Floer and Khovanov homologies of band sums

Joshua Wang

March 9, 2021
The cosmetic crossing conjecture

A nugatory crossing change:
The cosmetic crossing conjecture

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The cosmetic crossing conjecture: Every crossing change of an oriented knot that does not change the oriented knot type is nugatory.
Let $K_+$ and $K_-$ be oriented knots in $S^3$ differing by a crossing change, and let $L$ be the two-component link obtained by taking the oriented resolution.

\begin{center}
\begin{tikzpicture}

\draw[thick,->] (0,0) -- (1,1);
\draw[thick,-] (0,0) -- (1,-1);
\draw[thick,->] (2,0) -- (3,1);
\draw[thick,-] (2,0) -- (3,-1);
\draw[thick,->] (4,0) arc (270:90:1);
\draw[thick,-] (4,0) arc (90:270:1);
\node at (1.5,0.5) {$K_+$};
\node at (2.5,0.5) {$K_-$};
\node at (2.5,-0.5) {$L$};
\end{tikzpicture}
\end{center}
The cosmetic crossing conjecture

Let $K_+$ and $K_-$ be oriented knots in $S^3$ differing by a crossing change, and let $L$ be the two-component link obtained by taking the oriented resolution.

$K_-$ can be obtained from $L$ by band surgery along a band $b$. 
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Let $K_+$ and $K_-$ be oriented knots in $S^3$ differing by a crossing change, and let $L$ be the two-component link obtained by taking the oriented resolution.

$K_-$ can be obtained from $L$ by band surgery along a band $b$.

$K_+$ can then be obtained by adding a full twist to the band.
Cosmetic crossing conjecture (Problem 1.58 on Kirby’s list)

If $K_+$ and $K_-$ are isotopic as oriented knots, then the link $L$ is split and the band $b$ is trivial.
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Cosmetic crossing conjecture (Problem 1.58 on Kirby’s list)
If $K_+$ and $K_-$ are isotopic as oriented knots, then the link $L$ is split and the band $b$ is trivial.

Recall:

- $L$ is *split* if there exists an embedded sphere which separates its components.
- $b$ is *trivial* if there exists a splitting sphere for $L$ which intersects $b$ along a single arc.
The cosmetic crossing conjecture

Example: $L$ is split and $b$ is nontrivial.
The cosmetic crossing conjecture

The cosmetic crossing conjecture is true for:

- the unknot (Scharlemann-Thompson 1989)
- 2-bridge knots (Torisu 1999)
- composite knots, if the conjecture is true for prime knots (Torisu 1999)
- fibered knots (Kalfagianni 2012)
- genus 1 knots $K$ except when $K$ is algebraically slice and $H_1(\Sigma_2(K))$ is finite cyclic (Balm-Friedl-Kalfagianni-Powell 2012)
- Whitehead doubles of prime, non-cable knots (Balm-Kalfagianni 2016)
- knots $K$ for which $\Sigma_2(K)$ is an L-space and each cyclic summand of $H_1(\Sigma_2(K))$ has square-free order (Lidman-Moore 2017)
- genus 1 knots with nontrivial Alexander polynomial (Ito 2021).

In particular, all prime knots with crossing number $\leq 9$. 

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The cosmetic crossing conjecture

Fix a two-component link $L$ and a band $b$.

Let $K_b$ be the result of band surgery, and let $K_b + 1$ be obtained by adding a full twist to the band.

**Cosmetic crossing conjecture (restated)**

The oriented knots $K_b$ and $K_b + 1$ are distinct unless $L$ is split and $b$ is trivial.

**The cosmetic crossing conjecture for a two-component link $L$:**

- if $L$ is nonsplit, then $K_b$ and $K_b + 1$ are distinct for any band $b$.
- if $L$ is split, then $K_b$ and $K_b + 1$ are distinct for any nontrivial band $b$. 

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The cosmetic crossing conjecture for a two-component link $L$:

- if $L$ is *nonsplit*, then $K_b$ and $K_{b+1}$ are distinct for any band $b$.
- if $L$ is *split*, then $K_b$ and $K_{b+1}$ are distinct for any *nontrivial* band $b$. 
The **generalized** cosmetic crossing conjecture

Fix a two-component link $L$ and a band $b$. Let $K_b$ be the result of band surgery, and let $K_{b+1}$ be obtained by adding a full twist to the band.

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Fix a two-component link $L$ and a band $b$. Let $K_b$ be the result of band surgery, and let $K_{b+n}$ be obtained by adding $n$ full twists to the band.

