A case study in Interaction cohomology **Oliver Knill, 3/18/2016**

Simplicial Cohomology

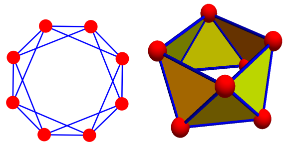
Simplicial cohomology is defined by an **exterior derivative** $dF(x) = F(dx)$ on **valuation forms** $F(x)$ on subgraphs x of a **finite simple graph** G, where dx is the **boundary chain** of a simplex x. Evaluation $F(A)$ is **integration** and $dF(A) = F(dA)$ is **Stokes**. Since $d^2 = 0$, the kernel of $d_p : \Omega^p \to \Omega^{p+1}$ contains the image of d_{p-1}. The vector space $H^p(G) = \ker(d_p)/\mathrm{im}(d_{p-1})$ is the p'th **simplicial cohomology** of G. The **Betti numbers** $b_p(G) = \dim(H^p(G))$ define $\sum_p (-1)^p b_p$ which is **Euler characteristic** $\chi(G) = \sum_x (-1)^{\dim(x)}$, summing over all complete subgraphs x of G. If T is an automorphism of G, the **Lefschetz number**, the super trace $\chi_T(G)$ of the induced map U_T on $H^p(G)$ is equal to the sum $\sum_{T(x)=x} i_T(x)$, where $i_T(x) = (-1)^{\dim(x)}\mathrm{sign}(T|x)$ is the **Brouwer index**. This is the **Lefschetz fixed point theorem**. The **Poincaré polynomial** $p_G(x) = \sum_{k=0} \dim(H^k(G))x^k$ satisfies $p_{G\times H}(x) = p_G(x)p_H(x)$ and $\chi(G) = p_G(-1)$ so that $\chi(G \times H) = \chi(G) \cdot \chi(H)$. For $T = Id$, the Lefschetz formula is **Euler-Poincaré**. With the **Dirac operator** $D = d + d^*$ and **Laplacian** $L = D^2$, discrete **Hodge** tells that $b_p(G)$ is the nullity of L restricted to p-forms. By **McKean Singer super symmetry**, the positive Laplace spectrum on even-forms is the positive Laplace spectrum on odd-forms. The super trace $\mathrm{str}(L^k)$ is therefore zero for $k > 0$ and $l(t) = \mathrm{str}(\exp(-tL)U_T)$ with **Koopman operator** $U_T f = f(T)$ is t-invariant. This heat flow argument proves Lefschetz because $l(0) = \mathrm{str}(U_T)$ is $\sum_{T(x)=x} i_T(x)$ and $\lim_{t\to\infty} l(t) = \chi_T(G)$ by Hodge.

Interaction Cohomology

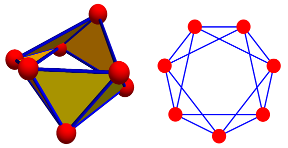
Super counting ordered pairs of intersecting simplices (x, y) gives the **Wu characteristic** $\omega(G) = \sum_{x\sim y}(-1)^{\dim(x)+\dim(y)}$. Like Euler characteristic it is **multiplicative** $\omega(G \times H) = \omega(G)\omega(H)$ and satisfies **Gauss-Bonnet** and **Poincaré-Hopf**. **Quadratic interaction cohomology** is defined by the **exterior derivative** $dF(x, y) = F(dx, y) + (-1)^{\dim(x)}F(x, dy)$ on functions F on ordered pairs (x, y) of intersecting simplices in G. If b_p are the Betti numbers of these interaction cohomology groups, then $\omega(G) = \sum_p(-1)^p b_p$ and the **Lefschetz formula** $\chi_T(G) = \sum_{(x,y)=(T(x),T(y))} i_T(x, y)$ holds where $\chi_T(G)$ is the **Lefschetz number**, the super trace of U_T on cohomology and $i_T(x, y) = (-1)^{\dim(x)+\dim(y)}\mathrm{sign}(T|x)\mathrm{sign}(T|y)$ is the **Brouwer index**. The heat proof works too. The **interaction Poincaré polynomial** $p_G(x) = \sum_{k=0} \dim(H^k(G))x^k$ again satisfies $p_{G\times H}(x) = p_G(x)p_H(x)$.

The Cylinder and the Möebius strip

The cylinder G and Möbius strip H are homotopic but not homeomorphic. As simplicial cohomology is a homotopy invariant, it can not distinguish H and G and $p_G(x) = p_H(x)$. But interaction cohomology can see it. The interaction Poincaré polynomials of G and H are $\boxed{p_G(x) = x^2 + x^3}$ and $\boxed{p_H(x) = 0}$. Like **Stiefel-Whitney classes**, interaction cohomology can distinguish the graphs. While Stiefel-Whitney is defined for vector bundles, interaction cohomologies are defined for all finite simple graphs. As it is invariant under Barycentric refinement $G \to G_1 = G \times K_1$, the cohomology works for continuum geometries like manifolds or varieties.



The Cylinder G is an orientable graph with $\delta(G) = C_4 \dot\cup C_4$.



