The Real Grassmannian $\text{Gr}(2, 4)$

We discuss the topology of the real Grassmannian $\text{Gr}(2, 4)$ of 2-planes in $\mathbb{R}^4$ and its double cover $\text{Gr}^+(2, 4)$ by the Grassmannian of oriented 2-planes. They are compact four-manifolds.

0. A Remark on Four-Manifolds

By applying the universal coefficients theorem and Poincaré duality to a general closed orientable four-manifold $M$, one finds that its homology and cohomology is restricted to take the following form:

$$
\begin{align*}
H_p(M, \mathbb{Z}) & \cong \mathbb{Z} \oplus T^a \oplus T^b \\
H^p(M, \mathbb{Z}) & \cong \mathbb{Z} \oplus T^a \oplus T^b \oplus T^a \oplus T^b \\
\end{align*}
$$

Here $a$ and $b$ are unknown integers and $T$ is a finite abelian group. They are determined by the fundamental group and the Euler characteristic.

$$
H_1 = \pi_1 \quad \chi = 2 - 2a + b.
$$

We will compute the homology and cohomology of $\text{Gr}(2, 4)$ by finding these two invariants.

1. A Cell Decomposition

Assume for now that the Grassmannian $\text{Gr}(2, 4)$ is orientable. Any 2-plane can be represented as the row space of a $2 \times 4$ matrix, and there is always a unique row-reduced representative. This decomposes $\text{Gr}(2, 4)$ into cells according to the shape of the row-reduced matrix.

$$
\begin{align*}
0\text{-cells} & : \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
1\text{-cells} & : \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
2\text{-cells} & : \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & * & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
3\text{-cells} & : \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \\
4\text{-cells} & : \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}
\end{align*}
$$

So the Euler characteristic is

$$
\chi = 1 - 1 + 2 - 1 + 1 = 2.
$$
Since $\pi_1$ is always generated by the 1-cells, we see that $\pi_1 = H_1$ is a cyclic group generated by the loop $\gamma$ corresponding to the 1-cell in the decomposition above. We will be finished if we can establish the following claim:

**Proposition.** The element $\gamma$ has order two.

**Proof 1.** The Grassmannian admits a connected double cover

$$\text{Gr}^+(2, 4) \rightarrow \text{Gr}(2, 4)$$

by the Grassmannian of oriented 2-planes. The existence of such a covering implies that $\pi_1$, and hence, $\gamma$ is nontrivial.

To see that $\gamma$ has order two, observe that it lies in the subspace

$$\text{Gr}(2, 3) = \{2\text{-planes contained in the hyperplane } (0, *, *, *)\} \subset \text{Gr}(2, 4)$$

The fundamental group of $\text{Gr}(2, 3) \cong \mathbb{RP}^2$ is $\mathbb{Z}/2\mathbb{Z}$, so $\gamma^2$ is nullhomotopic.

**Proof 2.** It is possible to circumvent the use of $\text{Gr}^+(2, 4)$ by identifying the attaching maps of the 2-cells directly. For example, the $\text{Gr}(2, 3)$ we just considered is a union of cells

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$ 

Therefore, the attaching map of the 2-cell in this decomposition has degree 2. For the other 2-cell, observe that

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the subspace of 2-planes containing $L$. This is the same as the space of lines in $\mathbb{R}^4/L$, which forms another $\mathbb{RP}^2 = \text{Gr}(1, 3)$. So the attaching map of this 2-cell has degree two as well.

In any case, we have determined the homology and cohomology of $\text{Gr}(2, 4)$.

$$\begin{array}{cccccc}
H_0(M, \mathbb{Z}) & Z & Z/2\mathbb{Z} & Z/2\mathbb{Z} & 0 & Z & 0 \\
H^0(M, \mathbb{Z}) & Z & 0 & Z/2\mathbb{Z} & Z/2\mathbb{Z} & Z & 0
\end{array}$$

