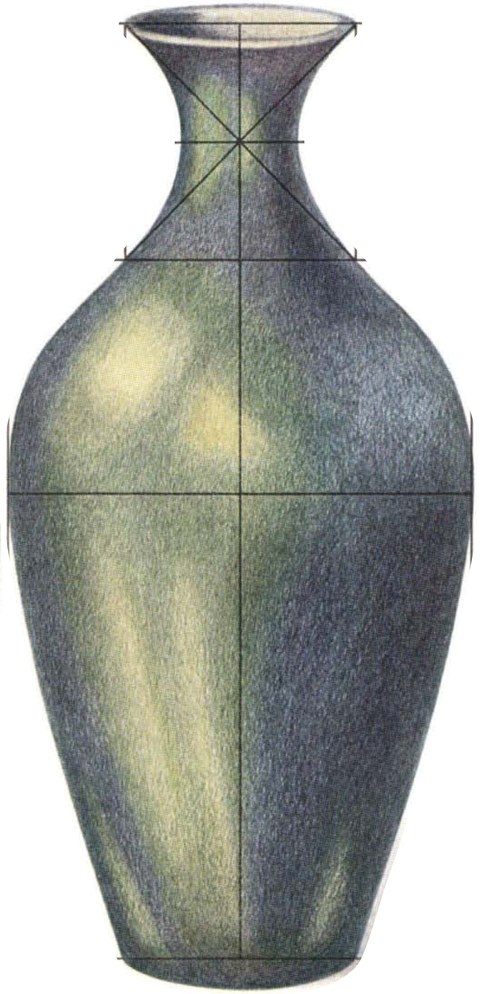


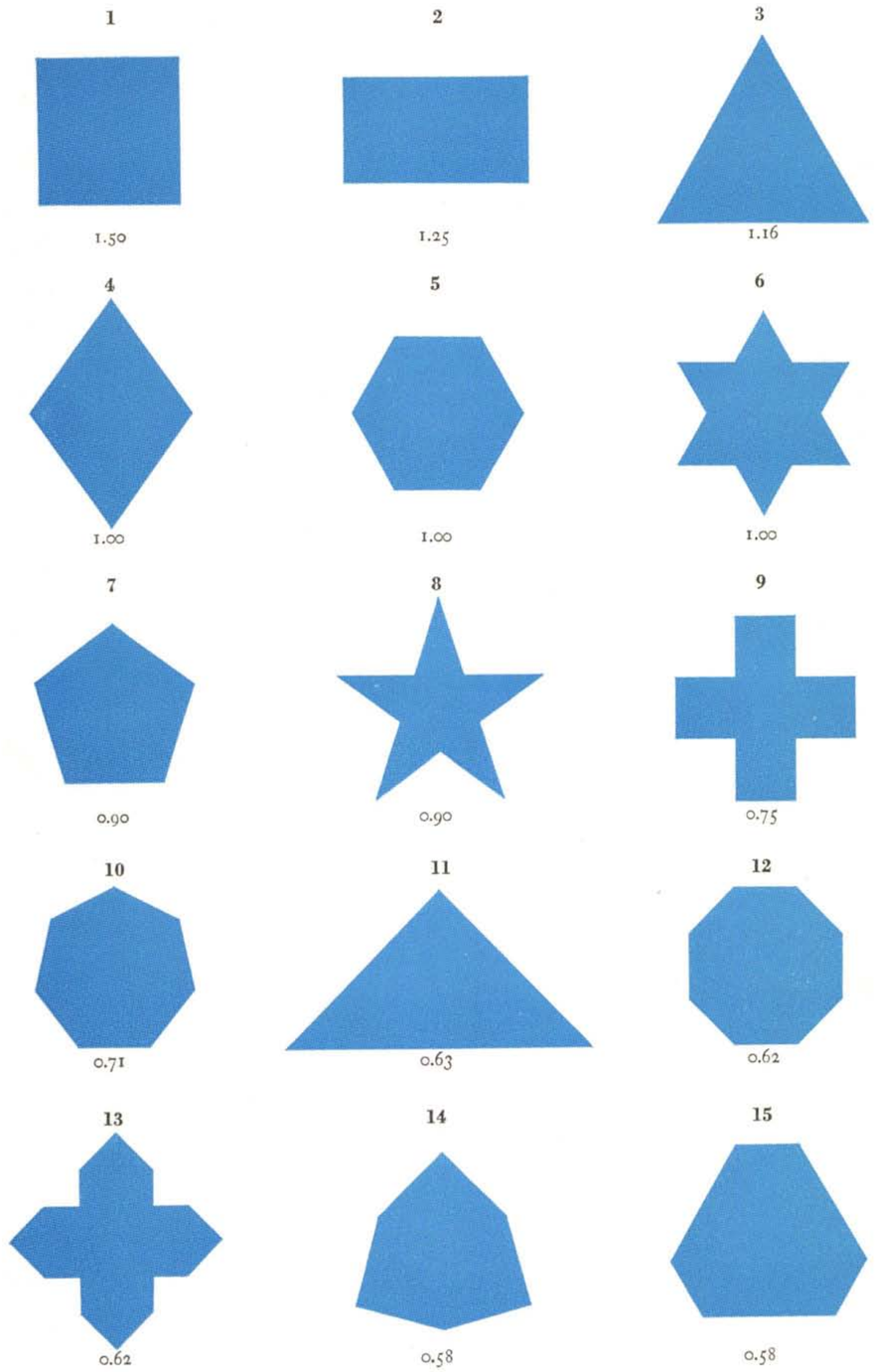


XXV. M = 0.58

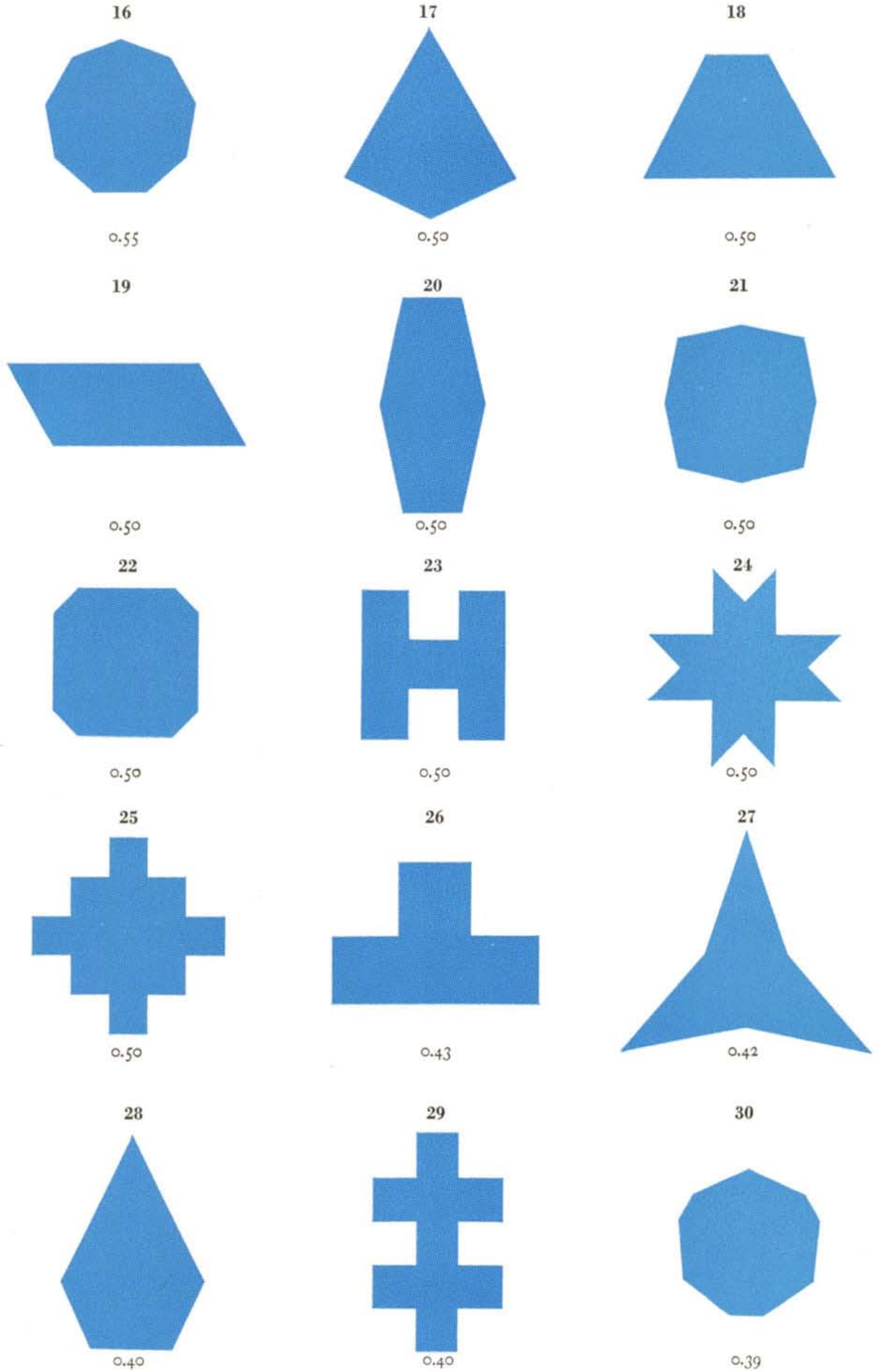


XXI. M = 0.50






Aesthetic Measures of 90 Polygonal Forms, Nos. 1–15




Aesthetic Measures of 90 Polygonal Forms, Nos. 16–30

Brought to you by | University of Arizona
Authenticated

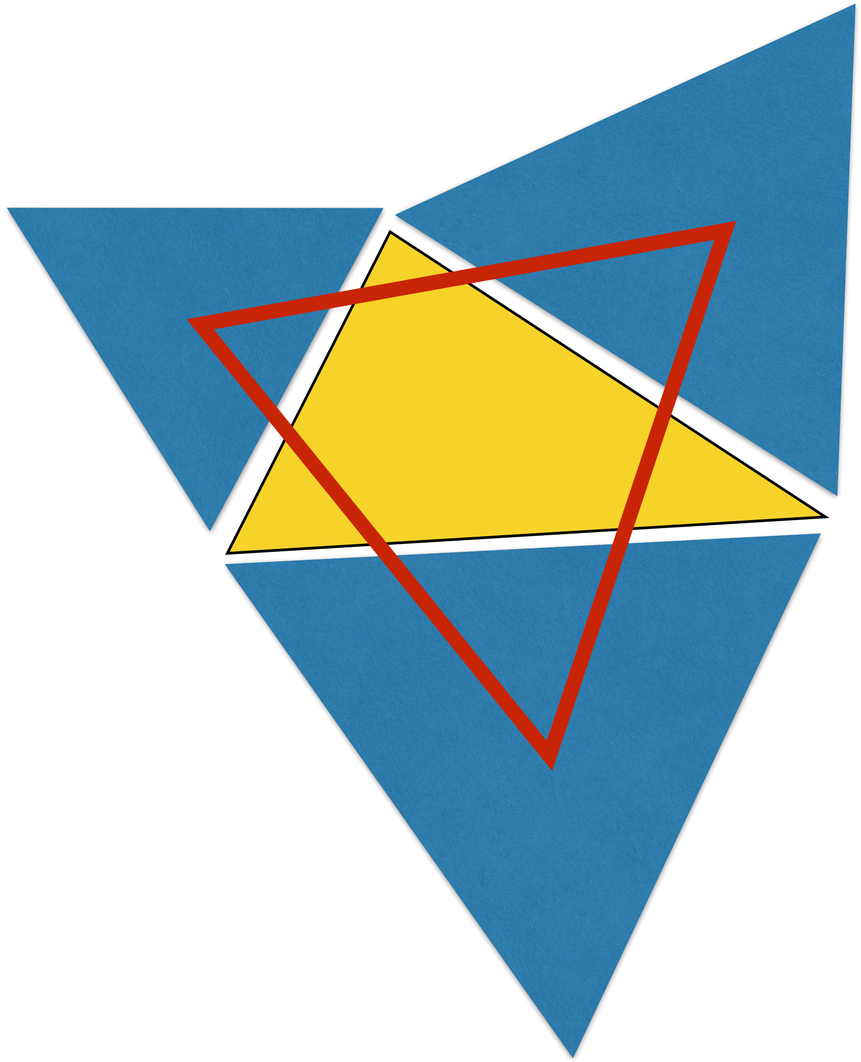
# SOCIAL CONTEXT

An injective holomorphic function from the unit disk to the complex plane with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

satisfies $|a_n| \leq n$

De Brange theorem
Biberbach conjecture



Napoleon's theorem
Most rediscovered

# PRETTY PICTURES

# Gallery of Singular Algebraic Surfaces

Herwig Hauser, Universities of Innsbruck & Vienna, Austria

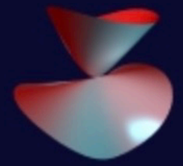
---

**Calyx**

$x^2+y^2z^3 = z^4$

**Calypso**

$x^2+y^2z = z^2$



**Columpius**

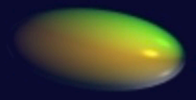
$x^3y+xz^3+y^3z+z^3+7z^2+5z=0$

**Cube**

$x^6+y^6+z^6=1$

**Dattel**

$3x^2+3y^2+z^2=1$

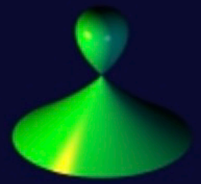