The Möbius strip H is non-orientable with $\delta(H) = C_7$.

Classical Calculus

Calculus on graphs either deals with **valuations** or **form valuations**. Of particular interest are **invariant linear valuations**, maps X on non-oriented subgraphs A of G satisfying the **valuation property** $X(A \cup B) + X(A \cap B) = X(A) + X(B)$ and $X(\emptyset) = 0$ and $X(A) = X(B)$ if A and B are isomorphic subgraphs. We don't assume invariance in general. By **discrete Hadwiger**, the vector space of invariant linear valuations has dimension $d+1$, where $d+1$ is the **clique number**, the cardinality of the vertex set a complete subgraph of G can have. Linear invariant valuations are of the form $\sum_{k=0}^{d} X_k v_k(G)$, where $v(G) = (v_0, v_1, \dots, v_d)$ is the f-vector of G. Examples of invariant valuations are $\chi(G)$ or $v_k(G)$ giving the number of k-simplices in G. An example of a non-invariant valuation is $\deg_a(A)$ giving the number of edges in A hitting a vertex a. To define **form valuations** which implements a discrete "integration of differential forms", one has to orient first simplices in G. No compatibility is required so that any graph admits a choice of orientation. The later is irrelevant for interesting quantities like cohomology. A **form valuation** X is a function on oriented subgraphs of G such that the valuation property holds and $X(A) = -X(\overline{A})$ if $\overline{A}$ is the graph A in which all orientations are reversed. Form valuations in particular change sign if a simplex changes orientation and when supported on p-simplices play the role of p-forms. The defining identity $dF(A) = F(dA)$ is already Stokes theorem. If A is a discrete p surface made up of $(p+1)$-simplices with cancelling orientation so that dA is a p-graph, then this discretizes the continuum Stokes theorem but the result holds for all subgraphs of any graph if one extends F to **chains** over G. For example, if $f_G = abc + ab + bc + ca + a + b + c$ is the algebraic description of a triangle then $f_{dG} = bc - ac + ab + b - a + c - b + a - c = ab + bc - ac$ is only a chain. With the orientation $f_G = abc + ba + bc + ca + a + b + c$ we would have got additionally the terms $2a - 2b$. The vector space $\Omega^p(G)$ of **all form valuations** on G has dimension $v_p(G)$ as we just have to give a value to each p-simplex to define F.

We use a cylinder G, with f-vector $(8, 16, 8)$ which super sums to the Euler characteristic $\chi(G) = 0$. An orientation on facets is fixed by giving graph algebraically like with $f_G = a + b + ab + c + bc + d + ad + cd + e + ae + be + abe + f + bf + cf + bcf + ef + bef + g + cg + dg + cdg + fg + cfg + h + ah + dh + adh + eh + aeh + gh + dgh$. The automorphism group $A = D_8$ of G has 16 elements. The Lefschetz numbers of the transformations are $(0, 2, 2, 0, 2, 0, 0, 2, 0, 2, 2, 0, 2, 0, 0, 2)$. The average Lefschetz number is $\chi(G/A) = 1$. The **gradient** d_0 and **curl** d_1 are

$$d_0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, d_1 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

The Laplacian $L = (d + d^*)^2$ has blocks $L_0 = d_0^* d_0$, which is the Kirchhoff matrix. The form Laplacian $L_1 = d_0 d_0^* + d_1^* d_1$ is a matrix on 1-forms. Together with the form Laplacian $L_2 = d_1 d_1^*$, the matrix L is

$$\begin{bmatrix}
0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 4 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 & 0 & -1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 3 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 3 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 3
\end{bmatrix}.$$

For the Möbius strip H with f-vector $(7,14,7)$ we could make an edge refinement to get a graph $\tilde{H}$ with f-vector $(8,16,8)$ which after an edge collapse goes back to H. The graphs H and $\tilde{H}$ are topologically equivalent in the discrete topological sense (there is a covering for both graphs for which the nerve graphs are identical and such that the elements as well as their intersections are all two dimensional.) Both simplicial as well as quadratic intersection cohomology are the same for H and $\tilde{H}$. The graph H has the automorphism group D_7 but $\tilde{H}$ only Z_2. The Lefschetz numbers of H are $(0,2,2,0,2,0,2,0,2,0,2,0,0,2)$. The Lefschetz numbers of $\tilde{H}$ are $(0,2)$. The automorphism $T=(1,8,7,6,5,4,3,2)$ which generates the automorphism group of $\tilde{H}$ has the Lefschetz number 2. There are 4 fixed points, they are the fixed vertices $1,5$ and the fixed edges $(1,5)$ and $(7,3)$. All Brouwer indices are 1 except for $(1,5)$ which has index -1. (It is not flipped by the transformation and has dimension 1). We continue to work with the graph H which has the algebraic description $f_H = a+b+ab+c+bc+d+ad+cd+e+ae+be+abe+de+ade+f+bf+cf+bcf+ef+bef+g+ag+cg+dg+adg+cdg+fg+cfg$.

