Lecture 12: Dynamical systems

12.1. Dynamical systems theory is the science of time. If time is continuous, the evolution is defined by a differential equation $\dot{x} = f(x)$. If time is discrete, then we look at the iteration of a map $x \rightarrow T(x)$. Here is the prototype of a differential equation in three dimensions:

$$
\dot{x} = \sigma(y - x) \\
\dot{y} = rx - y - xz \\
\dot{z} = xy - bz .
$$

the Lorenz system. There are three parameters $\sigma, r, b$. For $\sigma = 10, r = 28, b = 8/3$, one observes a strange attractor with fractal shape.

12.2. The goal is to predict the future of the system when the present state is known. A differential equation is an equation of the form $d/dt x(t) = f(x(t))$, where the unknown quantity is a path $x(t)$ in some “phase space”. We know the velocity $d/dt x(t) = \dot{x}(t)$ at all times and the initial configuration $x(0)$, we can to compute the trajectory $x(t)$. What happens at a future time? Does $x(t)$ stay in a bounded region or escape to infinity? Which areas of the phase space are visited and how often? Can we reach a certain part of the space when starting at a given point and if yes, when. An example of such a question is to predict, whether an asteroid located at a specific location will hit the earth or not. An other example is to predict the weather of the next week.
12.3. An example of a dynamical systems in one dimension is the differential equation

\[ x'(t) = x(t)(2 - x(t)), \quad x(0) = 1. \]

It is called the logistic system and describes population growth. This system has the solution

\[ x(t) = 2e^{t}/(1 + e^{2t}) \]

as you can see by computing the left and right hand side.

12.4. A map is a rule which assigns to a quantity \( x(t) \) a new quantity \( x(t+1) = T(x(t)) \). The state \( x(t) \) of the system determines the situation \( x(t+1) \) at time \( t+1 \). An example is is the Ulam map \( T(x) = 4x(1-x) \) on the interval \([0,1]\). This is an example, where we have no idea what happens after a few hundred iterates even if we would know the initial position with the accuracy of the Planck scale. We will experiment with that in class.

12.5. Dynamical system theory has applications in all fields of mathematics. We can use dynamical systems for example to find roots of equations. The Newton map

\[ T(x) = x - f(x)/f'(x) \]

is such a procedure. If we are close enough to the fixed point, applying \( T \) again and again will have us converge very fast to the fixed point.

12.6. Dynamical systems also appear in number theory. For large primes \( p \), nonlinear maps like \( T(x) = x^2 + c \mod p \) or \( T(x) = a^x \mod p \) behave rather erratically. And this is good so as the maps can be used for encryption.

12.7. A rather curious system of number theoretical nature is the Collatz map

\[ T(x) = \frac{x}{2} \text{ (even } x \text{)}, \quad 3x + 1 \text{ else}. \]

A system of geometric nature is the Pedal map which assigns to a triangle the pedal triangle.

12.8. Let’s look a bit at the history of chaos: about 100 years ago, Henry Poincaré was able to deal with chaos of low dimensional systems. While statistical mechanics had formalized the evolution of large systems with probabilistic methods already, the new insight was that simple systems like a three body problem or a billiard map can produce very complicated motion. It was Poincaré who saw that even for such low dimensional and completely deterministic systems, random motion can emerge.

12.9. While physicists have dealt with chaos earlier by assuming it or artificially feeding it into equations like the Boltzmann equation, the occurrence of stochastic motion in simple systems like double penduli, geodesic flows or billiards or restricted three body problems was a surprise. These findings needed half a century to sink in and only with the emergence of computers in the 1960ies, the awakening happened. Icons like Lorentz helped to popularize the findings and we owe them the ”butterfly effect” picture: a wing of a butterfly can produce a tornado in Texas in a few weeks.

12.10. The reason for this statement is that the complicated equations to simulate the weather reduce under extreme simplifications and truncations to a simple differential equation \( \dot{x} = \sigma(y - x), \dot{y} = rx - y - xz, \dot{z} = xy - bz \), the Lorenz system. For \( \sigma = 10, r = 28, b = 8/3 \), Ed Lorenz discovered in 1963 an interesting long time behavior and an aperiodic ”attractor”. Ruelle-Takens called it a strange attractor. It is a great moment in mathematics to realize that attractors of simple systems can become fractals on which the motion is chaotic. It suggests that such behavior is abundant. What is chaos? If a dynamical system shows sensitive dependence on initial conditions, we talk about chaos. We will experiment with the two maps \( T(x) = 4x(1 - x) \) and \( S(x) = 4x - 4x^2 \) which starting with the same initial conditions will produce different outcomes after a couple of iterations.
12.11. The sensitive dependence on initial conditions is measured by how fast the derivative \(dT^n\) of the \(n\)th iterate grows. The exponential growth rate \(\gamma\) is called the Lyapunov exponent. A small error of the size \(h\) will be amplified to \(he^{\gamma n}\) after \(n\) iterates. In the case of the Logistic map with \(c = 4\), the Lyapunov exponent is \(\log(2)\) and an error of \(10^{-16}\) is amplified to \(2^n \cdot 10^{-16}\). For time \(n = 53\) already the error is of the order 1. This explains the above experiment with the different maps. The maps \(T(x)\) and \(S(x)\) round differently on the level \(10^{-16}\). After 53 iterations, these initial fluctuation errors have grown to a macroscopic size.

12.12. Here is a famous open problem which has resisted many attempts to solve it: Show that the map

\[
T(x, y) = (c \sin(2\pi x) + 2x - y, x)
\]

with \(T^n(x, y) = (f_n(x, y), g_n(x, y))\) has sensitive dependence on initial conditions on a set of positive area. More precisely, verify that for \(c > 2\) and all \(n \frac{1}{n} \int_0^1 \int_0^1 \log |\partial_x f_n(x, y)| \, dx \, dy \geq \log(\frac{c}{2})\).

