Floer and Khovanov homologies of band sums

Joshua Wang

October 20, 2020
The cosmetic crossing conjecture

A nugatory crossing change:
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The cosmetic crossing conjecture: Every crossing change of an oriented knot that does not change the oriented knot type is nugatory.
The cosmetic crossing conjecture

Let $K_+$ and $K_-$ be oriented knots in $S^3$ differing by a crossing change, and let $L$ be the two-component link obtained by taking the oriented resolution.
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Let $K_+$ and $K_-$ be oriented knots in $S^3$ differing by a crossing change, and let $L$ be the two-component link obtained by taking the oriented resolution.

$K_+$ can be obtained from $L$ by band surgery along a band $b$.

$K_-$ can be obtained from $L$ by band surgery along a band $b$. 
Let $K_+$ and $K_-$ be oriented knots in $S^3$ differing by a crossing change, and let $L$ be the two-component link obtained by taking the oriented resolution.

$K_-$ can be obtained from $L$ by band surgery along a band $b$.

$K_+$ can then be obtained by adding a full twist to the band.
Cosmetic crossing conjecture (Problem 1.58 on Kirby’s list)

If $K_+$ and $K_-$ are isotopic as oriented knots, then the link $L$ is split and the band $b$ is trivial.
Cosmetic crossing conjecture (Problem 1.58 on Kirby’s list)

If \( K_+ \) and \( K_- \) are isotopic as oriented knots, then the link \( L \) is split and the band \( b \) is trivial.

Recall:

- \( L \) is *split* if there exists an embedded sphere which separates its components.
- \( b \) is *trivial* if there exists a splitting sphere for \( L \) which intersects \( b \) along a single arc.
The cosmetic crossing conjecture

Example: $L$ is split and $b$ is nontrivial.
The cosmetic crossing conjecture

The cosmetic crossing conjecture is true for:

- the unknot (Scharlemann-Thompson 1989)
- 2-bridge knots (Torisu 1999)
- composite knots, if the conjecture is true for prime knots (Torisu 1999)
- fibered knots (Kalfagianni 2012)
- all genus 1 knots except when $K$ is algebraically slice and $H_1(\Sigma_2(K))$ is finite cyclic (Balm-Friedl-Kalfagianni-Powell 2012)
- Whitehead doubles of prime, non-cable knots (Balm-Kalfagianni 2016)
- knots $K$ for which $\Sigma_2(K)$ is an L-space and each cyclic summand of $H_1(\Sigma_2(K))$ has square-free order (Lidman-Moore 2017)

In particular, all prime knots with crossing number $\leq 9$. 
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Fix a two-component link $L$ and a band $b$.

Cosmetic crossing conjecture (restated)

The oriented knots $K_b$ and $K_b + 1$ are distinct unless $L$ is split and $b$ is trivial.

The cosmetic crossing conjecture for a two-component link $L$:

• if $L$ is nonsplit, then $K_b$ and $K_b + 1$ are distinct for any band $b$.

• if $L$ is split, then $K_b$ and $K_b + 1$ are distinct for any nontrivial band $b$. 
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Fix a two-component link $L$ and a band $b$. Let $K_b$ be the result of band surgery, and let $K_{b+1}$ be obtained by adding a full twist to the band.
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The generalized cosmetic crossing conjecture

Fix a two-component link $L$ and a band $b$. Let $K_b$ be the result of band surgery, and let $K_{b+1}$ be obtained by adding a full twist to the band.

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Fix a two-component link $L$ and a band $b$. Let $K_b$ be the result of band surgery, and let $K_{b+n}$ be obtained by adding $n$ full twists to the band.

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The **generalized** cosmetic crossing conjecture

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The cosmetic crossing conjecture for a two-component link \( L \):

- if \( L \) is **nonsplit**, then \( K_b \) and \( K_{b+1} \) are distinct for any band \( b \).
- if \( L \) is **split**, then \( K_b \) and \( K_{b+1} \) are distinct for any **nontrivial** band \( b \).
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Fix a two-component link $L$ and a band $b$. Let $K_b$ be the result of band surgery, and let $K_{b+n}$ be obtained by adding $n$ full twists to the band.

**Generalized cosmetic crossing conjecture**

The oriented knots $K_{b+n}$ for $n \in \mathbb{Z}$ are distinct unless $L$ is split and $b$ is trivial.