Here are the gradient and curl of H:

$$d_0 = \begin{bmatrix}
-1 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}, d_1 = \begin{bmatrix}
1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1
\end{bmatrix}.$$

The Hodge Laplacian of H is the 28×28 matrix

$$\begin{bmatrix}
4 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & -1 & 0 & -1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 3 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 3
\end{bmatrix} .$$

It consists of three blocks. The first, $L_0 = d_0^* d_0$ is a 7×7 matrix which is the Kirchhoff matrix of H. The second, $L_1 = d_1^* d_1 + d_0 d_0^*$ is a 14×14 matrix which describes 1-forms, functions on ordered edges. Finally, the third $L_2 = d_1 d_1^*$ deals with 2-forms, functions on oriented triangles of H. Note that an orientation on the triangles of H is not the same than having H oriented. The later would mean that we find an orientation on triangles which is compatible on intersections. We know that this is not possible for the strip. The matrices d_0, d_1 depend on a choice of orientation as usually in linear algebra where the matrix to a transformation depends on a basis. But the Laplacian L does not depend on the orientation.

Quadratic interaction calculus

Quadratic interaction calculus is the next case after linear calculus. The later has to a great deal known by Kirchhoff and definitely by Poincaré, who however seems not have defined D, the form Laplacian L, nor have used Hodge or super symmetry relating the spectra on even and odd dimensional forms. Nor did Poincaré work with graphs but with simplicial complexes, a detail which is mainly language as the Barycentric refinement of any abstract finite simplicial complex is already the Whitney complex of a finite simple graph. Quadratic interaction calculus deals with functions $F(x, y)$, where x, y are both subgraphs. It should be seen as a form version of quadratic valuations [7]. The later were defined to be functions $X(A, B)$ on pairs of subgraphs of G such that for fixed B, the map $A \to X(A, B)$ and for fixed A, the map $B \to X(A, B)$ are both linear (invariant) valuations. Like for any calculus flavor, there is a valuation version which is orientation oblivious and a form valuation version which cares about orientations. When fixing one variable, we want $x \to F(x, y)$ or $y \to F(y, x)$ to be **valuations**: $F(x, y \cup z) + F(x, y \cap z) = F(x, y) + F(x, z)$ and $F(x, y) = -F(x, \overline{y}) = -F(\overline{x}, y)$ if $\overline{y}$ is the graph y for which all orientations are reversed and the union and intersection of graphs simultaneously intersect the vertex set and edge set. Note that we do **not** require $F(x, y) = -F(y, x)$. But unlike as for quadratic valuations (which are oblivious to orientations), we want the functions to change sign if one of the orientations of x or y changes. Again, the choice of orientation is just a choice of basis and not relevant for any results we care about. Calculus is defined as soon as an exterior derivative is given (integration is already implemented within the concept of valuation or form valuation). In our case, it is **defined** via Stokes theorem $dF(x, y) = F(dx, y) + (-1)^{\dim(x)} F(x, dy)$ for simplices first and then for general subgraphs x, y by the valuation properties. While in linear calculus, integration is the evaluation of the valuation on subgraphs, in quadratic interaction calculus, we evaluate=integrate on pairs of subgraphs. The reason for the name "interaction cohomology" is because for F to be non-zero, the subgraphs x, y are required to be **interacting** meaning having a non-empty intersection. (The word "Intersection" would also work but the name "Intersection cohomology" has been taken already in the continuum). We can think about $F(A, B)$ as a type of **intersection number** of two oriented interacting subgraphs A, B of the graph G and as $F(A, A)$ as the **self intersection number** $F(A, A)$. An other case is the intersection of the diagonal $A = \bigcup_x \{(x, x)\}$ and graph $B = \bigcup_x \{(x, T(x))\}$ in the product $G \times G$ of an automorphism T of G. The form version of the Wu intersection number $\omega(A, B)$ is then the Lefschetz number $\chi_T(G)$. A particularly important example of a self-intersection number is the Wu characteristic $\omega(G) = \omega(G, G)$ which is quadratic valuation, not a form valuation. Our motivation to define interaction cohomology was to get a Euler-Poincaré formula for Wu characteristic. It turns out that Euler-Poincaré automatically and always generalizes to the Lefschetz fixed point theorem, as the heat flow argument has shown; Euler-Poincaré is always just the case in which T is the identity automorphism.

Lets look now at **quadratic interaction cohomology** for the Cylinder graph G and the Möbius graph H. We believe this case demonstrates well already how quadratic interaction cohomology allows in an algebraic way to distinguish graphs which traditional cohomology can not. The f-matrices of the graphs are

$$V(G) = \begin{bmatrix} 8 & 32 & 24 \\ 32 & 112 & 72 \\ 24 & 72 & 40 \end{bmatrix}, V(H) = \begin{bmatrix} 7 & 28 & 21 \\ 28 & 98 & 63 \\ 21 & 63 & 35 \end{bmatrix}, V(\tilde{H}) = \begin{bmatrix} 8 & 32 & 24 \\ 32 & 114 & 74 \\ 24 & 74 & 42 \end{bmatrix}.$$