### 2. A Splitting

In this section, we will obtain identifications

$$\text{Gr}^+(2, 4) \cong S^2 \times S^2 \quad \text{Gr}(2, 4) \cong S^2 \times S^2/\langle (\tau, \tau) \rangle$$

where $\tau : S^2 \rightarrow S^2$ is the antipodal map. This justifies our previous assumption that $\text{Gr}(2, 4)$ is orientable. It is also gives another way to compute the homology and cohomology of $\text{Gr}(2, 4)$, since we can read off the invariants

$$\pi_1(\text{Gr}(2, 4)) = \mathbb{Z}/2\mathbb{Z} \quad \chi(\text{Gr}(2, 4)) = \frac{1}{2} \chi(S^2)^2 = 2.$$
Recall that $\text{SU}(2)$ can be identified with the group of unit norm quaternions. In what follows, we will not need to view elements of $\text{SU}(2)$ as unitary matrices, so we may as well take this to be the definition. This allows us to define a map

$$f : \text{SU}(2) \times \text{SU}(2) \to \text{SO}(4) \quad (p, q) \mapsto p(-)q^{-1}.$$ 

To be precise, we use the orthonormal basis $1, i, j, k$ for $\mathbb{H}$ to identify $\mathbb{H}$ with $\mathbb{R}^4$ and hence the group of orthogonal transformations of $\mathbb{H}$ with $\text{SO}(4)$.

**Lemma 1.** The map $f$ induces an isomorphism

$$\bar{f} : \text{SU}(2) \times \text{SU}(2)/\{\pm (I, I)\} \to \text{SO}(4).$$

**Proof.** If $(p, q)$ belongs to the kernel of $f$, then

$$x = f(p, q) \cdot x = px^{-1}$$

for all $x \in \mathbb{H}$. Taking $x = 1$ shows that $p = q$. Then this equation says that $p$ is in the center $Z(\mathbb{H}) = \mathbb{R}$. So the kernel of $f$ is $\pm (I, I)$.

But now $\bar{f}$ is an injective map between Lie groups of the same dimension, so it is open. Its image is closed because $\text{SU}(2) \times \text{SU}(2)$ is compact. Therefore, $\bar{f}$ is surjective because $\text{SO}(4)$ is connected. \hfill $\square$

**Lemma 2.** There is a homeomorphism

$$\text{SO}(4)/\text{SO}(2) \times \text{SO}(2) \to \text{Gr}^+(2, 4) \quad A \mapsto \langle Ae_1, Ae_2 \rangle,$$

where

$$\text{SO}(2) \times \text{SO}(2) \to \text{SO}(4) \quad (A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Now we identify the preimage of $\text{SO}(2) \times \text{SO}(2)$ in $\text{SU}(2) \times \text{SU}(2)$. We will make use of the identification

$$\text{U}(1) \cong \text{SO}(2) \quad \cos \theta + i \sin \theta \mapsto \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$ 

**Lemma 3.** We have a cartesian square

$$\begin{array}{ccc}
\text{U}(1) \times \text{U}(1) & \longrightarrow & \text{SU}(2) \times \text{SU}(2) \\
\downarrow & & \downarrow \\
\text{SO}(2) \times \text{SO}(2) & \longrightarrow & \text{SO}(4)
\end{array}$$

The upper horizontal map is the composition $\text{U}(1) \cong \text{SO}(2) \hookrightarrow \text{SU}(2)$ on each factor, and the vertical map on the left is

$$(z, w) \mapsto (zw^{-1}, zw).$$
Proof. The commutativity of the diagram amounts to the identity

\[
\begin{pmatrix} zw & 0 \\ 0 & zw \end{pmatrix} x = zx \bar{w}.
\]

In fact, this equality holds true for any pair of complex numbers \( z \) and \( w \). To verify this, note that it suffices to consider pairs of the form \((z, 1)\) and \((1, w)\). Furthermore, the equation is bilinear over \( \mathbb{R} \), so we only need to check \((i, 1)\) and \((1, i)\).

\[
f(i, 1) : 1 \mapsto i \quad i \mapsto -1 \quad j \mapsto k \quad k \mapsto -j
\]

\[
f(1, i) : 1 \mapsto -i \quad i \mapsto 1 \quad j \mapsto k \quad k \mapsto -j
\]

The square is cartesian because the arrow on the left is surjective and \( U(1) \times U(1) \) contains the kernel of \( f \).