The cosmetic crossing conjecture for a two-component link $L$:

- if $L$ is *nonsplit*, then $K_b$ and $K_{b+1}$ are distinct for any band $b$.
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The oriented knots $K_{b+n}$ for $n \in \mathbb{Z}$ are distinct unless $L$ is split and $b$ is trivial.
Theorem (W. 2020)

The generalized cosmetic crossing conjecture is true for split links.
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The generalized cosmetic crossing conjecture is true for split links. If \( L \) is a split two-component link, then \( K_{b+n} \) for \( n \in \mathbb{Z} \) are distinct for any nontrivial band \( b \).
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Let $K_b$ be a band sum of a split link $L$. Let $K\#$ be the connected sum.
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Let $K_b$ be a band sum of a split link $L$. Let $K_#$ be the connected sum. For a knot invariant $H$, two questions about band sums:

1. How are $H(K_b)$ and $H(K_{b+n})$ related?
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Band sums and the main topological result

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1. How are $H(K_b)$ and $H(K_{b+n})$ related?
2. How are $H(K_b)$ and $H(K_{\#})$ related?

In this talk: answers to these two questions for $H = \text{knot Floer homology, Khovanov homology, instanton knot homology.}$
Let $K_b$ be a band sum of a split two-component link $L$. 

The Alexander polynomial $\Delta$ satisfies the skein relation

$$\Delta(J^+) - \Delta(J^-) = (t - 1/2 - t^{-1/2}) \Delta(J^0).$$

Observation

$\Delta(K_b + n) = \Delta(K_b)$ for all $n \in \mathbb{Z}$.

Proof.

$\Delta(L) = 0$ when $L$ is split, so

$$\Delta(K_b + 1) - \Delta(K_b) = (t - 1/2 - t^{-1/2}) \Delta(L) = 0.$$

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![Diagram]

Observation $\Delta(K_b + n) = \Delta(K_b)$ for all $n \in \mathbb{Z}$.

Proof. $\Delta(L) = 0$ when $L$ is split so $\Delta(K_b + 1) - \Delta(K_b) = (t^{-1/2} - t^{1/2})\Delta(L) = 0$. $\square$
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**Observation**

$$\Delta(K_{b+n}) = \Delta(K_b) \text{ for all } n \in \mathbb{Z}.$$  

**Proof.**

$$\Delta(L) = 0 \text{ when } L \text{ is split so } \Delta(K_{b+1}) - \Delta(K_b) = (t^{-1/2} - t^{1/2})\Delta(L) = 0.$$
The knot Floer homology of $K_b$, denoted $\widehat{\text{HFK}}(K_b)$, is a vector space over $F = \mathbb{Z}/2$ with a $\mathbb{Z} \oplus \mathbb{Z}$ bigrading. It categorifies $\Delta(K_b)$. 

Theorem (W. 2020) $\widehat{\text{HFK}}(K_b + n) \cong \widehat{\text{HFK}}(K_b)$ as bigraded vector spaces over $F$. The same is true for $\text{HFK}^{-}(K_b)$. Hedden-Watson 2018 proved the special case of this result when the split link is the unlink.

The instanton knot Floer homology of $K_b$, denoted $\text{KHI}(K_b)$, is a vector space over $C$ with a $\mathbb{Z} \oplus \mathbb{Z}/2$ bigrading. It also categorifies $\Delta(K_b)$.

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$$\text{KHI}(K_{b+n}) \cong \text{KHI}(K_b)$$ as bigraded vector spaces over $\mathbb{C}$.
Alexander polynomial and knot Floer homology

There are two main ingredients to the proof.
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1. The skein exact triangle, which categorifies the skein relation:

\[ \widehat{\text{HFK}}(K_{b+1}) \rightarrow \widehat{\text{HFK}}(K_b) \rightarrow \widehat{\text{HFK}}(L) \]

The map \( \widehat{\text{HFK}}(K_{b+1}) \rightarrow \widehat{\text{HFK}}(K_b) \) preserves both gradings.
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2. Maps on \( \widehat{\text{HFK}} \) induced by ribbon concordances.
Ribbon concordances and knot Floer homology

Recall:

- A *concordance* $C : J_0 \to J_1$ between knots $J_i \subset S^3$ is a properly embedded annulus $C \hookrightarrow [0, 1] \times S^3$ with $\partial C = -J_0 \times 0 \sqcup J_1 \times 1$. 

Theorem (Miyazaki 1998)

There is a ribbon concordance $C : K_# \to K_{\text{b}}$.