The f-matrix is the matrix V for which V_{ij} counts the number of pairs (x, y), where x is an i-simplex, y is a j-simplex and x, y intersect. The Laplacian for quadratic intersection cohomology of the Cylinder is a 416×416 matrix because there are a total of $416 = \sum_{i,j} V_{ij}$ pairs of simplices which do intersect in the graph G. For the Möbius strip, it is a 364×364 matrix which splits into 5 blocks L_0, L_1, L_2, L_3, L_4. Block L_p corresponds to the interaction pairs (x, y) for which $\dim(x) + \dim(y)$ is equal to p. The scalar interaction Laplacian L_0 for the Cylinder is the diagonal matrix $\mathrm{Diag}(8, 8, 8, 8, 8, 8, 8)$. For the Möbius strip H, it is the matrix $L_0(H) = \mathrm{Diag}(8, 8, 8, 8, 8, 8, 8)$. The diagonal entries of $L_0(H)$ depend only on the vertex degrees. In general, for any graph, if all vertex degrees are positive, then the scalar interaction Laplacian L_0 of the graph is invertible. For the Cylinder, the block L_1 is an invertible 64×64 matrix because there are $V_{12} + V_{21} = 64$ vertex-edge or edge-vertex interactions. Its determinant is $2^{58} 3^8 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17^8 103^2 373^2 2089^2$. For the Möbius strip, the block L_1 is a 56×56 matrix with determinant $2^{46} 3^7 5 7^3 17^7 42924041^2$.

For the Cylinder graph G, the block $L_2(G)$ is a 160×160 matrix, which has a 1-dimensional kernel spanned by the vector [0, -5, -5, 10, 10, 2, 2, 5, 0, -5, 2, 10, 10, 2, 5, 0, -5, -2, -2, -10, -10, 5, 5, 0, 2, 10, 10, 2, -10, 2, 0, 9, -2, 9, 10, -10, -2, -9, 0, -9, -10, 2, -2, -10, 9, 0, 9, -10, -2, -10, -2, -9, 0, -2, -9, -10, 2, 10, 10, 2, 0, 5, 5, -2, -10, 9, 0, 9, -10, -2, 2, -10, -9, 0, -2, -9, -10, 2, 10, -5, 10, 2, 0, 5, -2, 10, -9, 0, -9, 10, -2, 10, -2, 9, 9, 0, 2, -10, -10, -2, -5, -10, -2, 0, 5, 2, 10, -5, 2, 10, -5, 0, -7, -7, -8, -7, -7, -8, 8, 7, 7, -7, -7, -8, 8, 7, 7, 7, 7, 8, -8, -7, -7, 8, 7, 7, -7, 7, -8, -7, -7, 8, -7, -7, 8, -7, 7, 8, -8, 7, -7, -8, 7, 7, -8, 7, 7, 8, -7, 7]. This vector is associated to edge-edge and triangle-vertex interactions. So far, harmonic forms always had physical relevance. We don't know what it is for this interaction calculus.

On the other hand, for the Möbius strip H, the interaction form Laplacian $L_2(H)$ is a 140×140 matrix which is invertible! Its determinant is $2^{42} 3^{14} 5^3 7^6 11 \cdot 17^{14} 11087^2 212633^2 42924041^4$.
The quadratic form Laplacian L_3 which describes the edge-triangle interactions. For the Cylinder G, the interaction form Laplacian $L_3(G)$ is a 144×144 matrix. It has a 1-dimensional kernel spanned by the vector [4, 3, -6, -3, 6, 3, 4, -6, 3, -6, -3, -3, 4, -6, 6, 3, 4, -6, -3, -6, 2, -3, -3, 2, -2, -3, 2, -3, 3, 2, -2, -3, -2, 3, -3, 2, 6, 6, -4, -3, 3, 3, 2, -2, -3, -2, 3, 3, 2, 6, -3, 6, -4, -3, -3, 2, -2, 3, 3, -2, 3, -2, -6, 3, 6, -4, 3, 6, -3, -6, 3, -4, -4, -3, 3, -2, 2, -3, -6, 3, 6, -3, -4, -3, 3, -2, 2, -6, -3, -6, 6, 6,

3, -2, 2, -3, 4, 3, -3, -3, 3, -4, 3, -2, 2, -6, -3, -6, 6, 6, 3, -2, 3, 2, -3, 4, 3, 3, -4, 3, 3, -3, -2, 2, -6, 6, -6, 6, -2, 3, -3, 2, -3, 4, -3, -6, 6, 3, -2, 3, -3, 2, -3, 4].

The surprise, which led us write this down quickly is that for the Möbius strip H, the interaction form Laplacian $L_3(H)$ is a 126×126 matrix which is **invertible** and has determinant $2^{28}3^7 5^3 7^3 11^2 17^7 11087^4 212633^4 42924041^2$. Since L_3 is invertible too, there is also no cohomological restrictions on the $p = 3$ level for the Möbius strip. The quadratic form Laplacian is invertible. This is in contrast to the cylinder, where cohomological constraints exist on this level. The interaction cohomology can detect topological features, which simplicial cohomology can not.

The fact that determinants of L_3, L_2 have common prime factors is not an accident and can be explained by **super symmetry**, as the union of nonzero eigenvalues of L_0, L_2, L_4 are the same than the union of the nonzero eigenvalues of L_1, L_3.