I have tried over a decade to prove this using methods from quantum mechanics, calculus of variations and complex analytic methods. The problem is open.

12.13. The left hand side converges to the average of the Lyapunov exponents which is in this case also the entropy of the map. For some systems, one can compute the entropy. The logistic map with \(c = 4\) for example, which is also called the Ulam map, has entropy \(\log(2)\). The cat map

\[
T(x, y) = (2x + y, x + y) \mod 1
\]

has positive entropy \(\log |(\sqrt{5} + 3)/2|\). This is the logarithm of the larger eigenvalue of the matrix implementing \(T\).

12.14. While questions about simple maps look artificial at first, the mechanisms prevail in other systems: in astronomy, when studying planetary motion or electrons in the van Allen belt, in mechanics when studying coupled pendulum or nonlinear oscillators, in fluid dynamics when studying vortex motion or turbulence, in geometry, when studying the evolution of light on a surface, the change of weather or tsunamis in the ocean.

12.15. Dynamical systems theory historically started with the problem to understand the motion of planets. Newton realized that this is governed by a differential equation, the \(n\)-body problem

\[
x''_j(t) = \sum_{i=1}^{n} \frac{c_{ij}(x_i - x_j)}{|x_i - x_j|^3},
\]

where \(c_{ij}\) depends on the masses and the gravitational constant. If one body is the sun and no interaction of the planets is assumed and using the common center of gravity as the origin, this reduces to the Kepler problem \(x''(t) = -Cx/|x|^3\), where planets move on ellipses, the radius vector sweeps equal area in each time and the period squared is proportional to the semi-major axes cubed. A great moment in astronomy was when Kepler derived these laws empirically. An other great moment in mathematics is Newton’s theoretically derivation from the differential equations.

**Work problems**

1) We experiment with simple transformations can produce chaotic outcome. Make sure your calculator is in the "Rad" mode. Remember that \(2\pi\) radians is equal to 360 degrees. You can check whether your calculator is in Radian mode, by computing \(\cos(\pi)\) and get the result \(-1\). Make sure your calculator is in rad mode. Use a scientific calculator. In the iphone calculator for example, turn the device to get to the scientific mode.
The Scientific Calculator built in by default in the Iphone/Ipod/Ipad appears when you turn the device.

a) Take a calculator, and pushing repetitively the button cos. What do you observe?

b) Now repeat pushing the sin button. What do you observe?

c) Now push $x^2$ repetitively.

d) Now push $\sqrt{x}$ repetitively.

e) What do you see if you push the buttons sin, then type $1/x$ and repeat this process again and again?

f) Experiment with the button tan. Also here, change from tan and cot (in the cot case, you might have to hit tan and $1/x$ buttons after each other).
g) Look for other "chaotic" key combinations? Experiment also with Deg and Rad changes and try especially the log functions.

2) We graphically compute a few iterates of one dimensional maps. This can be done on paper. One produces a so called cobweb.
3) We look at a dynamical system of number theoretical nature. In the **Collatz system**, we start with an integer and map it with the following rule:

\[
T(x) = \begin{cases} 
\frac{x}{2} & \text{x even} \\
3x + 1 & \text{x odd}
\end{cases}
\]

The question is whether the orbit always ends up with 1. For example: \( x = 7 \) produces 7, 22, 11, 34, 17.

a) Start with the initial condition 26:
b) Start with the initial condition 9:
c) Start with the initial condition 2048:
d) What is wrong with the following proof of the Collatz conjecture?

Proof. Consider only the odd numbers in the Collatz sequence. We show that each odd number is in average \( \frac{3}{4} \) times smaller than the previous one:

With probability \( \frac{1}{2} \) the number \( 3x + 1 \) is divisible by 2 and not 4: this increases \( x \) by \( \frac{3}{2} \)

With probability \( \frac{1}{4} \) the number \( 3x + 1 \) is divisible by 4 and not 8: this decreases \( x \) by \( \frac{3}{4} \)

With probability \( \frac{1}{8} \) the number \( 3x + 1 \) is divisible by 8 and not 16: this decreases \( x \) by \( \frac{3}{8} \)

To compute the probability, we take logarithms and compute

\[
a = \sum_{n=1}^{\infty} \frac{1}{2^n} \log(3/2^n).
\]

The average decay rate of the size of a number is the factor \( e^a = \frac{3}{4} \).
e) The Collatz system certainly can be modified. Can you find one, for which there is a nontrivial loop?

4) We look at a dynamical systems called Cellular automata. These are continuous maps on sequence spaces in which the evolution rule is translational invariant.

<table>
<thead>
<tr>
<th>neighborhood</th>
<th>new middle cell</th>
</tr>
</thead>
<tbody>
<tr>
<td>111</td>
<td>0</td>
</tr>
<tr>
<td>110</td>
<td>0</td>
</tr>
<tr>
<td>101</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
</tr>
<tr>
<td>011</td>
<td>0</td>
</tr>
<tr>
<td>010</td>
<td>0</td>
</tr>
<tr>
<td>001</td>
<td>1</td>
</tr>
<tr>
<td>000</td>
<td>0</td>
</tr>
</tbody>
</table>

to an offspring 1, we and $100 = 4$, $001 = 1$ in binary, we have $2^4 + 2^1 = 18$. 
a) run it:
Teaching Mathematics with a Historical Perspective

S. Knill, knill@math.harvard.edu Math E-320, Harvard Extension School, 2010-2021