**Daisy**

$(x^2 - y^3)^2 = (z^2-y^2)^3$

**Dingdong**

$x^2 +y^2 +z^3 = z^2$

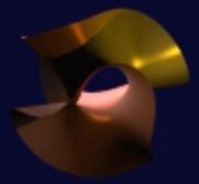


**Distel**

$x^2+y^2+z^2+1000 (x^2+y^2)(x^2+z^2)(y^2+z^2)=1$

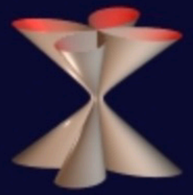
**Durchblick**

$x^3y+xz^3+y^3z+z^3+5z = 0$



**Eistüte**

$(x^2+y^2)^3 = 4x^2y^2(z^2+1)$



**Eve**

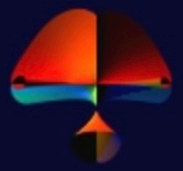
$0,5x^2 + 2xz^2 + 5y^6 + 15y^4 + 0,5z^2 = 15y^5 + 5y^3$

**Flirt**

$x^2-x^3+y^2+y^4+z^3-10z^4=0$

**Geisha**

$x^2yz + x^2z^2 = y^3z + y^3$



**Harlekin**

$x^3z + 10x^2y + xy^2 + yz^2 = z^3$

**Helix**

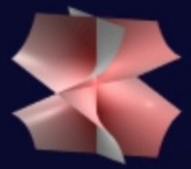
$6x^2 - 2x^4 = y^2z^2$

**Herz**

$y^2+z^3-z^4-x^2z^2 = 0$

**Himmel und Hölle**

$x^2-y^2z^2=0$



**Kolibri**

$x^3 + x^2z^2 - y^2=0$

**Leopold**

$100\ x^2y^2z^2+3x^2+3y^2+z^2=1$

**Octdong**

$x^2 + y^2 + z^4 = z^2$

Stern

$400(x^2y^2 + y^2z^2 + x^2z^2) + (x^2 + y^2 + z^2 - 1)^3 = 0$

Möbius

Sphäre

$x^2 + y^2 + z^2 = 1$

Limao

$x^2-y^3z^3 = 0$

Torus

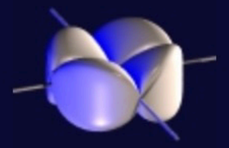
$(x^2+y^2+z^2+R^2-r^2)^2 = 4R^2(x^2+y^2)$