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Generalized cosmetic crossing conjecture

The oriented knots $K_{b+n}$ for $n \in \mathbb{Z}$ are distinct unless $L$ is split and $b$ is trivial.

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The oriented knots $K_{b+n}$ for $n \in \mathbb{Z}$ are distinct unless $L$ is split and $b$ is trivial.
Theorem (W. 2020)

*The generalized cosmetic crossing conjecture is true for split links.*
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The generalized cosmetic crossing conjecture is true for split links. If $L$ is a split two-component link, then $K_{b+n}$ for $n \in \mathbb{Z}$ are distinct for any nontrivial band $b$. 
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Let $K_b$ be a band sum of a split link $L$. Let $K\#$ be the connected sum.
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The generalized cosmetic crossing conjecture is true for split links. If \( L \) is a split two-component link, then \( K_{b+n} \) for \( n \in \mathbb{Z} \) are distinct for any nontrivial band \( b \).

Let \( K_b \) be a band sum of a split link \( L \). Let \( K_\# \) be the connected sum. For a knot invariant \( H \), two questions about band sums:

1. How are \( H(K_b) \) and \( H(K_{b+n}) \) related?
2. How are \( H(K_b) \) and \( H(K_\#) \) related?

In this talk: answers to these two questions for \( H = \) knot Floer homology, Khovanov homology, instanton knot homology.
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In this talk: answers to these two questions for $H =$ knot Floer homology, Khovanov homology, instanton knot homology.
Let $K_b$ be a band sum of a split two-component link $L$. 

Observation $\Delta(K_b + n) = \Delta(K_b)$ for all $n \in \mathbb{Z}$. 

Proof. $\Delta(L) = 0$ when $L$ is split so $\Delta(K_b + 1) - \Delta(K_b) = (t - 1/2 - t 1/2) \Delta(L) = 0$. $\square$
Let $K_b$ be a band sum of a split two-component link $L$. The Alexander polynomial $\Delta$ satisfies the skein relation

$$\Delta(J_+) - \Delta(J_-) = (t^{-1/2} - t^{1/2})\Delta(J_0).$$

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$\Delta(K_{b+n}) = \Delta(K_b)$ for all $n \in \mathbb{Z}$. 
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Observation

$\Delta(K_{b+n}) = \Delta(K_b)$ for all $n \in \mathbb{Z}$.

Proof.

$\Delta(L) = 0$ when $L$ is split so $\Delta(K_{b+1}) - \Delta(K_b) = (t^{-1/2} - t^{1/2})\Delta(L) = 0$.  \[\square\]
The knot Floer homology of $K_b$, denoted $\widehat{\text{HFK}}(K_b)$, is a vector space over $\mathbb{F} = \mathbb{Z}/2$ with a $\mathbb{Z} \oplus \mathbb{Z}$ bigrading. It categorifies $\Delta(K_b)$.

Theorem (W. 2020) $\widehat{\text{HFK}}(K_b + n)$ as bigraded vector spaces over $\mathbb{F}$.

The same is true for $\text{HFK}^-(K_b)$. Hedden-Watson 2018 proved the special case of this result when the split link is the unlink.

The instanton knot Floer homology of $K_b$, denoted $K\text{HI}(K_b)$, is a vector space over $\mathbb{C}$ with a $\mathbb{Z} \oplus \mathbb{Z}/2$ bigrading. It also categorifies $\Delta(K_b)$.

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$\widehat{\text{HFK}}(K_{b+n}) \cong \widehat{\text{HFK}}(K_b)$ as bigraded vector spaces over $F$.
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The instanton knot Floer homology of $K_b$, denoted $\text{KHI}(K_b)$, is a vector space over $C$ with a $\mathbb{Z} \oplus \mathbb{Z}/2$ bigrading. It also categorifies $\Delta(K_b)$. 
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Alexander polynomial and knot Floer homology

There are two main ingredients to the proof.
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1. The skein exact triangle, which categorifies the skein relation:

$$\widehat{\text{HFK}}(K_{b+1}) \rightarrow \widehat{\text{HFK}}(K_b)$$

The map $$\widehat{\text{HFK}}(K_{b+1}) \rightarrow \widehat{\text{HFK}}(K_b)$$ preserves both gradings.
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The map $\text{\widehat{HFK}}(K_{b+1}) \rightarrow \text{\widehat{HFK}}(K_b)$ preserves both gradings.