Theorem (Zemke 2019)

A ribbon concordance $C : J_0 \to J_1$ induces an injective map $\hat{HFK}(J_0) \to \hat{HFK}(J_1)$.

In fact, $\hat{HFK}(J_1) \cong \hat{HFK}(J_0) \oplus F$ for some bigraded vector space $F$. 

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- A concordance $C : J_0 \to J_1$ is ribbon if the projection to $[0, 1]$ is a Morse function on $C$ with no index 2 critical points.
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Ribbon concordance $K_\# \to K_b$ (Miyazaki 1998)
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Proof sketch of $\widehat{\text{HFK}}(K_{b+n}) \cong \widehat{\text{HFK}}(K_b)$.

Choose (compatible) ribbon concordances $C' : K\# \to K_{b+1}$ and $C : K\# \to K_b$. 
Proof sketch of $\widehat{\text{HFK}}(K_{b+n}) \cong \widehat{\text{HFK}}(K_b)$.

Choose (compatible) ribbon concordances $C' : K\# \to K_{b+1}$ and $C : K\# \to K_b$. There are induced splittings

$$\widehat{\text{HFK}}(K_{b+1}) \cong \widehat{\text{HFK}}(K\#) \oplus F_{b+1} \quad \widehat{\text{HFK}}(K_b) \cong \widehat{\text{HFK}}(K\#) \oplus F_b$$

for some bigraded vector spaces $F_{b+1}, F_b$ by Zemke’s inclusion maps.
Proof sketch of $\widehat{\text{HFK}}(K_{b+n}) \cong \widehat{\text{HFK}}(K_b)$.

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$$\begin{align*}
\widehat{\text{HFK}}(K_{b+1}) & \to \widehat{\text{HFK}}(K_b) & \widehat{\text{HFK}}(K#) & \to \widehat{\text{HFK}}(K#) & F_{b+1} & \to F_b \\
\Downarrow & & \Downarrow & & \Downarrow & \\
\text{HFK}(L) & & \text{HFK}(L) & & 0 & \\
\end{align*}$$
Proof sketch of $\widehat{\text{HFK}}(K_{b+n}) \cong \widehat{\text{HFK}}(K_b)$.

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$$\begin{align*}
\widehat{\text{HFK}}(K_{b+1}) &\to \widehat{\text{HFK}}(K_b) \\
\text{HFK}(L) &\to \text{HFK}(L)
\end{align*}$$

$$\begin{align*}
\widehat{\text{HFK}}(K\#) &\to \widehat{\text{HFK}}(K\#) \\
\text{HFK}(L) &\to 0
\end{align*}$$

for some bigraded vector spaces $F_{b+1}, F_b$ by Zemke’s inclusion maps. These splittings are compatible with the skein exact triangles
Jones polynomial and Khovanov homology

Let $K_b$ be a band sum of a split two-component link $L$, and let $K#$ be the connected sum.

Observation

Let $P_b$ satisfy $V(K_b) = V(K#) + P_b$. Then $V(K_b + n) = V(K#) + q^{4n} P_b$.

Proof.

Use the identity $V(L) = (q - 1 + q) V(K#)$ and the skein relation. □

Question

Does the Jones polynomial detect the trivial band? Is $P_b$, $0$ when $b$ is nontrivial?
Let $K_b$ be a band sum of a split two-component link $L$, and let $K#$ be the connected sum. The Jones polynomial $V$ satisfies the skein relation

$$q^{-2}V(J_+) - q^2V(J_-) = (q^{-1} - q)V(J_0).$$
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**Observation**

Let $P_b$ satisfy $V(K_b) = V(K#) + P_b$. Then $V(K_{b+n}) = V(K#) + q^{4n}P_b$. 

**Proof.** Use the identity $V(L) = (q^{-1} - q)V(K#)$ and the skein relation. □
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Use the identity $V(L) = (q^{-1} + q)V(K#)$ and the skein relation.

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FL and Kh homologies of band sums

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Use the identity $V(L) = (q^{-1} + q)V(K#)$ and the skein relation.

**Question**

Does the Jones polynomial detect the trivial band? Is $P_b \neq 0$ when $b$ is nontrivial?
The Khovanov homology of $K_b$, denoted $\text{Kh}(K_b)$, is a vector space over $\mathbb{F}$ with a $\mathbb{Z} \oplus \mathbb{Z}$ bigrading $(h, q)$. It categorifies $V(K_b)$. 
The Khovanov homology of $K_b$, denoted $\text{Kh}(K_b)$, is a vector space over $F$ with a $\mathbb{Z} \oplus \mathbb{Z}$ bigrading $(h, q)$. It categorifies $V(K_b)$.

