Aug 30, 2017

- Syllabus in my.harvard
- Click link to google doc for asking q’s in class
- Survey by Wed.

If we get “fundamental” prob wrong we have to go to office hrs and prove we learned how to do them.

Final exam & midterm 1 are take-home.

2 weeks 9 days

HW must be submitted electronically— if your handwriting is good you can just scan.

Optional problems are extra credit to make it more fun/ explore topics.

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Groups
(some symmetries)

We often consider symmetries of objects w/ some sort of structure.

\[ \begin{array}{ccc}
\circ & \square & \triangle \\
\text{very symmetric} & \text{quite symmetric} & \text{some rotations, shape can do anything, rotations & reflections if}
\end{array} \]

Groups
(some symmetries)

The structure of shapes w/ notion of distance & angles (that get preserved)

\[ \mathbb{R}^n = \text{n-tuples of real nums is a v. space } = \mathbb{R}^n = \{ (x_1, \ldots, x_n) | x_i \in \mathbb{R}^n \} \]

A sym. of \( \mathbb{R}^n \): Consider transforms of \( \mathbb{R}^n \) respecting v. space structure (\( f : \mathbb{R}^n \to \mathbb{R}^n \) linear) but we want them to be “undoable” so they can be symmetries so we consider \( \text{GL}_n(\mathbb{R}) = \{ \text{invertible n x n matrices} \} ^? \)

Formalize this “symmetry” idea: A collection of symmetries of some object \( X \) should satisfy:
1. “do nothing” is asymmetry (the identity)
2. Symmetries can be undone (睑 inverses)
3. “doing” should be associative

**Def**: A group is the data of a set \( G \) and a fn. \( m : G \times G \to G \) satisfying

1. \( \exists e \in G \) such that \( \forall g \in G \)
   \[ m(e, g) = g = m(g, e) \]
2. \( \forall g \in G, \exists g^{-1} \in G \)
   \[ m(g, g^{-1}) = e = m(g^{-1}, g) \]
3. \( m \) is associative:
   \[ m(m(g_3, g_2), g_1) = m(m(g_3, g_2), g_1) \]
Commutative diagram property 3 (any arrow path u follow gives same result)

\[
\begin{array}{ccc}
G \times G \times G & \xrightarrow{\text{id} \times m} & G \times G \\
\downarrow & & \downarrow \\
G \times G & \xrightarrow{m \times \text{id}_G} & G \\
\end{array}
\]

m's are annoying, we'll denote \( m(g, h) = gh \)

**Example**
\[
\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \}
\]
\[
m: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \\
(a, b) \mapsto a + b
\]

Claims the integers under addition is a group

1. \( e = 0 \)
2. \( a' = -a \)
3. \( a + (b + c) = (a + b) + c \)

**Examples**
* \( G = (\mathbb{R}, +) \) or \( (\mathbb{Q}, +) \) or \( (\mathbb{C}, +) \) or \( (\mathbb{R}^n, +) \)

**Non-example**
* \( G = \{0, 1, 2, \ldots \} \) under addition is not a group (no inverses)

**Example**
* \( G = \mathbb{Z} \times \mathbb{Z} \), \( m = \text{multiplication} \). This ex shows an elt. can b its own inverse (-1), also the underlying set of the grp. can be finite

* \( \mathbb{R} \times (0, 1) \) under mult.

**Less trivial ex**
Fix set \( S \). \( G = \{ \text{bijections \ } \phi: S \rightarrow S \} \), \( m = \text{composition} \)

**Exercise**
Let \( S = \{1, \ldots, n\} \), \( n \geq 1 \)

Define \( S_n = \text{group of bijections of } S \)

How many elements are in \( S_n \)? \( n! \) (set of all bijections = set of permutations)

"How are we allowed to relate these examples? When are groups essentially "the same?"

**Group homomorphisms**

**Defn.** Fix two groups \( G, H \). A function \( \phi: G \rightarrow H \) is called a group homomorphism if it "respects multiplication", i.e.

\[
\phi(g_1 g_2) = \phi(g_1) \phi(g_2)
\]

**Defn.** A group homomorphism \( \phi \) is a group isomorphism if \( \phi \) is a bijection

**Exercise** Find every group homomorphism from \( \mathbb{Z} \) to itself. (also group isomorphism)

* homos: \( a \mapsto na \) for \( n \in \mathbb{Z} \) (multiplying by an int)

* isos: \( n = \pm 1 \)

**Exercise** All group homomorphs \( S_n \rightarrow \mathbb{Z} \)

**Ans:** just take everything to 0 is the only one.

Why not others? \( S_n \) finite so \( 1 \) some power of \( g \) s.t. \( g^n = e \) but it would make no sense for \( a + a \ldots = na \neq 0 \) unless \( a = 0 \)