2. Maps on $\text{\widehat{HFK}}$ induced by ribbon concordances.
Recall:

- A *concordance* $C : J_0 \to J_1$ between knots $J_i \subset S^3$ is a properly embedded annulus $C \hookrightarrow [0, 1] \times S^3$ with $\partial C = -J_0 \times 0 \sqcup J_1 \times 1$. 

Theorem (Miyazaki 1998)

There is a ribbon concordance $C : K^\# \to K^b$.

Theorem (Zemke 2019)

A ribbon concordance $C : J_0 \to J_1$ induces an injective map $\hat{\text{HFK}}(J_0) \to \hat{\text{HFK}}(J_1)$.

In fact, $\hat{\text{HFK}}(J_1) \cong \hat{\text{HFK}}(J_0) \oplus F$ for some bigraded vector space $F$. 

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Ribbon concordances and knot Floer homology

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- A concordance $C : J_0 \to J_1$ is *ribbon* if the projection to $[0, 1]$ is a Morse function on $C$ with no index 2 critical points.

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Recall:

- A *concordance* $C : J_0 \to J_1$ between knots $J_i \subset S^3$ is a properly embedded annulus $C \hookrightarrow [0, 1] \times S^3$ with $\partial C = -J_0 \times 0 \sqcup J_1 \times 1$.

- A concordance $C : J_0 \to J_1$ is *ribbon* if the projection to $[0, 1]$ is a Morse function on $C$ with no index 2 critical points.

**Theorem (Miyazaki 1998)**

There is a ribbon concordance $C : K_\# \to K_b$.

**Theorem (Zemke 2019)**

A ribbon concordance $C : J_0 \to J_1$ induces an injective map

$$\widehat{\text{HFK}}(J_0) \to \widehat{\text{HFK}}(J_1).$$

In fact, $\widehat{\text{HFK}}(J_1) \cong \widehat{\text{HFK}}(J_0) \oplus F$ for some bigraded vector space $F$. 
Ribbon concordance $K_\# \rightarrow K_b$ (Miyazaki 1998)
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Proof sketch of $\widehat{\text{HFK}}(K_{b+n}) \cong \widehat{\text{HFK}}(K_b)$.

Choose (compatible) ribbon concordances $C' : K\# \to K_{b+1}$ and $C : K\# \to K_b$. 

There are induced splittings $\widehat{\text{HFK}}(K_{b+1}) \cong \widehat{\text{HFK}}(K\#) \oplus F_{b+1}$ for some bigraded vector spaces $F_b$ by Zemke's inclusion maps. These splittings are compatible with the skein exact triangles $\widehat{\text{HFK}}(K_{b+1}) \cong \widehat{\text{HFK}}(K\#) \oplus F_{b+1}$.
Proof sketch of $\hat{\mathit{HFK}}(K_{b+n}) \cong \hat{\mathit{HFK}}(K_b)$.

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Alexander polynomial and knot Floer homology

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$$\begin{align*}
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$$\begin{align*}
\widehat{\text{HFK}}(K_{b+1}) &\rightarrow \widehat{\text{HFK}}(K_b) \\
\& \downarrow \quad \downarrow \\
\widehat{\text{HFK}}(L) &\cong \widehat{\text{HFK}}(K_\#) & \widehat{\text{HFK}}(K_\#) &\rightarrow \widehat{\text{HFK}}(K_\#) \\
\& \downarrow \quad \downarrow \\
\widehat{\text{HFK}}(L) &\oplus \widehat{\text{HFK}}(L) & 0
\end{align*}$$
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$$\begin{align*}
\widehat{HFK}(K_{b+1}) \to \widehat{HFK}(K_b) \\
\downarrow \quad \downarrow \\
\widehat{HFK}(L) \quad &= \quad \widehat{HFK}(L) \\
\widehat{HFK}(K\#) \to \widehat{HFK}(K\#) \\
\downarrow \quad \downarrow \\
\widehat{HFK}(L) \quad &= \quad \widehat{HFK}(L) \\
F_{b+1} \quad \to \quad F_b \\
\oplus & \quad \downarrow \quad \downarrow \\
0 & \quad \quad \square
\end{align*}$$
Let $K_b$ be a band sum of a split two-component link $L$, and let $K\#$ be the connected sum.

**Observation**

Let $P_b$ satisfy $V(K_b) = V(K\#) + P_b$. Then $V(K_b + n) = V(K\#) + q^{4n}P_b$.

