Lecture 26: Pfaffians and the Euler class. Gauss-Bonnet-Chern Theorem.

1. Euler characteristic

Let M be a smooth, compact manifold. A theorem of Whitehead says that any such M can be given a triangulation—that is, any such M can be given a homeomorphism $K \to M$ from a simplicial complex K. Let K_i denote the number of i-simplices in K. This is necessarily finite since M is compact.

Definition 26.1. The *Euler characteristic* of M is the integer

$$\chi(M) := \sum_{i=0}^{\dim M} (-1)^i K_i.$$

What is non-trivial is that this is a number which remains the same under different triangulations.

2. Gauss-Bonnet-Chern Theorem

I will define the Euler class momentarily.

Theorem 26.2 (Gauss-Bonnet-Chern Theorem)**.** Let M be an smooth manifold which is

(1) oriented,
(2) dimension $2k$ (so it's even-dimensional), and
(3) compact.

Let $e(M) \in H^{2k}_{dR}(M)$ be the Euler class associated to the tangent bundle of M, together with its orientation. Then

$$\int_M e(M) = \chi(M).$$

Remark 26.3. What do I mean by integrating a cohomology class over a smooth manifold? We know what it means to integrate a differential *form*. Well, let α, α' be two forms of degree $2k$ in the same cohomology class. (They are automatically closed because $\Omega^{2k+1}_{dR}(M) = 0$.) This means there is some $2k-1$ form, β, such that $d\beta = \alpha - \alpha'$. Then Stokes's Theorem tells us that

$$\int_M \alpha - \alpha' = \int_M d\beta = \int_\emptyset \beta = 0.$$

So $\int_M \alpha = \int_M \alpha'$, so the integral is dependent only on the cohomology class of α.

Remark 26.4. You might be frustrated by all the requirements in the theorem. For instance, why does M have to be even-dimensional? Well, there are two related ideas: First, it turns out that for any compact, orientable manifold of odd dimension, the Euler characteristic is always zero. So the Euler characteristic is not an interesting invariant of odd-dimensional manifolds to begin with. Second, the Euler class in given in terms of the Pfaffian, which only exists in even-dimensional vector spaces.

Remark 26.5. You probably know that Gauss-Bonnet Theorem as something about integrating *curvature* over a 2-manifold to recover the Euler characteristic. We'll see the relationship later next week.

3. The Pfaffian

Let X be a $2k \times 2k$ matrix and assume it is skew-symmetric, so $X^T = -X$.

Definition 26.6. The *Pfaffian* is the polynomial

$$Pf = \frac{1}{2^k k!} \sum_{\sigma \in S_{2k}} x_{\sigma(1)\sigma(2)} \dots x_{\sigma(2k-1)\sigma(2k)}$$

where the variable x_{ij} corresponds to the i, j entry of a matrix. The *Pfaffian of* X, $Pf(X)$, is the number given by evaluating the polynomial on X.

The following Lemma is non-trivial, but we won't prove it.

Lemma 26.7. For any skew-symmetric, even-dimensional matrix X, the following holds:

(1) $Pf(X)^2 = \det(X)$.
(2) For any invertible matrix A,

$$Pf(AXA^{-1}) = \det(A)Pf(X).$$

Example 26.8. Let $k = 1$. Then any skew-symmetric matrix is of the form

$$\begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix}$$

where $x_{12} = -x_{21}$. Then the Pfaffian is given by
$$Pf(X) + (\frac{1}{2^1 \cdot 1!})(x_{12} - x_{21}) = x_{12}.$$
And indeed,
$$Pf(X)^2 = x_{12}^2 = \det(X).$$

4. The Euler form and Euler class

The game we played with characteristic classes is the following: Curvature looks a certain way locally, and we saw that it transformed by conjugation. So if we can construct an invariant polynomial, then that polynomial turns the curvature form (which is a 2-form with values in $End(E)$) into a globally defined differential form (with values in $\mathbb{R}$, as opposed to $End(E)$).

