BARYCENTRIC CHARACTERISTIC NUMBERS

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If $\mathcal{G}$ is the category of **finite simple graphs** $G = (V, E)$, the linear space $\mathcal{V}$ of **valuations** on $\mathcal{G}$ has a basis given by the **f-numbers** $v_k(G)$ counting complete subgraphs K_{k+1} in G. The **barycentric refinement** G_1 of $G \in \mathcal{G}$ is the graph with K_l subgraphs as vertex set where new vertices $a \neq b$ are connected if $a \subset b$ or $b \subset a$. Under refinement, the clique data transform as $\vec{v} \to A\vec{v}$ with the upper triangular matrix $A_{ij} = i!S(j,i)$ with **Stirling numbers** $S(j,i)$. The eigenvectors χ_k of A^T with eigenvalues $k!$ form an other basis in $\mathcal{V}$. The χ_k are normalized so that the first nonzero entry is > 0 and all entries are in $\mathbb{Z}$ with no common prime factor. χ_1 is the **Euler characteristic** $\sum_{k=0}^{\infty}(-1)^k v_k$, the homotopy and so cohomology invariant on $\mathcal{G}$. Half of the χ_k will be zero **Dehn-Sommerville-Klee invariants** like half the **Betti numbers** are redundant under **Poincaré duality**. On the set $\mathcal{G}_d \subset \mathcal{G}$ with clique number d, the valuations $\mathcal{V}$ have dimension $d+1$ by **discrete Hadwiger**. A basis is the eigensystem $\vec{\chi}$ of the upper left $(d+1) \times (d+1)$ sub-matrix A_d of A. The functional $\chi_{d+1}(G)$ is **volume**, counting the **facets** of G. For $x \in V$, define $V_{-1}(x) = 1$ and $V_k(x)$ as the number of complete subgraphs K_{k+1} of the unit sphere $S(x)$, the graph generated by the neighbors of x. The **fundamental theorem** of graph theory is the formula $\sum_{x \in V} V_{k-1}(x) = (k+1)v_k(G)$. For $k = 1$, it is the **Euler's handshake**. For a valuation $X(G) = \sum_{l=0}^{\infty} a(l)v_l(G)$, define **curvature** $K_X(x) = \sum_{l=0}^{\infty} a(l)V_{l-1}(x)/(l+1)$. Generalizing the fundamental theorem:

Theorem 1 (**Gauss-Bonnet-Chern-Levitt).** $\sum_{x \in V} K_X(x) = X(G)$.

Example. For an icosahedron with $\vec{v} = (12, 30, 20)$ and $\vec{v}(S(x)) = (5, 5)$, we have $a_1 = (1, -1, 1)$, $K_1(x) = 1 - 5/2 + 5/3 = 1/6$, $\chi_1 = 2$, $a_2 = (0, 2, -3)$, $K_2(x) = 10/2 - 15/3 = 0$, $\chi_2 = 0$, $a_3 = (0, 0, 1)$, $K_3(x) = 5/3$, $\chi_3 = 20$.

Let $\Omega(G)$ be the set of **colorings** of G, locally injective function f on $V(G)$. The **unit ball** $B(x)$ at x is the graph generated by the union of $\{x\}$ and the **unit sphere** $S(x) = \{y \in V \mid (x, y) \in E\}$ which is the **boundary** $\delta B(x)$. For $f \in \Omega$ and $X \in \mathcal{V}$ define the **index** $i_{X,f}(x) = X(B^-(x)) - X(S^-(x))$, where $B^-(x) = S^-(x) \cup \{x\} = \{y \in B(x) \mid f(y) \leq f(x)\}$ and $S^-(x) = \{f(y) < f(x)\}$. It is local and a **divisor**. Inductive attaching vertices gives:

Theorem 2 (**Poincaré-Hopf).** $\sum_{x \in V} i_{X,f}(x) = X(G)$.

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Let P be a **Borel probability measure** on $\Omega(G) = \mathbb{R}^{v_0(G)}$ and $E[\cdot]$ its **expectation**. Let $c(G)$ be the **chromatic number** of G. Assume either that P is the counting measure on the finite set of colorings of G with $c \geq c(G)$ real colors or that P is a product measure on Ω for which functions $f \to f(y)$ with $y \in V$ are independent identically distributed random variables with continuous probability density function. For all $G \in \mathcal{G}$ and $X \in \mathcal{V}$:

Theorem 3 (**Banchoff index expectation**). *$E[i_{X,f}(x)] = K_X(x)$.*

The **empty graph** $\emptyset$ is a (-1)-graph and (-1)-sphere. Inductively, a **d-graph** is a $G \in \mathcal{G}$ for which the unit spheres are $(d-1)$-spheres. An **Evako d-sphere** is a d-graph which when punctured becomes contractible. Inductively, G is **contractible** if there exists $x \in V(G)$ such that both $S(x)$ and the graph without x are contractible. The graph K_1 is contractible. Given $f \in \Omega$ and $c \notin f(V)$, define the graph $\{f = c\}$ in the refinement of G consisting of vertices, where $f - c$ changes sign. In that case, at every vertex x, there is a $(d-2)$-graph $R_f(x)$ defined as the **level surface** $\{f(y) = f(x)\}$ in $S(x)$. The next **Sard** result belongs to **discrete multivariable calculus**:

Theorem 4 (**Implicit function theorem**). *For a d-graph and $f \in \Omega$ and $c \notin f(V)$, the hyper surface $\{f = c\}$ is a $(d-1)$-graph.*

The **symmetric index** of f at x is defined as $2j_{X,f}(x) = i_{X,f}(x) + i_{X,-f}(x)$.