**Proof.**

Use the identity $V(L) = (q - 1 + q) V(K\#)$ and the skein relation. □

**Question**

Does the Jones polynomial detect the trivial band? Is $P_b$, $0$ when $b$ is nontrivial?
Let $K_b$ be a band sum of a split two-component link $L$, and let $K\#$ be the connected sum. The Jones polynomial $V$ satisfies the skein relation

$$q^{-2}V(J_+) - q^2V(J_-) = (q^{-1} - q)V(J_0).$$
Jones polynomial and Khovanov homology

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The Khovanov homology of $K_b$, denoted $\text{Kh}(K_b)$, is a vector space over $\mathbb{F}$ with a $\mathbb{Z} \oplus \mathbb{Z}$ bigrading $(h, q)$. It categorifies $V(K_b)$.
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A ribbon concordance $K_\# \to K_b$ induces a splitting $\text{Kh}(K_b) \cong \text{Kh}(K_\#) \oplus H_b$ (Levine-Zemke 2019). The graded Euler characteristic of $H_b$ is $P_b$. 

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Recall: $V(K_{b+n}) = V(K_\#) + q^{4n}P_b$. 
Jones polynomial and Khovanov homology

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**Theorem (W. 2020)**

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**Theorem (W. 2020)**

$\text{Kh}(K_{b+n}) \cong \text{Kh}(K_\#) \oplus h^{2n}q^{4n}H_b$

In fact, $\text{Kh}(K_{b+m/2}) \cong \text{Kh}(K_\#) \oplus h^m q^{2m}H_b$. 

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Proof sketch of $\text{Kh}(K_{b+n}) \cong \text{Kh}(K_{\#}) \oplus h^{2n}q^{4n}H_b$.

There are unoriented skein exact triangles

$$
\begin{align*}
\text{Kh}(K_b) & \rightarrow \text{Kh}(K_{b+1/2}) \rightarrow \text{Kh}(K_{b+1}) \\
\text{Kh}(L) & \leftarrow \text{Kh}(L) \\
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\end{align*}
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\begin{array}{c}
\text{Kh}(K_b) & \text{Kh}(K_{b+1/2}) & \text{Kh}(K_{b+1}) \\
\downarrow & \downarrow & \downarrow \\
\text{Kh}(L) & \text{Kh}(L) & \\
\end{array}
$$

compatible with ribbon concordance splittings (Levine-Zemke 2019)

$$
\begin{array}{c}
\text{Kh}(K_{#}) & \text{Kh}(K_{#}) & \text{Kh}(K_{#}) \\
\downarrow & \downarrow & \downarrow \\
\text{Kh}(L) & \text{Kh}(L) & \\
\end{array}
\oplus
\begin{array}{c}
H_b & H_{b+1/2} & H_{b+1} \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & \\
\end{array}
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\begin{array}{ccc}
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\downarrow & & \downarrow & & \downarrow \\
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\end{array}
\]

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\begin{array}{cccc}
\text{Kh}(K_\#) & \rightarrow & \text{Kh}(K_\#) & \rightarrow & \text{Kh}(K_\#) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Kh}(L) & \rightarrow & \text{Kh}(L) & \rightarrow & \text{Kh}(L)
\end{array} \oplus
\begin{array}{ccc}
H_b & \rightarrow & H_{b+1/2} & \rightarrow & H_{b+1} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

The isomorphisms $H_b \rightarrow H_{b+1/2} \rightarrow H_{b+1}$ each shift bigradings by $(1, 2)$. 

□
Jones polynomial and Khovanov homology

To show the groups $\text{Kh}(K_{b+n}) \cong \text{Kh}(K_{\#}) \oplus h^{2n}q^{4n}H_b$ for $n \in \mathbb{Z}$ are distinct, it suffices to show that $H_b \neq 0$ whenever $b$ is nontrivial.
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**Theorem (W. 2020)**

$\dim \text{Kh}(K_b) = \dim \text{Kh}(K_{\#})$ if and only if $b$ is trivial.

In other words, $H_b = 0$ if and only if $b$ is trivial.
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*The generalized cosmetic crossing conjecture is true for split links.*
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**Corollary (Miyazaki 2020)**

*If $K_b$ is isotopic to $K_\#$, then $b$ is trivial.*
Proposition

Let $H$ be a knot invariant taking the form of a vector space over $\mathbb{F}$ with a functorial spectral sequence $\text{Kh} \Rightarrow H$. If $\dim H$ detects the trivial band, then $\dim \text{Kh}$ detects the trivial band.