Combining these lemmata gives a chain of homeomorphisms

\[
\text{Gr}^+(2, 4) \cong \text{SO}(4)/\text{SO}(2) \times \text{SO}(2) \cong \text{SU}(2) \times \text{SU}(2)/U(1) \times U(1).
\]

Of course

\[
\text{SU}(2) \longrightarrow \text{SU}(2)/U(1) \cong \mathbb{P}^1
\]

is the Hopf fibration, so

\[
\text{Gr}^+(2, 4) \cong S^2 \times S^2.
\]

We can go further and identify explicitly the involution giving the double cover

\[
\text{Gr}^+(2, 4) \longrightarrow \text{Gr}(2, 4).
\]

**Proposition.** We have

\[
\text{Gr}(2, 4) \cong S^2 \times S^2/\langle (\tau, \tau) \rangle
\]

where \( \tau \) is the antipodal map on \( S^2 \).

**Proof.** Observe that

\[
\text{Gr}(2, 4) = (J)\backslash \text{SO}(4)/\text{SO}(2) \times \text{SO}(2) \quad J = \text{diag}(1, -1, 1, -1).
\]

The pre-image of \( (J) \) in \( \text{SU}(2) \times \text{SU}(2) \) is the subgroup generated by \((j, j)\).

\[
f(j, j) : 1 \mapsto 1 \quad i \mapsto -i \quad j \mapsto j \quad k \mapsto -k
\]

So we need to identify the action of \( j \) on \( \text{SU}(2)/U(1) \) as the antipodal action.

Viewing \( \mathbb{H} = \mathbb{C} \oplus \mathbb{C} j \) as a vector space over \( \mathbb{C} \), we have an explicit formula for the Hopf fibration

\[
\text{SU}(2) \longrightarrow \mathbb{P}^1 \quad x_0 + x_1 j \longmapsto [x_0 : x_1].
\]

We compute that

\[
j(x_0 + x_1 j) = -\overline{x}_1 + \overline{x}_0 j,
\]

so multiplication by \( j \) descends to the antipodal map \([x_0 : x_1] \mapsto [-\overline{x}_1 : \overline{x}_0] \).
3. Lie Theory

The result

\[ \text{SO}(4)/\text{SO}(2) \times \text{SO}(2) \cong \mathbb{P}^1 \times \mathbb{P}^1 \]

of the previous section can be obtained without calculation if we are willing to use the theory of compact Lie groups. The subgroup \( T = \text{SO}(2) \times \text{SO}(2) \) is a maximal torus of \( G = \text{SO}(4) \), and we need to identify the space \( G/T \).

Since \( \text{SU}(2) \times \text{SU}(2) \) is simply connected and has the same complexified Lie algebra as \( G \), it must be the universal cover \( \tilde{G} \) of \( G \). Since \( \tilde{T} = \text{U}(1) \times \text{U}(1) \) is a maximal torus of \( \tilde{G} \),

\[ \frac{G}{T} \cong \frac{\tilde{G}}{\tilde{T}} \cong \frac{\text{SU}(2)/\text{U}(1) \times \text{SU}(2)/\text{U}(1)}{\cong \mathbb{P}^1 \times \mathbb{P}^1}. \]

We could also use the fact that \( G/T \cong G_C/B \) is the flag variety of the complexified group. In our case, we the Borel subgroups of \( \text{SO}(4, \mathbb{C}) \) are the stabilizers of flags taking the form

\[ 0 \subset L \subset W = W^\bot \subset L^\bot \subset \mathbb{C}^4. \]

Every line with \( L \subset L^\bot \) determines two such flags. Furthermore, two flags have the same stabilizer precisely when they start with the same line \( L \). Therefore, the flag variety \( \text{SO}(4, \mathbb{C})/B \) is the space of lines in \( \mathbb{C}^4 \) such that \( L \subset L^\bot \). This is a smooth quadric surface in \( \mathbb{P}^3 \).