Whitney

$x^2-y^2z=0$

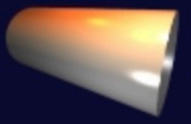
Buggle

$x^4y^2+y^4x^2-x^2y^2+z^6=0$



Zylinder

$y^2 + z^2 = 1$

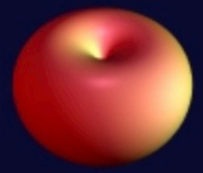


Diabolo

$x^2 = (y^2 + z^2)^2$

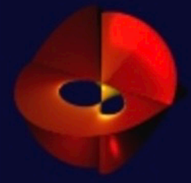
Dullo

$(x^2+y^2+z^2)^2-(x^2+y^2) = 0$



Miau

$x^2yz + x^2z^2 + 2y^3z + 3y^3 = 0$

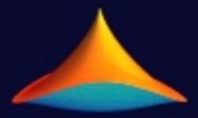


Trichter

$x^2 + z^3 = y^2z^2$

Nepali

$(xy-z^3-1)^2 + (x^2+y^2-1)^3 = 0$

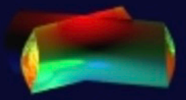


Pilzchen

$(z^3 - 1)^2 + (x^2+y^2-1)^3=0$

Subway

$x^2y^2 + (z^2-1)^3=0$



Polsterzipf

$(x^3-1)^2 + (y^3-1)^2+ (z^2-1)^3 = 0$

Crixxi

$(y^2+z^2-1)^2 + (x^2+y^2-1)^3 = 0$

Berg

$x^2+y^2z^2+z^3 = 0$

Gupf

$x^2+y^2+z=0$

Kegel

$x^2+y^2-z^2=0$

Wigwam

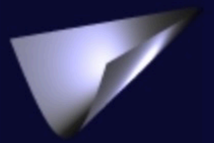
$x^2+y^2z^3=0$

Tuelle

$yz(x^2+y-z) = 0$

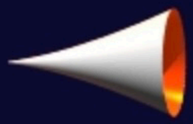
Pipe

$x^2-z=0$



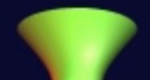
Fanfare

$-x^3+z^2+y^2=0$

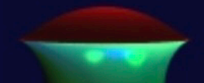


Kreuz

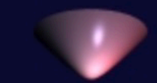
Spindel



Twilight



Ufo



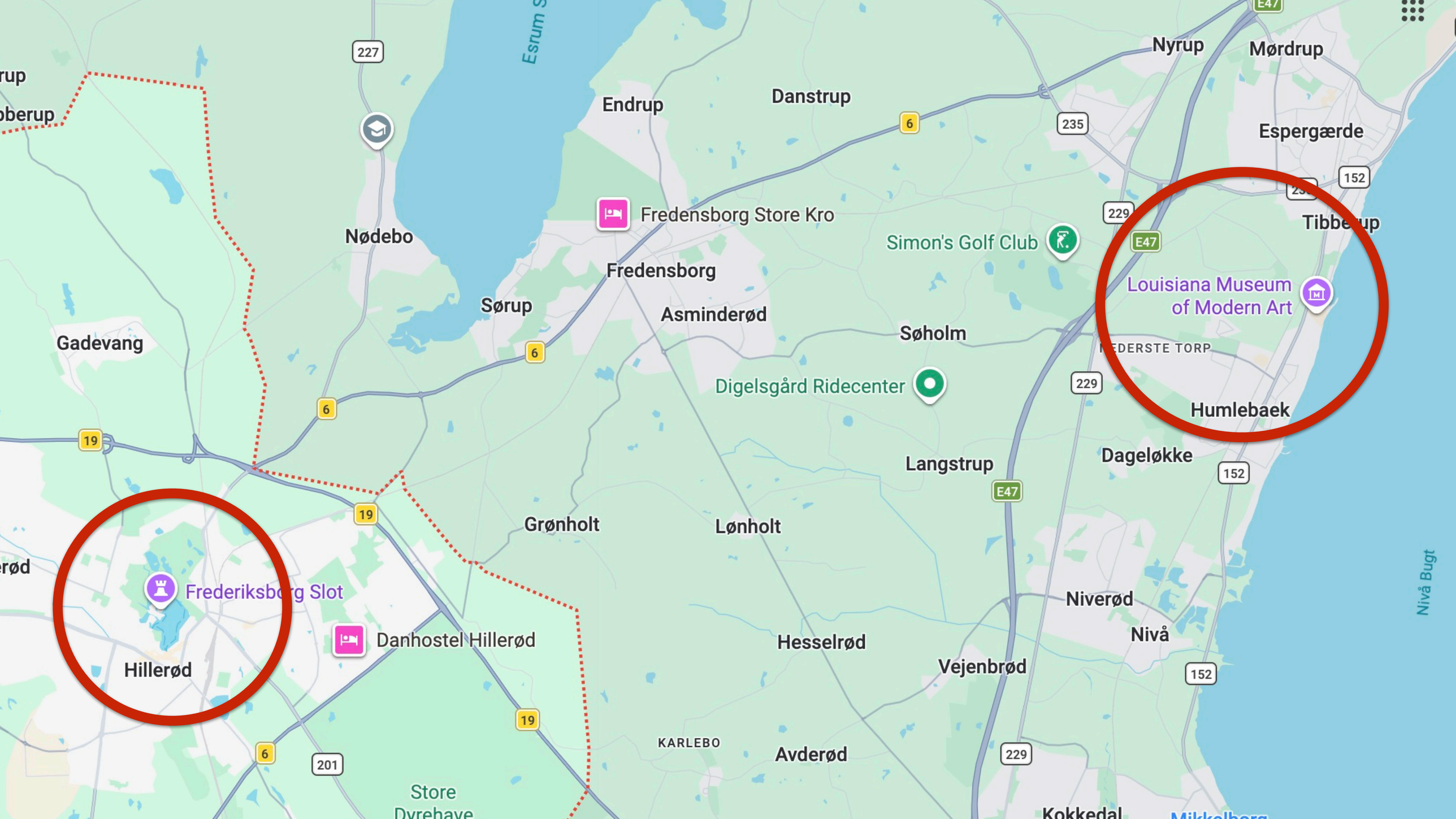
AN IMPORTANT INGREDIENT IS STRUCTURE

| | | | |
|---|---|---|---|
| Algebraic | field | group | monoid |
| Topological | manifold | metric space | topology |
| Order | total order | poset | quasiorder |
| Measure | Lebesgue | sigma algebra | dynkin set |
| Simplex | Whitney | simplicial complex | delta set |
| Sheave | Principal | fiber bundle | pre sheaf |

# ORIGINALITY

Meret Oppenheim, 1936

Hillerod, 6/20-7/2/1993


227
Esrum S
Endrup
Danstrup
6
235
Nyrup
Mørdrup
Espergærde
152
Nødebo
Fredensborg Store Kro
229
Simon's Golf Club
E47
Tibberup
Fredensborg
Louisiana Museum
of Modern Art
Sørup
Asminderød
Søholm
Gadevang
6
Digelsgård Ridecenter
229
Humlebaek
19
Langstrup
Dageløkke
152
E47
19
Grønholt
Lønholt
Frederiksborg Slot
Niverød
Nivå
Danhostel Hillerød
Hesselrød
Vejenbrød
Hillerød
152
19
6
KARLEBO
Avderød
201
229
Store
Nivå Bugt

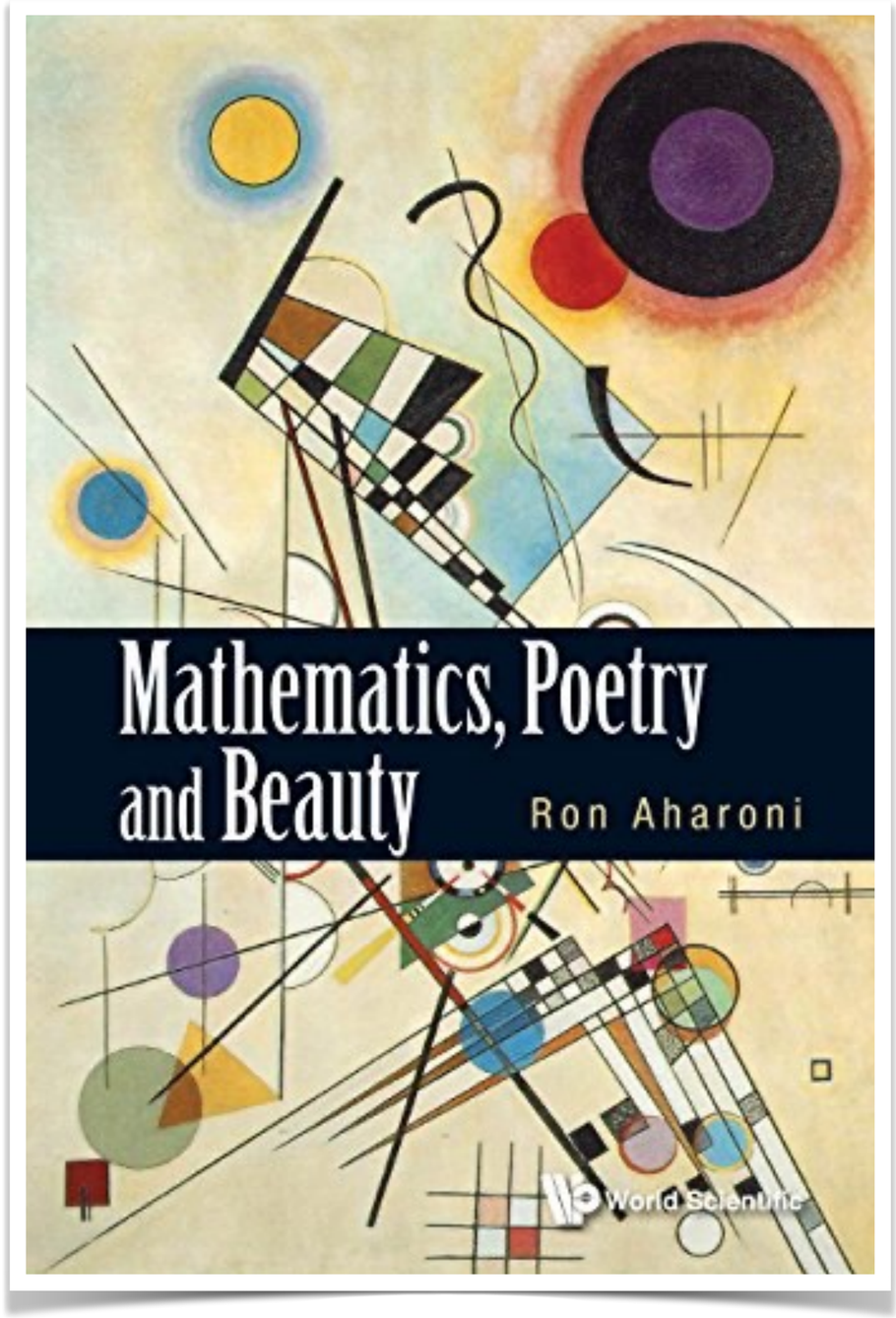
Louisiana Museum of Modern Art, Humlebaek, Denmark