A ribbon concordance $K_# \to K_b$ induces a splitting $\text{Kh}(K_b) \cong \text{Kh}(K#) \oplus H_b$ (Levine-Zemke 2019). The graded Euler characteristic of $H_b$ is $P_b$. 

Recall: $V(K_b + n) = V(K#) + q^{4n} P_b$. 

Theorem (W. 2020) $\text{Kh}(K_b + m/2) = \text{Kh}(K#) \oplus h^m q^{2m} H_b$. 

In fact, $\text{Kh}(K_b + m/2) = \text{Kh}(K#) \oplus h^m q^{2m} H_b$. 

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Recall: $V(K_{b+n}) = V(K_\#) + q^{4n}P_b$.

**Theorem (W. 2020)**

$$\text{Kh}(K_{b+n}) \cong \text{Kh}(K_\#) \oplus h^{2n}q^{4n}H_b$$

In fact, $\text{Kh}(K_{b+m/2}) \cong \text{Kh}(K_\#) \oplus h^{m}q^{2m}H_b$. 
Proof sketch of $\text{Kh}(K_{b+n}) \cong \text{Kh}(K_{\#}) \oplus h^{2n}q^{4n}H_b$.

There are unoriented skein exact triangles

$$\begin{align*}
\text{Kh}(K_b) &\longrightarrow \text{Kh}(K_{b+1/2}) \longrightarrow \text{Kh}(K_{b+1}) \\
\text{Kh}(L) &\quad \quad \text{Kh}(L)
\end{align*}$$
Jones polynomial and Khovanov homology

Proof sketch of $\text{Kh}(K_{b+n}) \cong \text{Kh}(K_{\#}) \oplus h^{2n} q^{4n} H_b$.

There are unoriented skein exact triangles

\[
\text{Kh}(K_b) \rightarrow \text{Kh}(K_{b+1/2}) \rightarrow \text{Kh}(K_{b+1}) \\
\text{Kh}(L) \leftarrow \text{Kh}(L) \leftarrow \text{Kh}(L)
\]

compatible with ribbon concordance splittings (Levine-Zemke 2019)

\[
\text{Kh}(K_{\#}) \rightarrow \text{Kh}(K_{\#}) \rightarrow \text{Kh}(K_{\#}) \\
\text{Kh}(L) \leftarrow \text{Kh}(L) \leftarrow \text{Kh}(L)
\]

\[
\text{Kh}(K_{\#}) \rightarrow \text{Kh}(K_{\#}) \rightarrow \text{Kh}(K_{\#}) \\
\text{Kh}(L) \leftarrow \text{Kh}(L) \leftarrow \text{Kh}(L)
\]

\[
H_b \rightarrow H_{b+1/2} \rightarrow H_{b+1} \oplus 0 \leftarrow 0 \leftarrow 0
\]
Proof sketch of $\text{Kh}(K_{b+n}) \cong \text{Kh}(K_{\#}) \oplus h^2n q^{4n} H_b$.

There are unoriented skein exact triangles

\[
\begin{align*}
\text{Kh}(K_b) & \to \text{Kh}(K_{b+1/2}) \to \text{Kh}(K_{b+1}) \\
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\[
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\text{Kh}(K_{\#}) & \to \text{Kh}(K_{\#}) \to \text{Kh}(K_{\#}) \\
& \text{Kh}(L) & \text{Kh}(L)
\end{align*}
\]

\[
\begin{align*}
H_b & \to H_{b+1/2} \to H_{b+1} \\
\oplus & 0 & 0
\end{align*}
\]

The isomorphisms $H_b \to H_{b+1/2} \to H_{b+1}$ each shift bigradings by $(1, 2)$. \qed
Jones polynomial and Khovanov homology

To show the groups $\text{Kh}(K_{b+n}) \cong \text{Kh}(K_{\#}) \oplus h^{2n}q^{4n}H_b$ for $n \in \mathbb{Z}$ are distinct, it suffices to show that $H_b \neq 0$ whenever $b$ is nontrivial.
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**Theorem (W. 2020)**

$$\dim \text{Kh}(K_b) = \dim \text{Kh}(K_\#) \text{ if and only if } b \text{ is trivial.}$$

In other words, $H_b = 0$ if and only if $b$ is trivial.
Jones polynomial and Khovanov homology

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**Corollary (W. 2020)**

*The generalized cosmetic crossing conjecture is true for split links.*
To show the groups $\text{Kh}(K_{b+n}) \cong \text{Kh}(K_{\#}) \oplus h^{2n} q^{4n} H_b$ for $n \in \mathbb{Z}$ are distinct, it suffices to show that $H_b \neq 0$ whenever $b$ is nontrivial.