Finally, we look at the L_4 block, which describes triangle-triangle interactions. For the Cylinder G, this interaction Laplacian is a 40×40 matrix which has the determinant $2^{26}3^3 5 \cdot 11^3 23^2 29^2 71^2 241^2$. For the Möbius strip, it is a 35×35 matrix which has determinant $2^{11} 5 \cdot 11 \cdot 11087^2 212633^2$.

The derivative d_0 for the Moebius strip H is the 56×7 matrix

$$d_0 = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix} .$$

It takes as arguments a list of function values on pairs of self-interacting vertices (x, x) and gives back a function on the list of pairs of interacting vertex-edge or edge-vertex pairs (x, y). The exterior derivative d_1 for H is a 140×56 matrix. We use now the notation $a = -1$ for formatting reasons:

$$
\begin{bmatrix}
a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{bmatrix}
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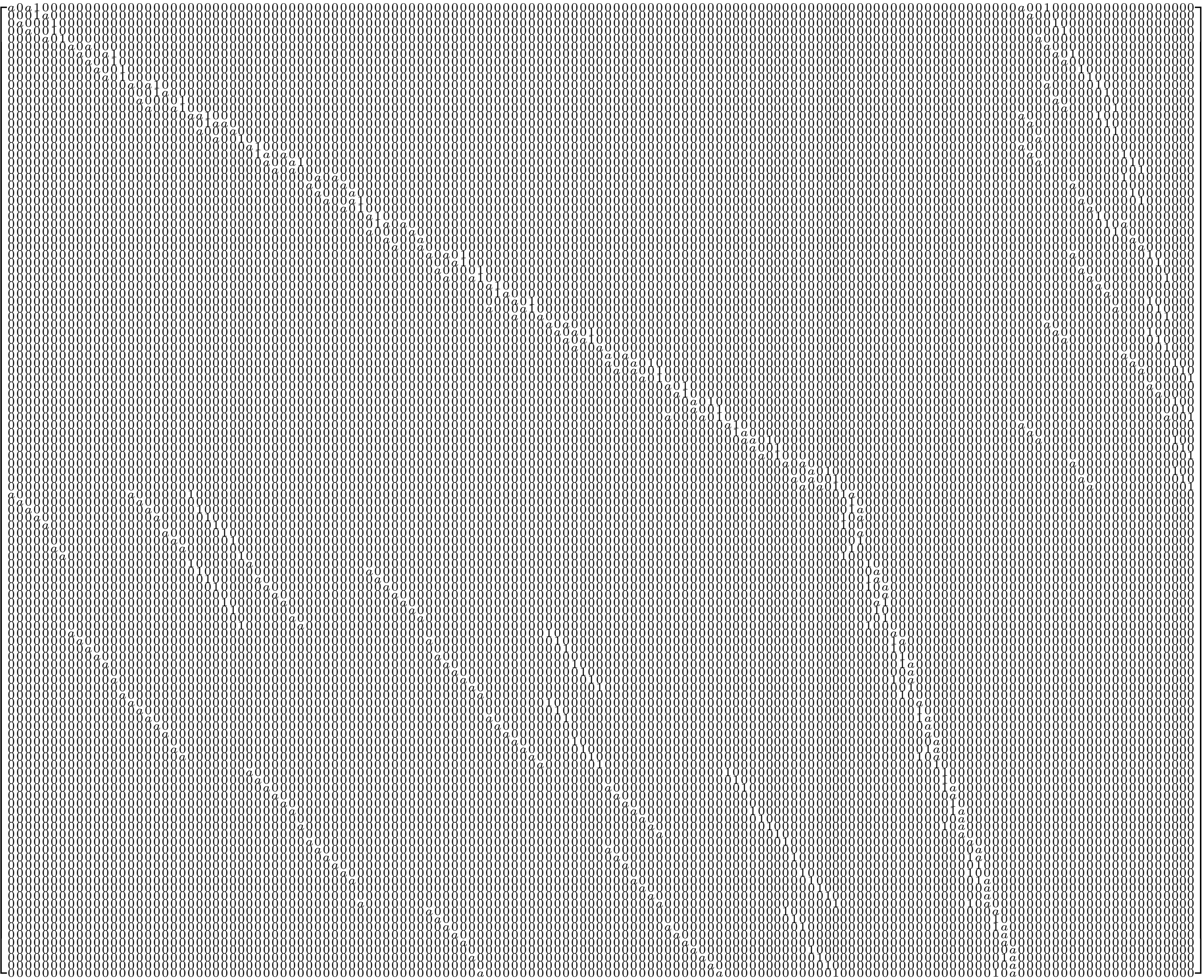
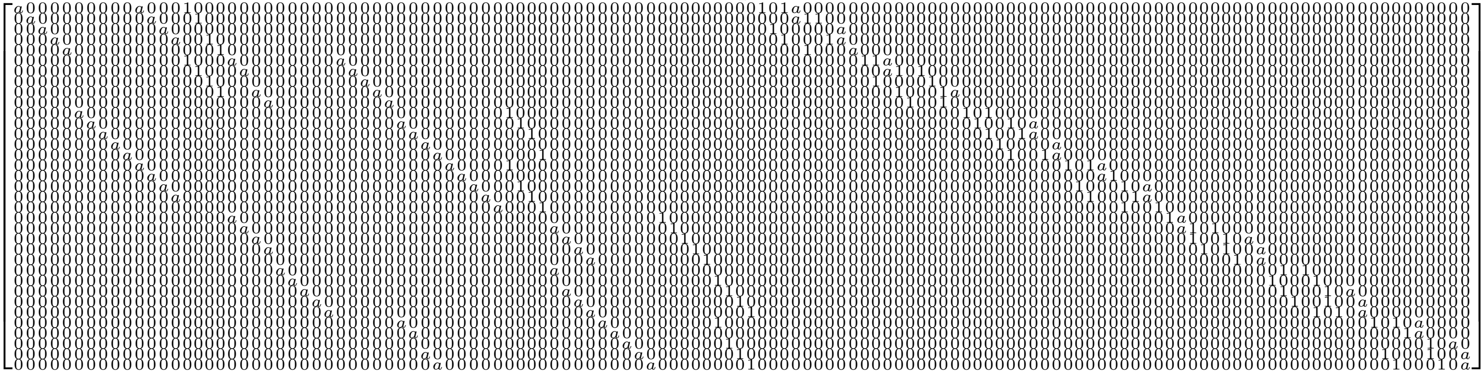
The derivative d_2 for H is a 126×140 matrix:



The exterior derivative d_3 for the Möbius strip H is the 35×126 matrix



It takes as an argument a 3-form F, a function on pairs (x, y) of simplices for which the dimension adds to three. These are the 63 intersecting edge-triangle pairs together with the 63 intersecting triangle-edge pairs, in total 126 pairs. The derivative result dF is a 4-form, a function on the 35 triangle-triangle pairs.

It is rare to have trivial quadratic cohomology. We let the computer search through Erdös-Renyi spaces of graphs with 10 vertices and found one in maybe 5000 graphs. Interestingly, so far, all of them were homeomorphic to the Moebius strip.

Remarks

- Quadratic cohomology is the second of an infinite sequence of interaction cohomologies. In general, we can work with k-tuples of pairwise intersecting simplices. The basic numerical value in the k-linear case is the **k'th Wu characteristic** $\omega_k(G) = \sum_{x_1 \sim \dots \sim x_k} (-1)^{\sum_j \dim(x_j)}$ which sums over all pairwise intersecting ordered k-tuples $(x_1, \dots, x_k)$ of simplices in the graph G. The example of odd-dimensional discrete manifolds shows that it is possible that all higher cohomologies depend on simplicial cohomology only. So, even when including all cohomologies, we never can get a complete set of invariants. There are always non-homeomorphic graphs with the same higher cohomology data.

- In the case of the example used here, the cubic interaction cohomology of the graph is $(0, 0, 0, 0, 1, 1, 0)$. It is the same than the cubic interaction cohomology of the Möbius strip. The Laplacian L of the cohomology already has 3824 entries in the Cylinder case with $|V| = 8$ and 3346 entries in the Möbius case with $|V| = 7$. After a Barycentric refinement, the cylinder is a graph with 32 vertices and 80 edges. The cubic interaction Laplacian is then already a 45280×45280 matrix. For the refinment of the Möbius strip, a graph with 28 vertices and 70 edges, the cubic interaction Laplacian is a 39620×39620 matrix. This is already tough for computers. In the quadratic case, where we have a 2856×2856 Laplacian matrix in the Möbius case, we can still determine that the determinant is invertible. The determinant is a 1972 digit number but it appears already too large to be factored (we can factor $\det(L_0), \det(L_1)$ but no more $\det(L_2)$.

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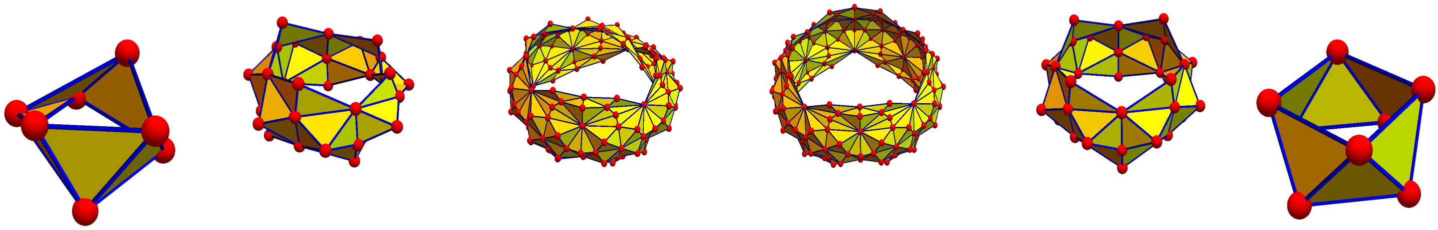
- The interaction Betti numbers satisfy the Whitney-Cartan type formula $b_p(G \times H) = \sum_{k+l=p} b_k(G) b_l(H)$ like Stiefel-Whitney classes but seem unrelated to Stiefel-Whitney classes.