# Modern math

---

A perverse scheme of sheaves
chews on a stalk of stacks.
Is it etale or crystal?
We are all chained to a ringed
topos in Serres saga of gaga.

# APROPOS POETRY




I will have to touch upon that most elusive of philosophical questions - what is beauty?

There is one word that captures its essence:

Magic

4

# PART 4: HOW DOES THE WORLD JUDGE?

LET'S GOOGLE

India Soale, Medium, 2023

Medium Search Get app Write

# Top 10 Most Beautiful Mathematical Equations

India Soale Follow 6 min read · Mar 9, 2023

# India Soale, Medium, 2023

$$F = ma$$

$$e^{i\pi} + 1 = 0$$

$$E = mc^2$$

$$\aleph_0 = |\mathbb{N}|$$

$$\frac{\mathrm{d}}{\mathrm{d}x} x^x = x^x(1 + \ln x)$$

$$G = (\star, E)$$

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

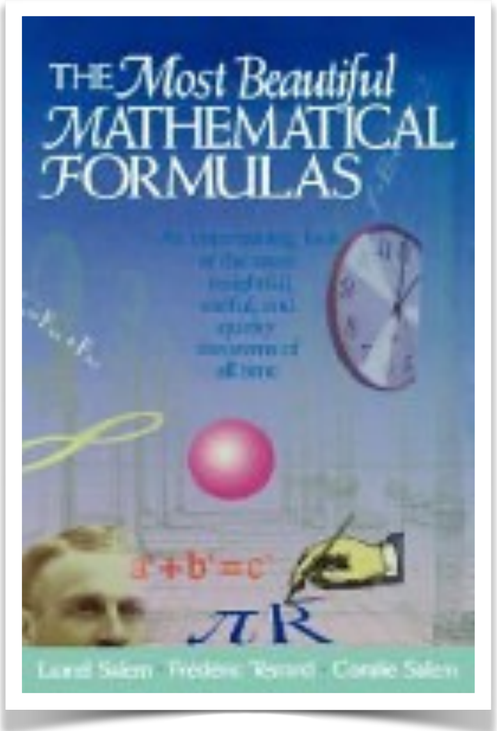
$$1 + 1 = 2$$

$$x = e^{\ln x}$$

$$i = \sqrt{-1}$$

Lionel Salem
Frederic Testard
Coralie Salem
1992

49 formulas





15. $\pi \approx \frac{355}{113}$ 38

**Quadratic Equations**

16. The Roots of a Quadratic Equation 42
17. The Golden Ratio: $\frac{1+\sqrt{5}}{2}$ 44
18. Imaginary Numbers: $i = \sqrt{-1}$ 48

**Logarithms and Exponentials**

19. The Discovery of Logarithms 52
20. The Wonderful Law of Logarithms: $\log(ab) = \log a + \log b$ 56
21. $1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n$ converges to 0.577... 58
22. $\log(1+x) \approx x$ ($x$ small) 62
23. The Number $e$ 64
24. The Number $e$ Raised to a Real Power 66
25. Derivatives and Integrals: Areas Viewed from Two Different Perspectives 68
26. The Number $e$ Raised to an Imaginary Power: $e^{i\alpha} = \cos\alpha + i\sin\alpha$ 70
27. $e^{i\pi} = -1$ 72
28. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$; $\sin 2\alpha = 2\sin\alpha\cos\alpha$ 74

**Series of Numbers**

29. $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ 78
30. The Fibonacci Sequence: $F_n = F_{n-1} + F_{n-2}$ 80
31. The Number of Ways of Arranging $n$ Objects is $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ 82
32. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$ 84
33. $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$ (for $|x| < 1$) 86
34. A Few Other Sums 88

**Objects in Space**

35. Euler's Theorem: $f - e + v = 2$ 92
36. The Surface Area of a Sphere Equals $4\pi R^2$ 94
37. The Volume of a Sphere Equals $\frac{4}{3}\pi R^3$ 96
38. The Angle at the Center of a Regular Tetrahedron Equals 109° 28′ 98
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**Whole Numbers, Prime Numbers**

40. Lagrange's Theorem: Every Whole Number Is the Sum of Four Squares 104
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42. Prime Numbers Are Indivisible 108
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**Chance**

45. The Chance of Winning the Lottery 116
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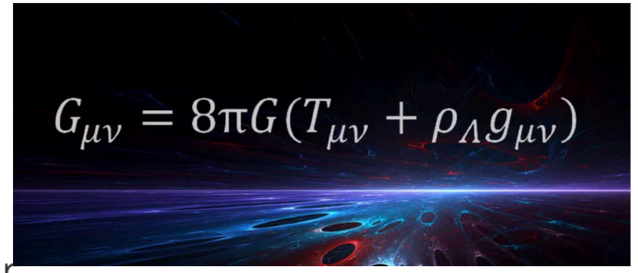
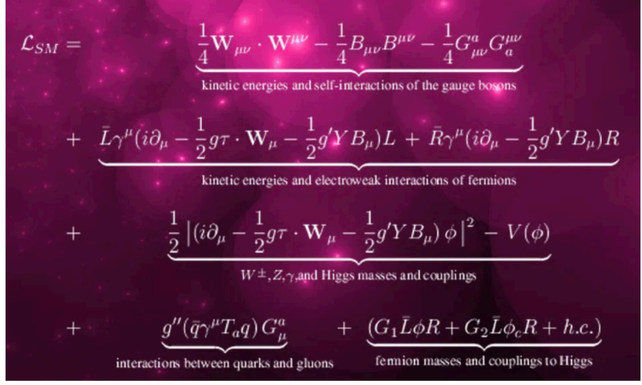
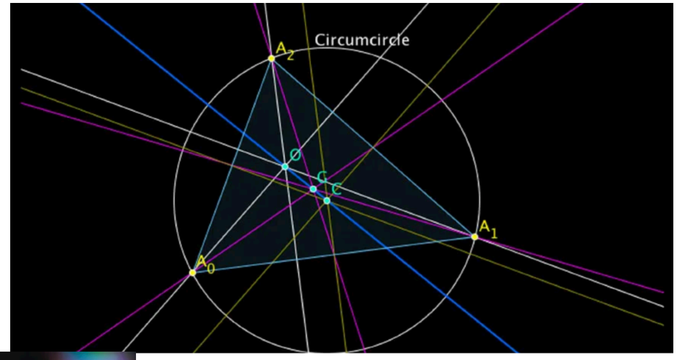
# The 11 most beautiful mathematical equations

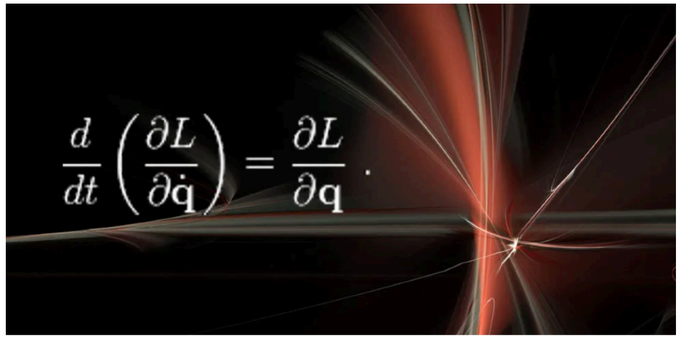
Countdowns By Clara Moskowitz published June 1, 2017

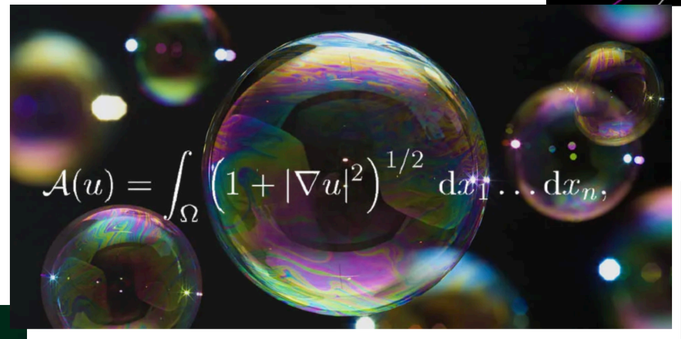
Live Science asked physicists, astronomers and mathematicians for their favorite equations. Here's what we found.