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The generalized cosmetic crossing conjecture is true for split links.

Hedden-Watson 2018 showed that the Khovanov homology groups of $K_{b+n}$ for $n \in \mathbb{Z}$ are distinct in the case where the split link is the unlink.
Jones polynomial and Khovanov homology

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**Corollary (Miyazaki 2020)**

*If $K_b$ is isotopic to $K\#$, then $b$ is trivial.*
Proposition

Let $H$ be a knot invariant taking the form of a vector space over $\mathbb{F}$ with a functorial spectral sequence $\text{Kh} \Rightarrow H$. If $\dim H$ detects the trivial band, then $\dim \text{Kh}$ detects the trivial band.

$H$ could be the Heegaard Floer homology of the double branched cover, singular instanton homology $I^\#$, etc. (Baldwin-Hedden-Lobb 2019).

Proof.

Suppose $\dim \text{Kh}(K^b) = \dim \text{Kh}(K^\#)$. Let $C: K^\# \to K^b$ be a ribbon concordance. Then $C$ induces a map of spectral sequences $\text{Kh} \Rightarrow H$ which is an isomorphism on the $E^2$-page. It is therefore an isomorphism on the $E^\infty$-page, so $\dim H(K^b) = \dim H(K^\#)$. Thus $b$ is trivial. $\square$
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Let $H$ be a knot invariant taking the form of a vector space over $F$ with a functorial spectral sequence $\text{Kh} \Rightarrow H$. If $\dim H$ detects the trivial band, then $\dim \text{Kh}$ detects the trivial band.

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Proof.

Suppose $\dim \text{Kh}(K_b) = \dim \text{Kh}(K^\#)$.
Let $H$ be a knot invariant taking the form of a vector space over $\mathbb{F}$ with a functorial spectral sequence $\text{Kh} \Rightarrow H$. If $\dim H$ detects the trivial band, then $\dim \text{K}h$ detects the trivial band.

$H$ could be the Heegaard Floer homology of the double branched cover, singular instanton homology $I^\#$, etc. (Baldwin-Hedden-Lobb 2019).

**Proof.**

Suppose $\dim \text{K}h(K_b) = \dim \text{K}h(K^\#)$. Let $C : K^\# \to K_b$ be a ribbon concordance.
Jones polynomial and Khovanov homology

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Jones polynomial and Khovanov homology

**Proposition**

Let $H$ be a knot invariant taking the form of a vector space over $\mathbb{F}$ with a functorial spectral sequence $\text{Kh} \Rightarrow H$. If $\dim H$ detects the trivial band, then $\dim \text{Kh}$ detects the trivial band.

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There is a functorial spectral sequence $\text{Kh} \Rightarrow I^\#$.

**Theorem (W. 2020)**

Singular instanton homology detects the trivial band:

$\dim I^\#(K^b) = \dim I^\#(K^#)$ if and only if $b$ is trivial.

The proof involves showing that $\dim KHI$ detects the trivial band.

Dowlin 2018 constructed a spectral sequence from Khovanov homology to knot Floer homology.

**Theorem (W. 2020)**

Knot Floer homology detects the trivial band:

$\dim \hat{HFK}(K^b) = \dim \hat{HFK}(K^#)$ if and only if $b$ is trivial.