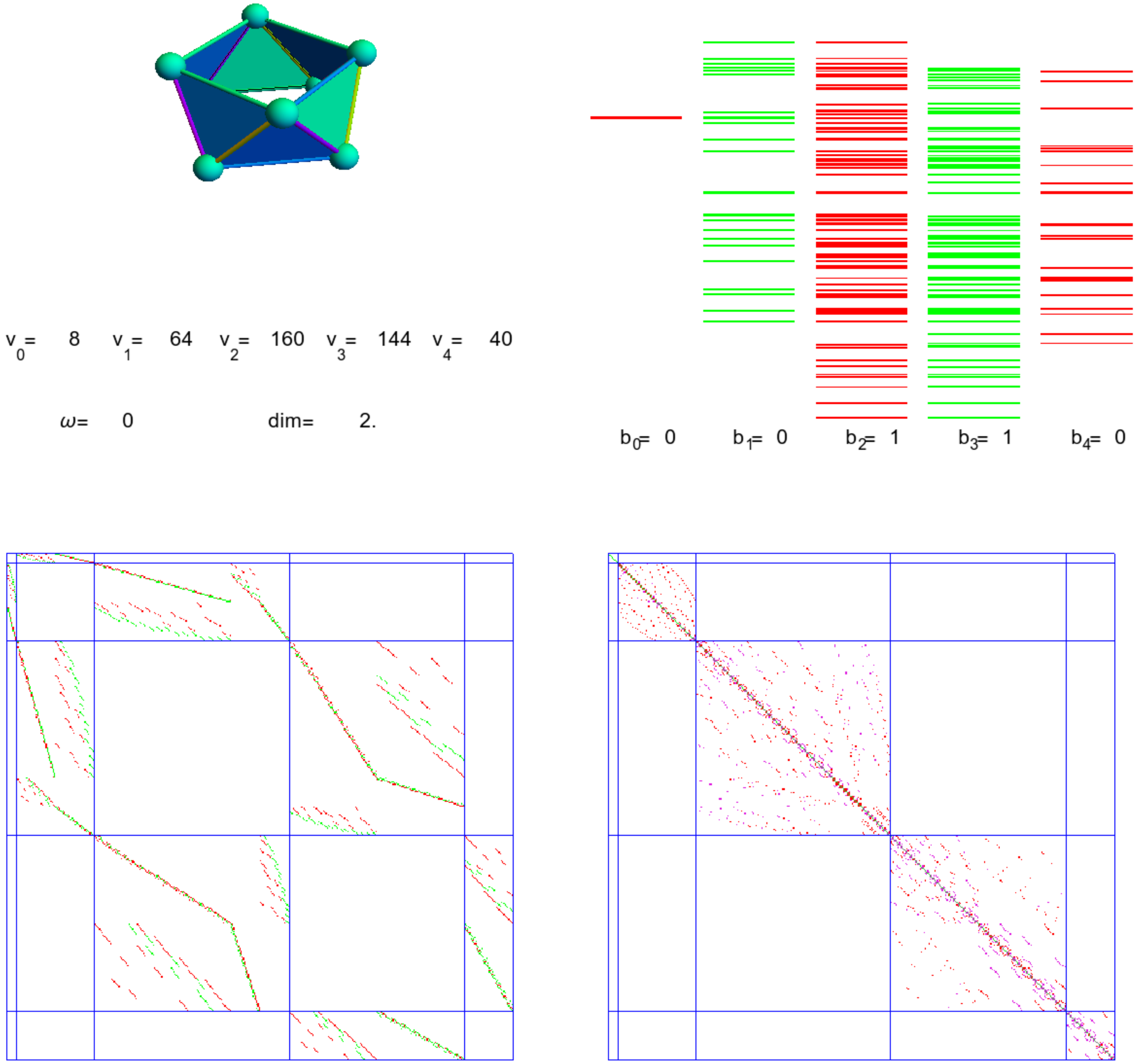
- The Cylinder G and Möbius strip H can be distinguished topologically: one tool is orientation, an other is the connectedness of the boundary. Why is an algebraic description via Betti numbers interesting? Because they are purely algebraically defined.

- Using computer search we have found examples of finite simple graphs for which the f-vector and f-matrix are both the same, where the singular cohomologies are the same and where the Wu characteristic is the same, but where the Betti vector for quadratic cohomology is different.

