Lecture 24/25: Riemannian metrics

An inner product on $\mathbb{R}^n$ allows us to do the following: Given two curves intersecting at a point $x \in \mathbb{R}^n$, one can determine the angle between their tangents:

$$\cos \theta = \frac{\langle u, v \rangle}{|u||v|}.$$ 

On a general manifold, we would again like to have an inner product on each $T_pM$. Then if two curves intersect at a point $p$, one can measure the angle between them. Of course, there are other geometric invariants that arise out of this structure.

1. Riemannian and Hermitian metrics

**Definition 24.2.** A Riemannian metric on a vector bundle $E$ is a section

$$g \in \Gamma(\text{Hom}(E \otimes E, \mathbb{R}))$$

such that $g$ is symmetric and positive definite. A Riemannian manifold is a manifold together with a choice of Riemannian metric on its tangent bundle.

**Chit-chat 24.3.** A section of $\text{Hom}(E \otimes E, \mathbb{R})$ is a smooth choice of a linear map

$$g_p : E_p \otimes E_p \to \mathbb{R}$$

at every point $p$. That $g$ is symmetric means that for every $u, v \in E_p$, we have

$$g_p(u, v) = g_p(v, u).$$

That $g$ is positive definite means that

$$g_p(v, v) \geq 0$$

for all $v \in E_p$, and equality holds if and only if $v = 0 \in E_p$.

**Chit-chat 24.4.** As usual, one can try to understand $g$ in local coordinates. If one chooses a trivializing set of linearly independent sections $\{s_i\}$, one obtains a matrix of functions

$$g_{ij} = g_p((s_i)_p, (s_j)_p).$$

By symmetry of $g$, this is a symmetric matrix.
Example 24.5. $T\mathbb{R}^n$ is trivial. Let $g_{ij} = \delta_{ij}$ be the constant matrix of functions, so that $g_{ij}(p) = I$ is the identity matrix for every point. Then on every fiber, $g$ defines the usual inner product on $T_p\mathbb{R}^n = \mathbb{R}^n$.

Example 24.6. Let $j : M \to N$ be a smooth immersion and let $h$ be a metric on $N$. Then one can define a Riemannian metric on $TM$ by setting

$$g_p(u, v) = h_{j(p)}(Tj(u), Tj(v)).$$

We call this the induced or inherited metric. As an example, the standard sphere $j : S^2 \hookrightarrow \mathbb{R}^3$ inherits a Riemannian metric from $\mathbb{R}^3$ in this way.

Proposition 24.7. For any vector bundle $E$, a Riemannian metric exists.

Proof. Partitions of unity. □

Definition 24.8. A Hermitian metric on a complex vector bundle $E$ is a choice of Hermitian inner product $g$ on each fiber $E_p$.

As with above, one can prove a Hermitian metric exists on any complex vector bundle $E$.

Definition 24.9. Two Riemannian manifolds $(M, g)$ and $(N, h)$ are isometric if there is a diffeomorphism $f : M \to N$ for which $g(u, v) = h(Tu, Tv)$.

Definition 24.10. Sections $s_i$ are called orthonormal if

$$g(s_i, s_j) = \delta_{ij}.$$ 

If an orthonormal collection $\{s_i\}$ also spans $E_p$ for every $p$, then we call $\{s_i\}$ an orthonormal frame.

Proposition 24.11. For any Riemannian metric on $E$, and for any $p \in M$, there exists a neighborhood $U$ of $p$ on which one can find an orthonormal frame of $E|_U$. Likewise for Hermitian metrics on a complex vector bundle.

Warning 24.12. Let $g$ be a Riemannian metric on $M$. The above proposition does not imply that one can find a coordinate chart for $M$ on which $g$ looks like the identity matrix. One can find sections of $TM$ for which this is true, but these sections are not induced by a coordinate chart $\mathbb{R}^n \to M$ in general. Indeed, when one can find orthonormal sections $s_i$ such that $s_i = Tf(\partial/\partial x_i)$ for some open embedding $f : U \hookrightarrow M$, we say that the metric is flat on $f(U)$.
2. Levi-Civita Connection and metric connections

Given two sections $s_1, s_2$ of $E$, one can try to measure the rate of change of the function $g(s_1, s_2)$.

We say that a connection on $E$ is compatible with $g$ if

$$d(g(s_1, s_2)) = g(\nabla s_1, s_2) + g(s_1, \nabla s_2)$$

for all $s_i \in \Gamma(E)$. Note that this is an equality of 1-forms. The same equation defines the notion of compatibility of $\nabla$ with a Hermitian connection, in the case that $E$ is complex.

Put another way, for any pair $s_i \in \Gamma(E)$ and any vector field $X$, we must have

$$X(g(s_1, s_2)) = g(\nabla X s_1, s_2) + g(s_1, \nabla X s_2).$$

When $E = TM$, we further say that $\nabla$ is torsion-free if

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Proposition 24.13. For any Riemannian metric $g$ on $E$, there exists some connection that is compatible with $g$. If $E = TM$, there is a unique connection which is both compatible with the metric and torsion-free.

Definition 24.14. This unique connection on $TM$ is called the Levi-Civita connection. For an arbitrary $E$, $\nabla$ may not be unique, but is still called a metric connection.

3. Christoffel Symbols

Let $\nabla$ be a connection on $E$. Given a local frame $s_i$ and a local chart for the manifold, one can write

$$\nabla s_b = \Gamma^c_{ab} dx_a \otimes s_c.$$  

Or, if one likes,

$$\nabla_{\partial x_a}s_b = \Gamma^c_{ab} s_c.$$ 

In the case $E = TM$, of course, a local chart for $M$ induces a local frame $s_i$ on $TM$, and one can write

$$\nabla_{\partial x_a} \partial x_b = \Gamma^c_{ab} \partial x_c.$$ 

The $\Gamma^c_{ab}$ are called the Christoffel symbols for the connection $\nabla$. If $\nabla$ is torsion-free, we have that

$$\Gamma^c_{ab} = \Gamma^c_{ba}.$$