Functoriality of Dowlin's spectral sequence has not been established.
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Functoriality of Dowlin’s spectral sequence has not been established.
\[ \dim \widehat{\text{HFK}}(K_b) > \dim \widehat{\text{HFK}}(K_{\#}) \text{ when } b \text{ is nontrivial} \]
\[ \dim \widehat{\text{HFK}}(K_b) > \dim \widehat{\text{HFK}}(K\#) \text{ when } b \text{ is nontrivial} \]

A diagram for \( K_b \cup C \) has basepoints \( w_K, z_K, w_C, z_C \).
\[ \dim \widehat{\text{HFK}}(K_b) > \dim \widehat{\text{HFK}}(K\#) \text{ when } b \text{ is nontrivial} \]

A diagram for \( K_b \cup C \) has basepoints \( w_K, z_K, w_C, z_C \). Let \( \text{CFL}^{-}(K_b \cup C, \sigma) \) over \( F[U] \) count discs blocked by \( w_K, z_K, w_C \) and record intersection with \( z_C \) in \( U \).
dim $\widehat{\text{HFK}}(K_b) > \dim \widehat{\text{HFK}}(K_\#)$ when $b$ is nontrivial

A diagram for $K_b \cup C$ has basepoints $w_K, z_K, w_C, z_C$. Let $\text{CFL}^{-}(K_b \cup C, \sigma)$ over $F[U]$ count discs blocked by $w_K, z_K, w_C$ and record intersection with $z_C$ in $U$.

Claim: $\text{rank } \text{HFL}^{-}(K_b \cup C, \sigma) = 2 \cdot \dim \widehat{\text{HFK}}(K_b)$. 
\[
\dim \widehat{\text{HFK}}(K_b) > \dim \widehat{\text{HFK}}(K\#) \text{ when } b \text{ is nontrivial}
\]

A diagram for \(K_b \cup C\) has basepoints \(w_K, z_K, w_C, z_C\). Let \(\text{CFL}^-(K_b \cup C, \sigma)\) over \(\mathbb{F}[U]\) count discs blocked by \(w_K, z_K, w_C\) and record intersection with \(z_C\) in \(U\).

**Claim:** \(\text{rank } \text{HFL}^-(K_b \cup C, \sigma) = 2 \cdot \dim \widehat{\text{HFK}}(K_b)\).

\[
0 \longrightarrow \text{CFL}^-(K_b \cup C, \sigma) \xrightarrow{U-\text{Id}} \text{CFL}^-(K_b \cup C, \sigma) \longrightarrow \frac{\text{CFL}^-(K_b \cup C, \sigma)}{U-\text{Id}} \longrightarrow 0
\]

\[
\cdots \longrightarrow \text{HFL}^-(K_b \cup C, \sigma) \xrightarrow{U-\text{Id}} \text{HFL}^-(K_b \cup C, \sigma) \longrightarrow \widehat{\text{HFK}}(K_b) \otimes \mathbb{F}^2 \longrightarrow \cdots
\]
\[ \dim \widehat{\text{HFK}}(K_b) > \dim \widehat{\text{HFK}}(K_{\#}) \text{ when } b \text{ is nontrivial} \]

**Goal:** \( \text{rank } \text{HFL}^{-}(K_b \cup C, \sigma) > \text{rank } \text{HFL}^{-}(K_{\#} \cup C, \sigma) \) when \( b \) is nontrivial.
dim $\widehat{\text{HF}}K(K_b) > \dim \widehat{\text{HF}}K(K\#)$ when $b$ is nontrivial

**Goal:** rank $\text{HFL}^-(K_b \cup C, \sigma) > \text{rank} \: \text{HFL}^-(K\# \cup C, \sigma)$ when $b$ is nontrivial.

A ribbon concordance $K\# \cup C \to K_b \cup C$ gives an inclusion

$$\text{HFL}^-(K\# \cup C, \sigma) \hookrightarrow \text{HFL}^-(K_b \cup C, \sigma)$$

onto an $\mathbb{F}[U]$-module summand.
\[ \dim \widehat{\text{HFK}}(K_b) > \dim \widehat{\text{HFK}}(K_\#) \text{ when } b \text{ is nontrivial} \]

**Goal:** \( \text{rank } \text{HFL}^-(K_b \cup C, \sigma) > \text{rank } \text{HFL}^-(K_\# \cup C, \sigma) \) when \( b \) is nontrivial.

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Sutured manifold decompositions along surfaces \( S_i \) disjoint from \( \partial \mathcal{N}(C) \):

\[ S^3(K_b \cup C) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} S^3(\text{Hopf link}) \]

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If the generalized cosmetic crossing conjecture is false for $L$, then the polynomial invariants of $L$ look like those of a split link.

- The Alexander polynomial of $L$ vanishes.
- The Jones polynomial of $L$ is divisible by $q - 1 + q^2$.
- The HOMFLYPT polynomial of $L$ is divisible by $(\ell - 1 + \ell^2)/m$. 
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Thanks for listening!