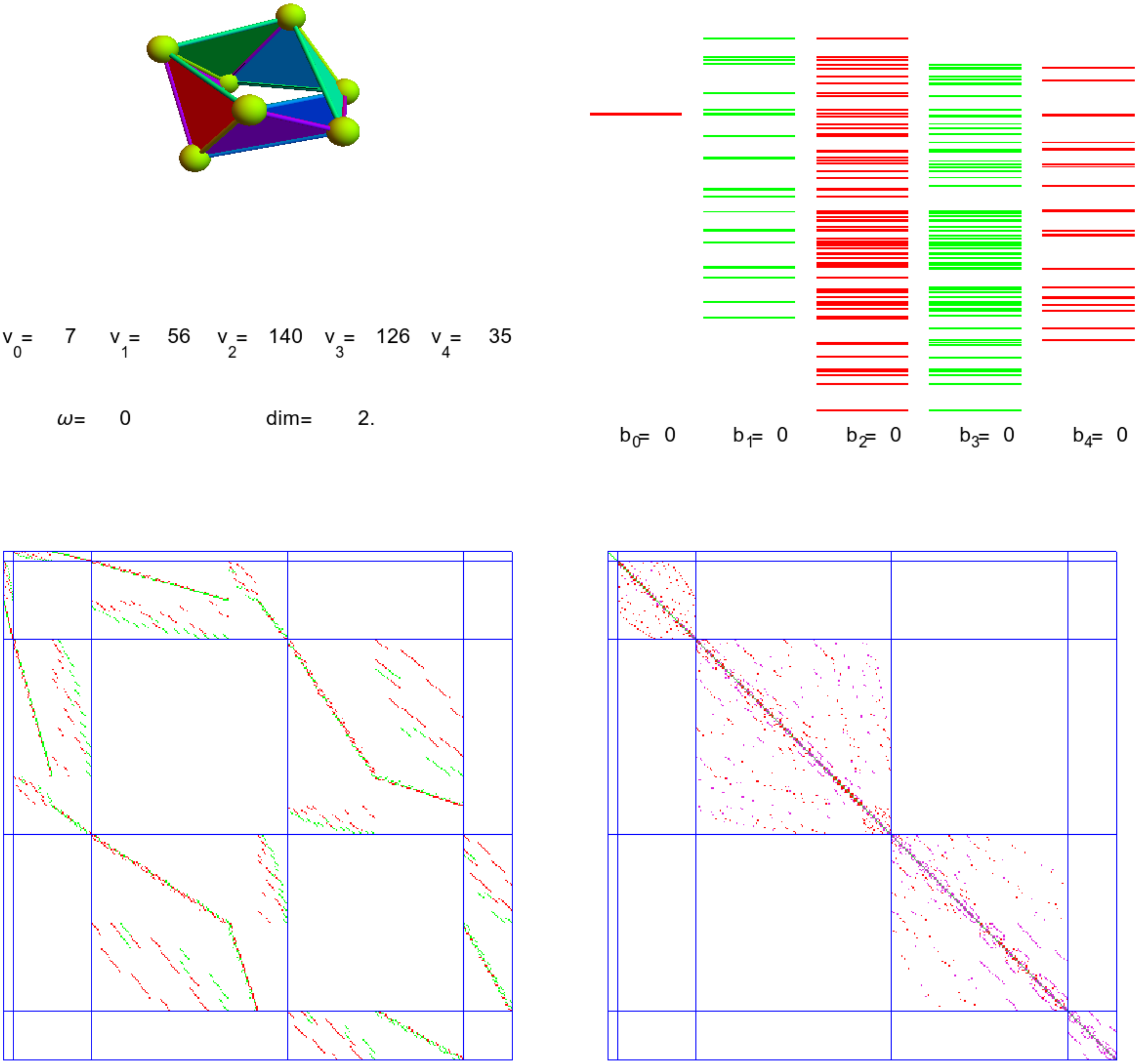
- For complicated networks, the computations of interaction cohomology become heavy, already in cubic cases we reach limits of computing rather quickly. It can be useful therefore to find first a smaller homeomorphic graph and compute the cohomology there.

- Homeomorphic graphs have the same simplicial cohomologies. This is true also for higher cohomologies but it is less obvious. One can see it as follows; after some Barycentric refinements, every point is now either a point with manifold neighborhood or a singularity, where various discrete manifolds with boundary come together. For homeomorphic graphs, the structure of these singularities must correspond. It follows from Gauss-Bonnet formulas that the higher Wu-characteristics are the same. To see it for cohomology, we will have to look at Meyer-Vietoris type computations for glueing cohomologie or invariance under deformations like edge collapse or refinement.
- We feel that the subject of interaction cohomology has a lot of potential also to distinguish graphs embedded in other graphs. While Wu characteristic for embedded knots in a 3-sphere is always 1, the cohomology could be interesting.
- We have not yet found a known cohomology in the continuum which comes close to interaction cohomology. Anything we looked at looks considerably more complicated. Interaction cohomology is elementary: one just defines some concrete finite matrices and computes their kernel.
- For more, see [6, 5, 3, 4, 8, 1, 2, 7].





The figure shows the Dirac and Laplacian of the Cylinder graph G. We see the spectrum of the interaction Laplacian on each of the 5 interaction form sectors $p = 0, \dots, p = 4$. We see also the super symmetry: the union of the spectra on even dimensional forms $p = 0, 2, 4$ is the same than the union of the spectra on odd-dimensional forms $p = 1, 3$.



The figure shows the Dirac and Laplacian of the Möbius graph G. We see the spectrum of the interaction Laplacian on each of the 5 interaction sectors $p = 0, \dots, p = 4$. There is one important difference however: the Laplacian is invertible.

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