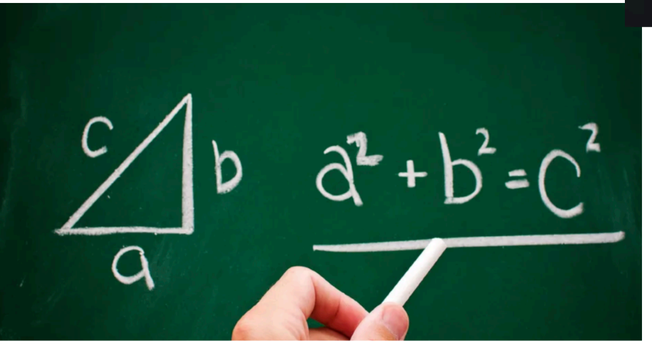


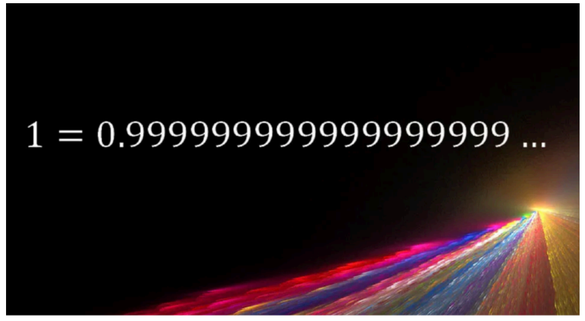



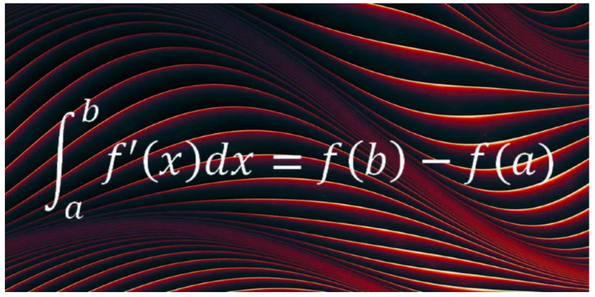


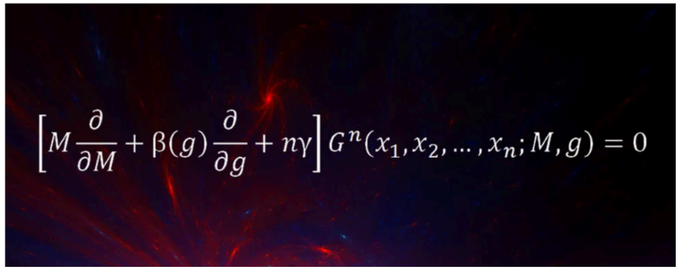



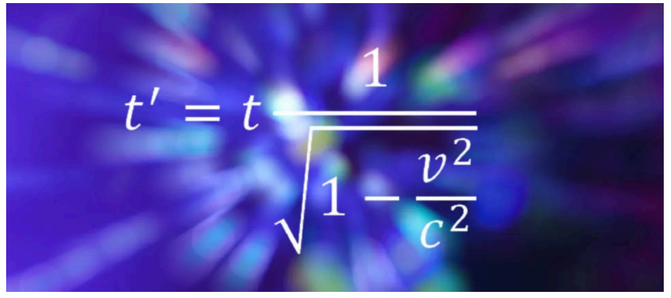




**The Callan-Symanzik equation**



The Pragmatic Programmers

# Good Math

A Geek's Guide to the Beauty of Numbers, Logic, and Computation

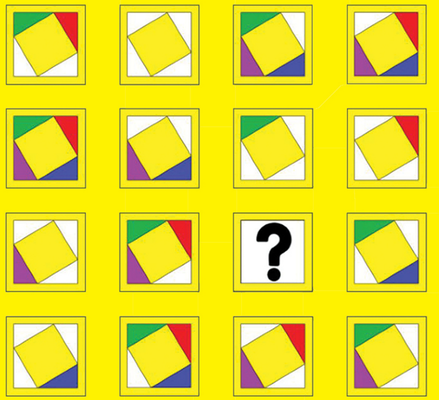
Mark C. Chu-Carroll

*Edited by John Osborn*

Problem Solving in Mathematics and Beyond

Volume 38

# Mathematics: The Quest for Truth and Beauty




James D Stein

World Scientific

Jörg Neunhäuserer

# Schöne Sätze der Mathematik

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*2. Auflage*

Springer Spektrum


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## Logic

An axiom system is never complete nor provably consistent

## Calculus

The derivative of an anti-derivative of f is again f.

## Algebra

Every non-constant polynomial has a root in the complex plane

## Geometry

In a right angle triangle $a^2 + b^2 = c^2$

## Computability

General recursive functions = Turing computatable functions

# Top 10 Fundamental Theorems

Oliver Knill July 23, 2018

## Analysis

Every normal operator can be diagonalized unitarily.

## Arithmetic

Every integer n>1 has a unique prime factorization.

## Counting

$\chi(A \times B) = \chi(A)\,\chi(B)$

where $\chi$ is Euler characteristic

## Combinatorics

If placing n things in m boxes and n is larger than m, then one box will have two tings.

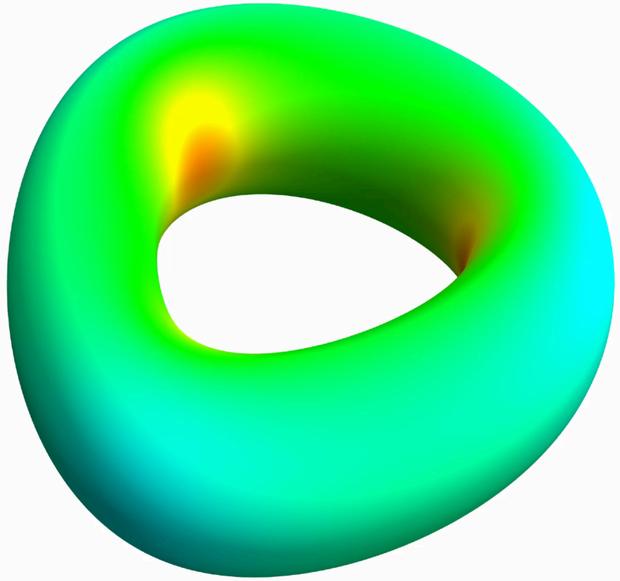
## Probability

A normalized sum of IIID random variables converges to a Gaussian in distribution.

1. Fundamental theorem of arithmetic
2. Pythagorean theorem in geometry
3. Incompleteness theorem of Goedel
4. The rule of product in topology
5. Fundamental theorem of computation
6. Fundamental theorem of calculus
7. Pigeonhole principle in combinatorics
8. Spectral theorem in analysis
9. Fundamental theorem of algebra
10. Central limit theorem in probability

knill youtube 2018

1. Euler' s identity $e^{i\pi} + 1 = 0$
2. The Pythagorean theorem $a^2 + b^2 = c^2$
3. The fundamental theorem of calculus
4. The fundamental theorem of arithmetic
5. The Gauss-Bonnet theorem
6. The spectral theorem
7. The central limit theorem
8. The prime number theorem
9. Gödel's incompleteness theorem
10. Cantor' s theorem