Theorem 5 (Index formula). *For $G \in \mathcal{G}$ and $X \in \mathcal{V}$, then*

$$2j_{X,f}(x) = X(B(x)) + X(\{x\}) - X(S(x)) - X(R_f(x)).$$

It allows fast recursive computation of X for most G in $\mathcal{G}_d$ quantified using **Erdös-Renyi** measures. For Euler characteristic, $j_f(x) = (1 - \chi(S(x))/2) - X(R_f(x))/2$. For d-graphs it is $-\chi(R_f(x))/2$ if d is odd and $1 - \chi(R_f(x))/2$ if d is even. If $k + d$ is even, we have $\chi_k(B(x)) = \chi_k(S(x))$, as curvature is supported on δG. Furthermore, we have $\chi_k(\{x\}) = 0$ if $k > 1$.

Theorem 6 (**Dehn-Sommerville-Klee**). *For a d-graph and even $d + k$, the functions K_k are supported on δG. If $\delta G = \emptyset$, then $\chi_k(G) = 0$.*

The χ_k span the classical DSK-space. The condition $K_k(x) = 0$ for all $x \in V$ follows from $\chi_k(G) = 0$ because $K(x)$ is the same for G or for the suspension of $S(x)$. If $K(x) \neq 0$, then by Gauss-Bonnet, there is $y \in S(x)$ with $K(y) \neq 0$. Repeated suspension lead to a constant curvature case. Curvature functionals are linear combination of Barycentric functionals for $d-1$.

Illustration: $\chi_2(G) = 0$ on 4-graphs shows that a 4-graph **triangulation** G of a compact 4-manifold with v vertices, e edges, f triangles, t tetrahedra, and p pentatopes satisfies $22e + 40t = 33f + 45p$. Examples: for the 4-crosspolytop G, a 4-sphere with $\vec{v} = (10, 40, 80, 80, 32)$, we get $\vec{\chi}(G) = (2, 0, 240, 0, 32)$. For a discrete $G = S^2 \times T^2$, a product graph constructed using the **Stanley-Reisner ring**, with $\vec{v} = (1664, 23424, 77056, 92160, 36864)$, we get $\vec{\chi}(G) = (0, 0, -10496, 0, 36864)$. For a discrete $G = P^2 \times S^2$ with $\vec{v} = (1898, 26424, 86736, 103680, 41472)$ we get $\vec{\chi}(G) = (2, 0, -10896, 0, 41472)$.

1. About the literature

The results are formulated in the language of graph theory [15, 9, 10], topological graph theory [51] or algebraic graph theory [14]. There is overlap with work on polytopes [127, 52], simplicial complexes [120, 121] or algebraic topology like [53, 118]. See [30] for history. Various flavors of discrete topologies have emerged: digital topology [91, 54, 34], discrete calculus [49], Fisk theory [2, 38, 36, 37, 39] to which we got in the context of graph colorings [81, 84] leading to the notion of spheres which appeared in [33] which is based on homotopy [61, 19], based on notions put forward in [65, 125], networks [104, 103, 124, 23, 104, 124, 60], physics [28, 112, 24, 46, 100, 115], computational geometry [29, 12, 27, 106, 126] discrete Morse theory [40, 41, 42, 44], eying classical Morse theory [97], discrete differential geometry in relation to classical differential geometry [13, 47, 64]. We got to into the subject through [69] and generalized it to [68] and summarized in [74] after [70]. The general Gauss-Bonnet-Chern result appeared in [68] but was predated in [92]. It seems Gauss-Bonnet for graphs has been rediscovered a couple of times like [62, 43]. We noticed the first older appearance [62] in [85] and found [92, 43] while working on the present topic. Various lower dimensional versions of curvature have appeared [50, 109, 110, 56, 102, 123]. The first works on Gauss-Bonnet in arbitrary dimensions include [58, 4, 35, 3, 20, 21]. For modern proofs, see [25, 114]. The story of Euler characteristic is told in [45, 113, 94]. A historical paper is [32]. The first works on Poincare-Hopf were [107, 58, 99]. For more history, see [119, 98, 16, 48, 57]. Poincaré-Hopf indices are central in discrete approaches to Riemann-Roch [6, 5]. The index expectation result is [72]. The closest related work is Banchoff [8, 7]. For integral geometry and geometric probability, see [66, 116]. The index formula for Euler characteristic appeared in [71] which proves a special case of the result here along the same lines. See also [77] for the recursion. The Sard approach in [87] simplified this. That paper gives a discrete version of [117]. We got into the Barycentric invariants through [83, 88] after introducing a graph product [86] which was useful in [85], a paper exploring topology of graphs [82, 63, 75]. Originally we were interested in the spectral theory of graphs [22, 111, 1, 26, 108, 122] which parallels the continuum [18, 114, 13]. The linear algebra part of networks was explored in [73] which is a discrete version of [96]. See also [76, 80, 79, 78, 89, 90] and [59, 95] for discrete combinatorial Laplacians. For the Dehn-Sommerville relations, see [67, 105, 101, 93, 17, 55, 31]. In [105, 31] appear Klee-Dehn-Sommerville equations for discrete for manifolds with boundary. In [93], it was noted that the Euler characteristic is the only invariant, using the operator A. The combinatorics of the Barycentric operator was studied in[11] in the case $d = 2$. The explicit formula using Stirling numbers appeared in [17].

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