We want to play the same game with the Pfaffian, but there are a few issues:

(1) The Pfaffian is only defined for even-dimensional vector spaces, so the vector bundle had better be even rank.
(2) Second, the Pfaffian satisfies a nice property—namely, it's a square root for the determinant—but this only holds for skew-symmetric matrices, so we need to be able to guarantee a situation in which the curavture form is skew-symmetric. Well, we know how to do that: Impose a metric. (This was part of your homework.) To be specific, fixing a metric, there exists an orthonormal basis around any point, so we can choose trivializing neighborhoods and trivializations for which the curvature form is skew-symmetric.
(3) Lastly, the Pfaffian isn't an invariant polynomial. If $g_{\alpha\beta}$ is a transition matrix from one trivialization to another, the metric only guarantees that $g_{\alpha\beta}$ is a function into $O(n)$. (After all, a trivialization respecting the metric has to send one orthonormal basis to another.) So there is a sign ambiguity in the Pfaffian when we change trivializations. To get rid of this ambiguity, we should find a situation in which we can guarantee that each $g_{\alpha\beta}$ lands not in $O(n)$, but in $SO(n)$. Well, we can do that if we can guarantee that each orthonormal basis is *compatible* with an orientation on E.

Definition 26.9. An orientation on a vector bundle E is a nowhere vanishing section of $\Lambda^{\dim E} E$. Or, equivalently, a nowhere vanishing section of $\Lambda^{\dim E} E^*$. (The two are isomorphic by choosing a Riemannian metric.)

Definition 26.10. Let E be an oriented vector bundle of rank $2k$. Fix a metric g and a compatible connection ∇. By the discussion above, the form

$Pf(\Omega_\nabla)$ is a globally defined differential form of degree $2k$. The *Euler form* is defined to be

$$eu(E, g, \nabla) = \frac{1}{(2\pi)^k} Pf(\Omega_\nabla).$$

We again do not prove the following lemma, but it is the natural result to pursue if inspired by other characteristic classes.

Lemma 26.11. The Euler form is closed for any choice of g, ∇, and its cohomology class is unchanged by changing g or ∇.

Definition 26.12. The *Euler class* of an oriented vector bundle is

$$e(E) := [eu(E, g, \nabla)] \in H^{2k}_{dR}(M).$$

For an oriented manifold M, we write $e(M) = e(TM)$.

5. Geometric interpretation of Euler class via Poincaré Duality

There's a very satisfying geometric interpretation of the Euler class. Let E be an oriented vector bundle over an oriented manifold M. I claim that since E is oriented as a vector bundle, and since M is oriented, the *manifold* E is oriented as well. Thus it makes sense to talk about oriented intersections inside of E.

Let $s : M \to E$ be a section. By Sard's Theorem, we can assume it is generic—i.e., that s is transverse to the zero section $i : M \to E$. Then by transversality, $s(M) \cap i(M)$ is a smooth submanifold of E. Since it is a subset set of the zero section $i(M)$, one can also think of it as a smooth submanifold of M.

Now, Poincaré Duality tells you there is a duality between smooth submanifolds of M and deRham cohomology classes of M. A rough picture is as follows: A differential k-form is something that you can integrate over smooth manifolds, so it's a way of taking a smooth k-dimensional manifold and spitting out a number. What's a systematic way of doing such a thing? Take a codimension k submanifold of M, and compute its intersection number against a k-dimensional submanifold.

(We haven't talked about Poincaré Duality yet, so it's okay if this is unfamiliar.)

The Euler class is then the Poincaré dual to the intersection $s(M) \cap i(M)$. Let's do a quick dimension count: By transversality, the intersection $s(M) \cap i(M)$ must be of dimension $M - \text{rank}(E)$. That is, it is a codimension $\text{rank}(E)$ submanifold of M. Poincaré Duality assigns to it a degree $\text{rank}(E) = 2k$ cohomology class, and indeed this is the degree of the Euler class.

Example 26.13. If $E = TM$, then the intersection $s(M) \cap i(M)$ of a generic section with the zero section is some collection of points. If M is compact, this set is finite, and if M is oriented, we can count this intersection number. The intersection number—according to the Gauss-Bonnet-Chern Theorem—is the Euler characteristic of M.