GPT 5.4 from 5/4/26

Gauss-Bonnet

Riemann-Hurwitz

Poincare-Hopf

Euler-Poincare

# My work Favorites

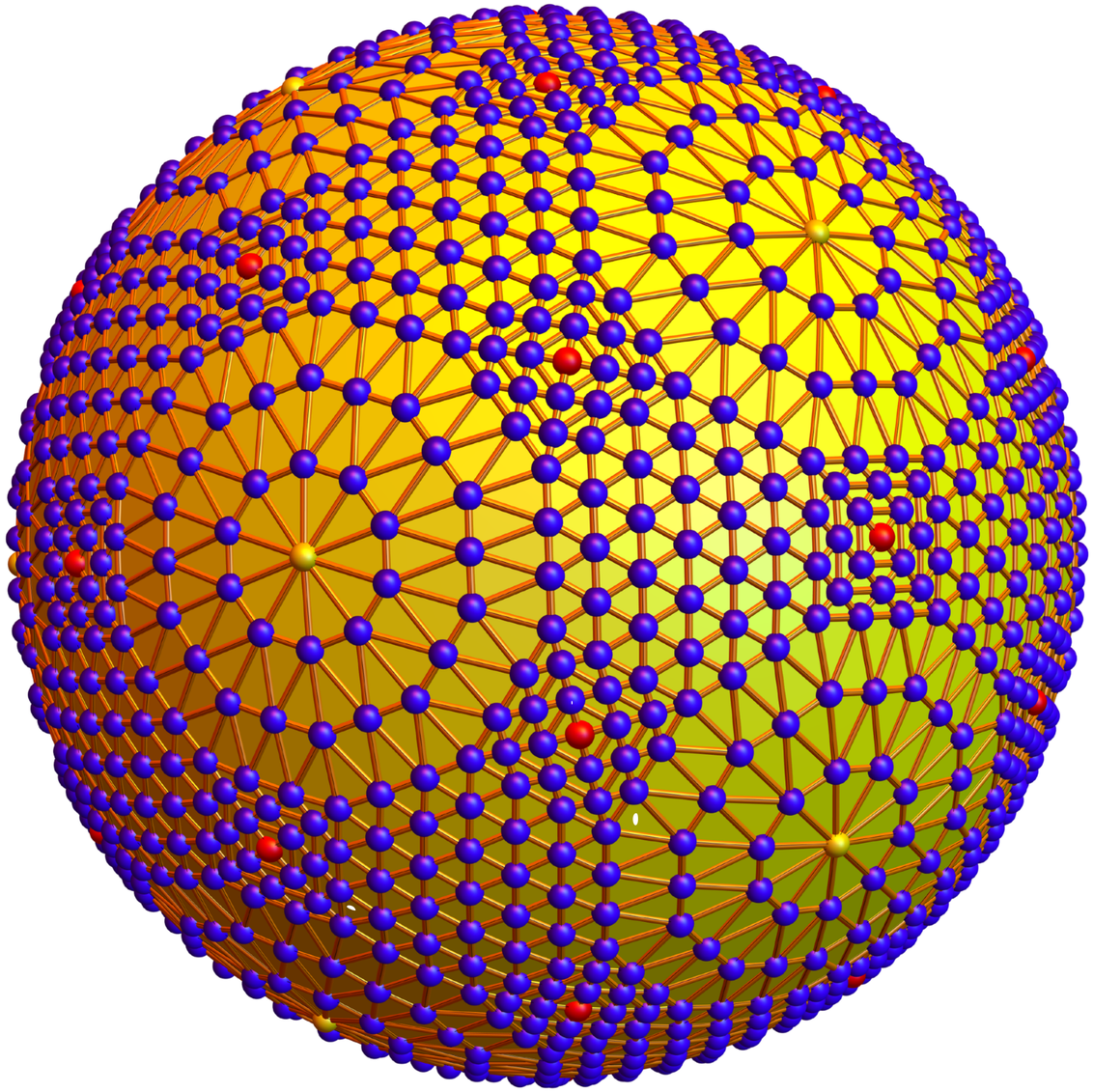
Riemann Roch

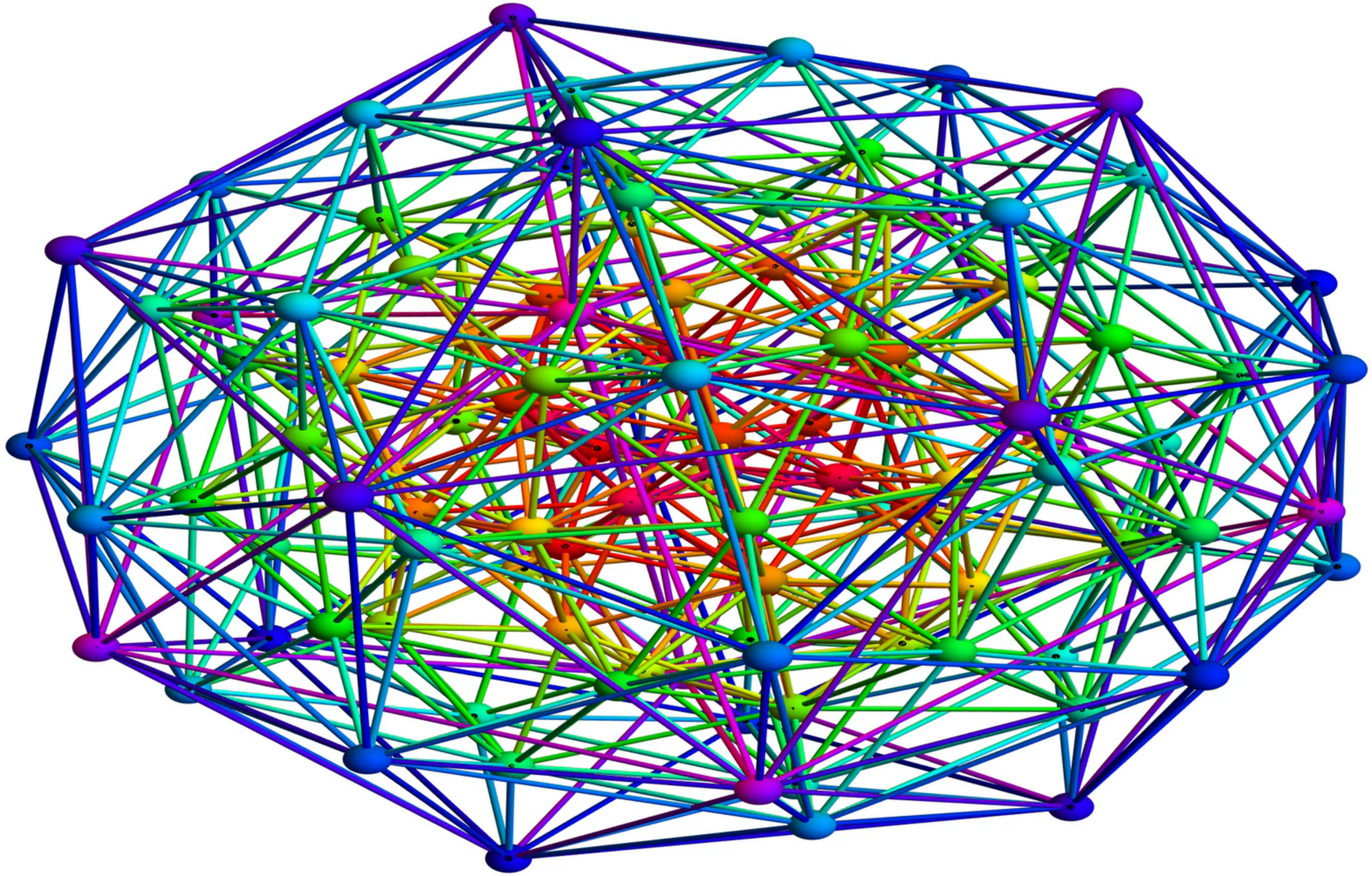
Brouwer Lefschetz

McKean-Singer

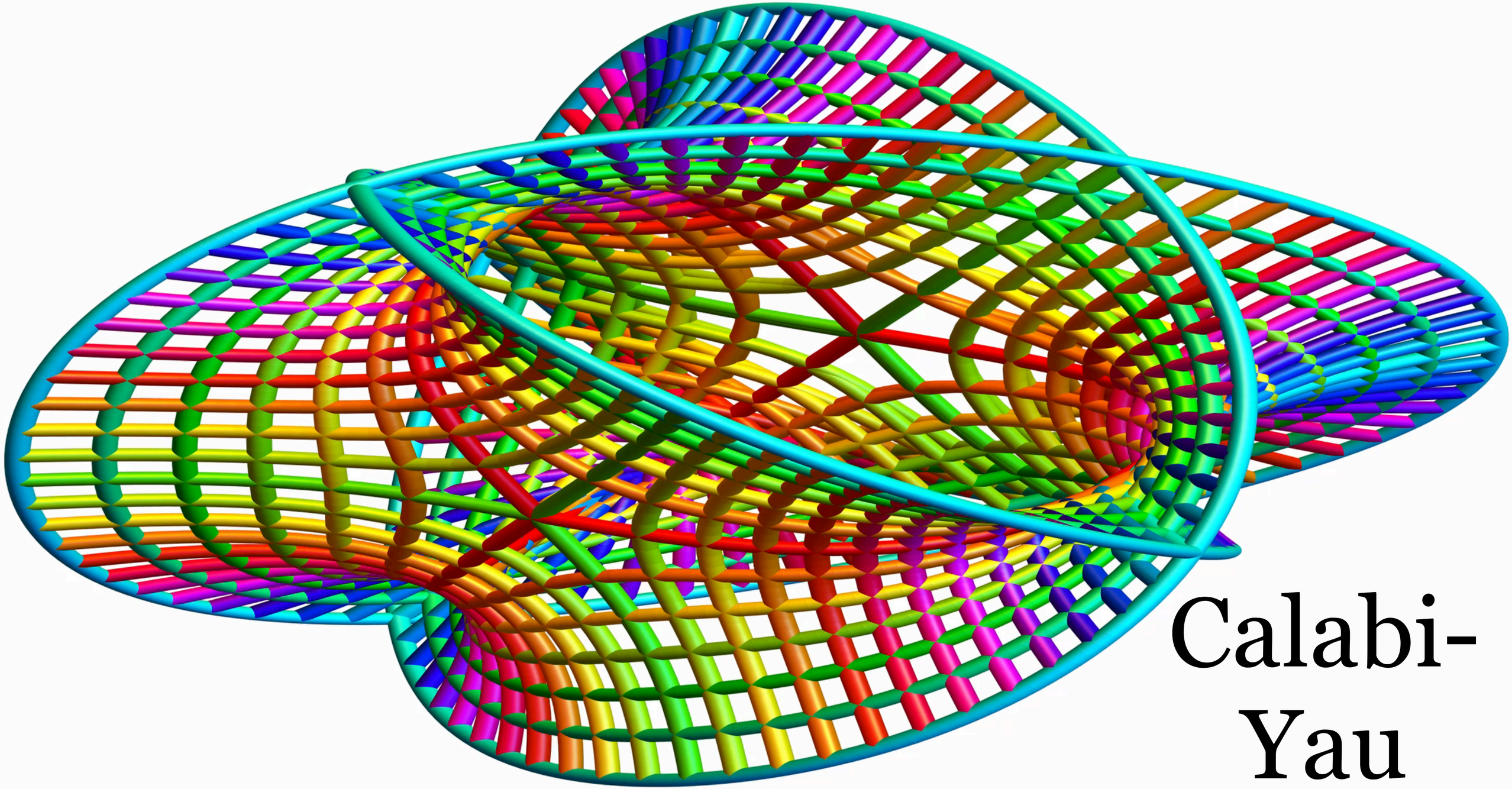
Atiah-Singer

Product Formula





600 Cell


Calabi-
Yau

# Ordinary People Can See Beauty in Complex Math, New Research Reveals

Sep 6, 2019 by News Staff « Previous | Next »

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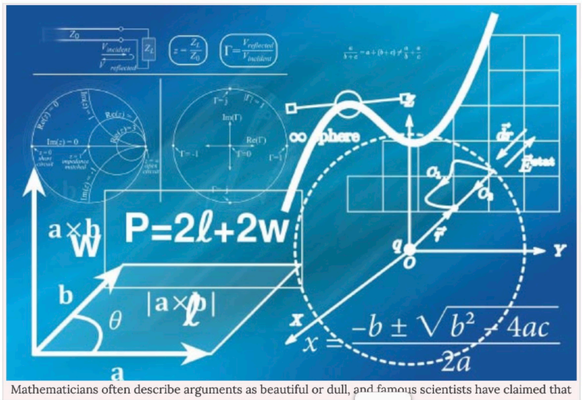
**According to a series of experiments by Yale University and University of Bath researchers, ordinary people see beauty in complex mathematical arguments in the same way they can appreciate a piano sonata.**




Mathematicians often describe arguments as beautiful or dull, and famous scientists have claimed that mathematical beauty is a guide toward the truth. Do laypeop hematicians and scientists,

Ordinary people see beauty in complex mathematical arguments in the same way they can appreciate a beautiful landscape painting or a piano sonata -- and you don't need to be a mathematician to get it, a new study by Yale University and the University of Bath has revealed.

Science Daily, 1919:


Contents lists available at ScienceDirect

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Original Articles

# Intuitions about mathematical beauty: A case study in the aesthetic experience of ideas


Samuel G.B. Johnson[a,*], Stefan Steinerberger[b]

[a] Division of Marketing, Business, & Society, University of Bath School of Management, United Kingdom
[b] Department of Mathematics, Yale University, United States

Cognition 189 (2019) 242–259

Contents lists available at ScienceDirect

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Original Articles

# Intuitions about mathematical beauty: A case study in the aesthetic experience of ideas

Samuel G.B. Johnson[a,*], Stefan Steinerberger[b]

[a] Division of Marketing, Business, & Society, University of Bath School of Management, United Kingdom
[b] Department of Mathematics, Yale University, United States

ARTICLE INFO

*Keywords:*
Psychology of mathematics
Explanation
Aesthetics
Reasoning
STEM education

ABSTRACT

Can an idea be beautiful? Mathematicians often describe arguments as "beautiful" or "dull," and famous scientists have claimed that mathematical beauty is a guide toward the truth. Do laypeople, like mathematicians and scientists, experience mathematics aesthetically? Three studies suggest that they do. When people rated the similarity of simple mathematical arguments to landscape paintings (Study 1) or pieces of classical piano music (Study 2), their similarity rankings were internally consistent across participants. Moreover, when participants rated beauty and various other potentially aesthetic dimensions for artworks and mathematical arguments, they relied mainly on the same three dimensions for judging beauty—elegance, profundity, and clarity (Study 3). These aesthetic judgments, made separately for artworks and arguments, could be used to predict similarity judgments out-of-sample. These studies also suggest a role for expertise in sharpening aesthetic intuitions about mathematics. We argue that these results shed light on broader issues in how and why humans have aesthetic experiences of abstract ideas.