$H$ could be the Heegaard Floer homology of the double branched cover, singular instanton homology $\text{I}^{\#}$, etc. (Baldwin-Hedden-Lobb 2019).

Proof.

Suppose $\dim \text{Kh}(K^b) = \dim \text{Kh}(K^\#)$. Let $C: K^\# \to K^b$ be a ribbon concordance. Then $C$ induces a map of spectral sequences $\text{Kh} \Rightarrow H$ which is an isomorphism on the $E_2$-page. It is therefore an isomorphism on the $E_\infty$-page, so $\dim H(K^b) = \dim H(K^\#)$. Thus $b$ is trivial. □
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Let $H$ be a knot invariant taking the form of a vector space over $F$ with a functorial spectral sequence $Kh \Rightarrow H$.

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Jones polynomial and Khovanov homology

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There is a functorial spectral sequence $\text{Kh} \Rightarrow \text{I}^\#$.

**Theorem (W. 2020)**

Singular instanton homology detects the trivial band:

$$\dim \text{I}^\#(K_b) = \dim \text{I}^\#(K#)$$

if and only if $b$ is trivial.

The proof involves showing that $\dim K\text{HI}$ detects the trivial band.

Dowlin 2018 constructed a spectral sequence from Khovanov homology to knot Floer homology.

**Theorem (W. 2020)**

Knot Floer homology detects the trivial band:

$$\dim \hat{HFK}(K_b) = \dim \hat{HFK}(K#)$$

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Functoriality of Dowlin's spectral sequence has not been established.
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\[ \dim \widehat{\text{HFK}}(K_b) > \dim \widehat{\text{HFK}}(K\#) \text{ when } b \text{ is nontrivial} \]
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A diagram for $K_b \cup C$ has basepoints $w_K, z_K, w_C, z_C$. 
\[ \dim \widehat{\text{HFK}}(K_b) > \dim \widehat{\text{HFK}}(K\#) \text{ when } b \text{ is nontrivial} \]

A diagram for \( K_b \cup C \) has basepoints \( w_K, z_K, w_C, z_C \). Let \( \text{CFL}^-(K_b \cup C, \sigma) \) over \( \mathbb{F}[U] \) count discs blocked by \( w_K, z_K, w_C \) and record intersection with \( z_C \) in \( U \).
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**Claim:** \( \text{rank } \text{HFL}^-(K_b \cup C, \sigma) = 2 \cdot \dim \widehat{\text{HFK}}(K_b) \).
$\dim \widehat{HFK}(K_b) > \dim \widehat{HFK}(K\#)$ when $b$ is nontrivial

A diagram for $K_b \cup C$ has basepoints $w_K, z_K, w_C, z_C$. Let $\text{CFL}^-(K_b \cup C, \sigma)$ over $F[U]$ count discs blocked by $w_K, z_K, w_C$ and record intersection with $z_C$ in $U$.

**Claim:** $\text{rank } \text{HFL}^-(K_b \cup C, \sigma) = 2 \cdot \dim \widehat{HFK}(K_b)$. 
\[ \dim \widehat{\text{HFK}}(K_b) > \dim \widehat{\text{HFK}}(K#) \text{ when } b \text{ is nontrivial} \]

**Goal:** \( \text{rank } \text{HFL}^{-}(K_b \cup C, \sigma) > \text{rank } \text{HFL}^{-}(K# \cup C, \sigma) \) when \( b \) is nontrivial.
\[
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Sutured manifold decompositions along surfaces \( S_i \) disjoint from \( \partial N(C) \):

\[ S^3(K_b \cup C) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} S^3(\text{Hopf link}) \]

\[ \text{HFL}^-(K_b \cup C, \sigma) \leftrightarrow \text{SFH}^-(M_1, \gamma_1, \sigma) \leftrightarrow \cdots \leftrightarrow \text{HFL}^-(\text{Hopf link}, \sigma) \]
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Nonsplit links?

If the generalized cosmetic crossing conjecture is false for $L$, then the polynomial invariants of $L$ look like those of a split link.

- The Alexander polynomial of $L$ vanishes.
- The Jones polynomial of $L$ is divisible by $q - 1$.
- The HOMFLYPT polynomial of $L$ is divisible by $\frac{\ell - 1}{m}$.

Potential proof strategy: show that a categorified invariant of $L$ looks like that of a split link, then prove that the categorified invariant detects splitness.

Theorem (Lipshitz-Sarkar 2019)
The module structure on $\text{Kh}(L)$ detects if $L$ is split.

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Thanks for listening!