## 1. Introduction

> "Beauty is the first test. There is no permanent place in the world for ugly mathematics."
>
> G.H. Hardy

> "Mathematics is the music of reason."
>
> James Joseph Sylvester

Can an idea be beautiful? Scientists and mathematicians seem to think so—they often imbue explanations with aesthetic qualities. Proofs can be "elegant" or "beautiful"; they can be "dull" or "trivial." Albert Einstein "was quite convinced that beauty was a guiding principle in the search for important results in theoretical physics" (Zee, 1999), while physicist Paul Dirac (1963) even claimed that "it is more important to have beauty in one's equations than to have them fit experiment."

In this paper, we explore the aesthetics of ideas by testing laypeople's aesthetic experiences of mathematical arguments. In particular, we test whether laypeople have nonrandom degrees of consensus in their judgments of aesthetic similarity between mathematical proofs and art objects such as paintings and music (Studies 1 and 2), whether there is consensus about which proofs are most beautiful (Study 3), and what properties contribute to these judgments of aesthetic similarity and pleasure (Study 3). Overall, these findings reveal a degree of consensus about the aesthetics of even the most abstract kind of ideas. We argue that these findings contribute to our understanding of how and why ideas can evoke aesthetic experiences.

### 1.1. Aesthetics: the true, the adaptive, and the ugly

To judge by the practitioners quoted above, aesthetic experiences not only imbue their interactions with scientific and mathematical ideas, but guide them toward the truth. On the face of it, this is quite a strange (if not ugly) idea. When we seek knowledge, we strive to employ tools, such as deductive logic and the scientific method, that reliably lead to justified true beliefs. Aesthetic experiences, such as beauty, could hardly be more different. Indeed, beauty is commonly said to be in the eye of the beholder. David Hume (1985/1757) put this point (how else but) elegantly:

> Beauty is no quality in things themselves: It exists merely in the mind which contemplates them; and each mind perceives a different beauty. One person may even perceive deformity, where another is sensible of beauty; and every individual ought to acquiesce in his own sentiment, without pretending to regulate those of others.

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https://doi.org/10.1016/j.cognition.2019.04.008
Received 14 December 2018; Received in revised form 3 April 2019; Accepted 10 April 2019
Available online 20 April 2019
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**Pigeonhole:** Pigeonhole principle

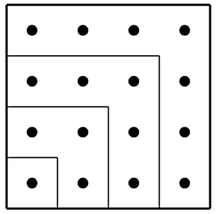
> In any group of 5 people, there are two who have the same number of friends within the group.
>
> We can see this as follows: suppose there exist somebody who is friends with everybody else. Then every person in the group has either 1,2,3 or 4 friends. Since there are 5 people, one number has to appear twice. If nobody is friends with everybody else, then everybody has either 0,1,2 or 3 friends; again, since there are 5 people, one number has to appear twice.

**Faulhaber:** Geometric proof of a Faulhaber formula

The sum of consecutive odd numbers always adds up to a square number:

$$1 = 1^2$$
$$1 + 3 = 2^2$$
$$1 + 3 + 5 = 3^2$$
$$1 + 3 + 5 + 7 = 4^2$$

The reason is explained in the picture below: adding the next odd number creates a suitable layer for the next square.



## Appendix B. Study 1 results by subgroup

See Tables B1–B3.

**Table B1**
Similarity judgments in Study 1 – Mechanical Turk sample *without* higher mathematics training.

| | Yosemite | Rockies | Suffolk | Andes |
|---|---|---|---|---|
| *Raw means* | | | | |
| Geometric | 3.41 | 2.88 | 3.08 | 2.88 |
| Gauss | 2.31 | 2.15 | 2.28 | 1.88 |
| Pigeonhole | 2.23 | 1.99 | 2.09 | 2.37 |
| Faulhaber | 2.77 | 2.64 | 3.06 | 2.35 |
| *Ranks* | | | | |
| Geometric | 16 | 12.5 | 15 | 12.5 |
| Gauss | 7 | 4 | 6 | 1 |
| Pigeonhole | 5 | 2 | 3 | 9 |
| Faulhaber | 11 | 10 | 14 | 8 |
| *Z-scored means* | | | | |
| Geometric | 0.48 | 0.23 | 0.33 | 0.20 |
| Gauss | −0.12 | −0.24 | −0.12 | −0.38 |
| Pigeonhole | −0.10 | −0.38 | −0.31 | −0.02 |
| Faulhaber | 0.19 | 0.09 | 0.33 | −0.17 |
| *Frequency top-ranked* | | | | |
| Geometric | 39% | 18% | 23% | 20% |

(*continued on next page*)





judgments about a domain as abstract as mathematics, it is likely that aesthetic experiences in less abstract domains of ideas would be even more powerful. Scientists often invoke aesthetic considerations in justifying their preference for one theory over another, which perhaps helps to guide scientists toward truth (if we believe Dirac, Einstein, and Keats). Social engineers may experience aesthetic pleasure at contemplating particularly beautiful social arrangements, and such considerations may play a role in political decision-making, particularly in regimes unconstrained by democratically expressed preferences. Consumers may be attracted to particular ideas expressed in the marketplace, especially in the less tangible domain of services, complementing insights about physical products in consumer behavior and product design (e.g., Desmet & Hekkert, 2007; Hoyer & Stokburger-Sauer, 2012). Understanding the aesthetics of ideas could prove to be a critical link between the affective and cognitive sciences.

**Acknowledgements**

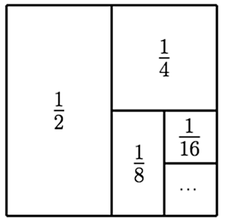
We thank Woo-kyoung Ahn, Lance Rips, and Nadya Vasilyeva for discussion, and Asher Auel, Miki Havlickova, John Hall, Brett Smith, and Sarah Vigliotta for assistance with data collection. These studies were presented at the 40th Conference of the Cognitive Science Society and a subset of the findings were discussed in an informal commentary in the *Mathematical Intelligencer* magazine; we thank the conference attendees and *Intelligencer* editor for their interest (Johnson & Steinerberger, 2018, 2019). This work was supported by a grant from the American Psychological Association awarded to the first author.

## Appendix A. Text of arguments

**Geometric:** Sum of an infinite geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = 1.$$

We can see this by cutting a square with total area 1 into little pieces.




**Gauss:** Gauss's summation trick for positive integers[3]

A quick way of computing

$$1 + 2 + 3 + 4 + \cdots + 98 + 99 + 100 = 5050$$

is as follows: write the total sum twice and add the columns

$$\begin{array}{rrrrrrrrrrrr} 1+ & 2+ & 3+ & 4+\cdots+ & 98+99+100 \\ 100+99+98+98+\cdots+ & 3+ & 2+ & 1 \\ \hline 101 & 101 & 101 & 101 & \ldots & 101 & 101 & 101 \end{array}$$

This yields a total of 100 times 101 (giving 10100) and half of that is exactly 5050.

[3] The minor typo in this item (can you spot it?) was present in the experimental materials. However, it is unlikely that this substantially influenced the results of the study, as (a) it scores as well as or better than the other proofs on the dimensions measured in Study 3, and (b) Study 1 participants had the opportunity to write comments at debriefing, and no participant mentioned the typo.

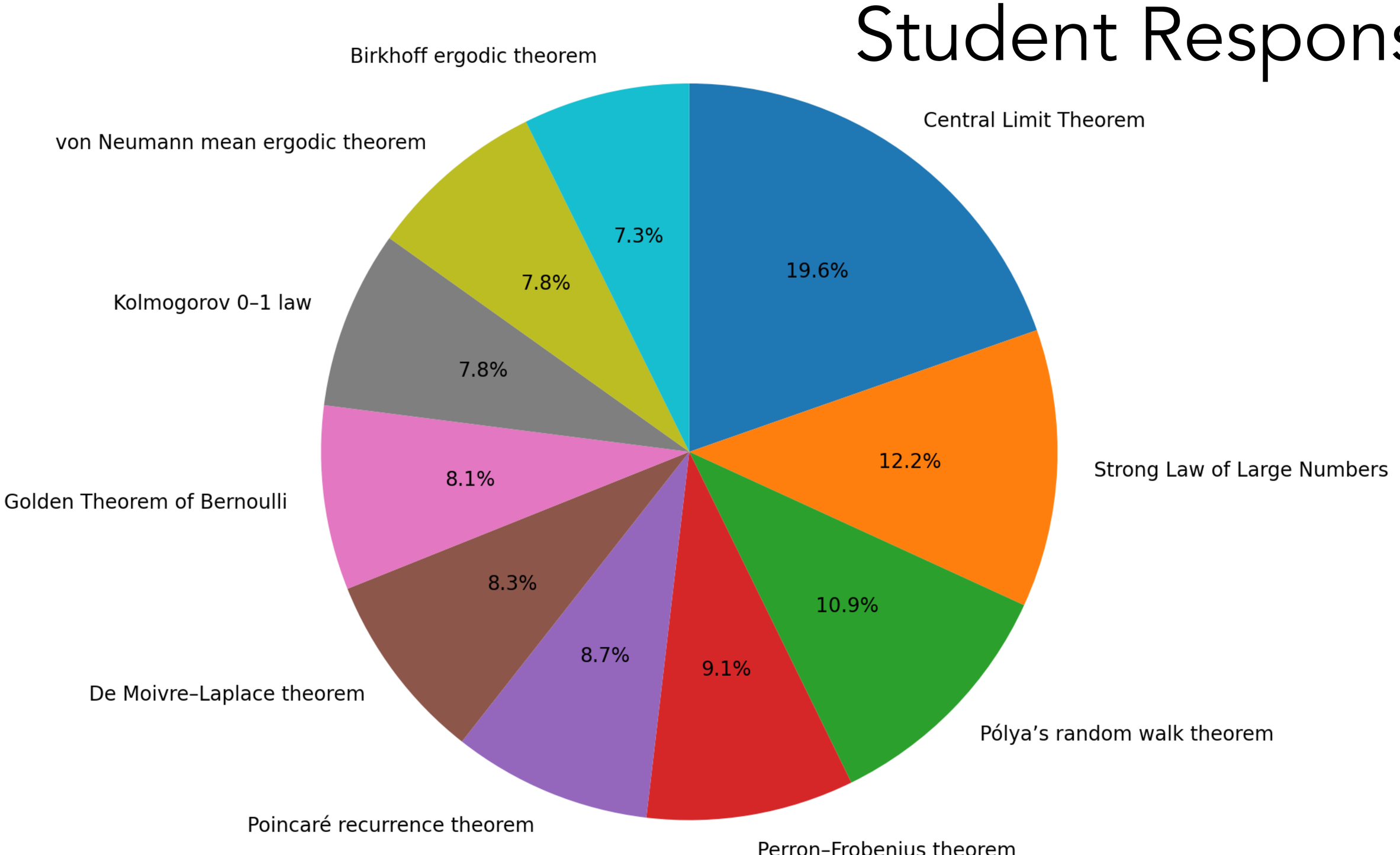
# IT IS PERSONAL:

# From Final Exam 2026

10) Write down the names of 10 theorems we have proven in this course, but order according to your personal preferences! This is a little competition for the theorems. Oliver will announce which theorem is the winner.

| Theorem | Give the precise name and if possible the person/persons |
|---|---|
| Number 1 | |
| Number 2 | |
| Number 3 | |
| Number 4 | |
| Number 5 | |

# Student Response



# THE END