BIRTHING OPERS

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1. Introduction

1.1. Let G be a simply connected semisimple group with Borel subgroup B, $N = [B, B]$ and let $H = B/N$. Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{n}$ and $\mathfrak{h}$ be the respective Lie algebras of these groups. Let ${}^LG$ be the Langlands dual group with dual Borel ${}^LB$, etc. Let X be a smooth curve or the formal disc $\mathcal{D}$ or the formal punctured disc $\mathcal{D}^\times$.

1.2. Recall that a ${}^L\mathfrak{g}$-oper over X is a ${}^LG$-bundle $\mathfrak{F}_{{}^LG} = \mathfrak{F}$ with a reduction $\mathfrak{F}_{{}^LB}$ to ${}^LB$ and a connection ∇ on $\mathfrak{F}$ satisfying a property that we recall. There is an obstruction $c(\nabla) \in ({}^L\mathfrak{g}/{}^L\mathfrak{b})_{\mathfrak{F}_{{}^LB}} \otimes \omega_X$ to the preservation of ∇ under the reduction $\mathfrak{F}_{{}^LB}$ and we demand that 1) $c(\nabla) \in ({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-1}_{\mathfrak{F}_{{}^LB}} \otimes \omega_X$ with $({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-1} = \oplus_{\check{\alpha}} ({}^L\mathfrak{g}/{}^L\mathfrak{b})^{\check{\alpha}}$ the space spanned by the negative simple coroots of $\mathfrak{g}$ and 2) the projection of $c(\nabla)$ to $({}^L\mathfrak{g}/{}^L\mathfrak{b})^{\check{\alpha}}_{\mathfrak{F}_{{}^LB}}$ is nowhere vanishing on X for each negative simple coroot $\check{\alpha}$.

1.3. This definition can be said in D-families, i.e., for a D_X-scheme Y, there is a notion of a ${}^LG$-oper on Y which is a ${}^LG$-torsor on Y with a connection along the vector fields coming from X with a reduction to ${}^LB$ satisfying some properties. We denote by $\mathrm{DOp}_{{}^LG}(X)$ the (affine) D-scheme of opers on X and let $\mathrm{Op}_{{}^LG}(X) = H_\nabla(\mathrm{DOp}_{{}^LG}(X))$.

1.4. Let $\widehat{\mathfrak{g}}_{crit}$ be the Kac-Moody algebra at critical level, i.e., with $\kappa = -\frac{1}{2}\kappa_{killing}$. The Feigin-Frenkel isomorphism says that the space of opers $\mathrm{Op}_{{}^LG}(\mathcal{D})$ on $\mathcal{D}$ is isomorphic to the spectrum of the (commutative) algebra $\mathfrak{z}_\mathfrak{g} = \mathfrak{z}$ of endomorphisms of the vacuum module $\mathbb{V}_{crit}$.

The goal for this lecture is to formulate a theorem of [BD1] describing this isomorphism in terms of the affine Grassmannian. We will more or less construct a map from $\mathrm{Spec}(\mathfrak{z})$ to $\mathrm{Op}_{{}^LG}(\mathcal{D})$ and the theorem (which we do not address) will say that this map is the Feigin-Frenkel isomorphism.

1.5. To construct a map from $\mathrm{Spec}(\mathfrak{z})$ to $\mathrm{Op}_{{}^LG}(\mathcal{D})$, it suffices to construct a ${}^LG$-bundle $\mathfrak{F}_{\mathfrak{z},{}^LG} = \mathfrak{F}_\mathfrak{z}$ on $\mathrm{Spec}(\mathfrak{z})\widehat{\times}\mathcal{D}$ with a reduction $\mathfrak{F}_{\mathfrak{z},{}^LB}$ to ${}^LB$ and a connection ∇ along $\mathcal{D}$ which satisfies the oper properties. In Section 2 we will construct (modulo Theorem 2.1) a ${}^LG$-bundle $\mathfrak{F}^0_\mathfrak{z}$ on $\mathrm{Spec}(\mathfrak{z})$ corresponding to the pull-back of $\mathfrak{F}_{\mathfrak{z},{}^LG}$ along the zero map $\mathrm{Spec}(\mathfrak{z}) \longrightarrow \mathrm{Spec}(\mathfrak{z})\widehat{\times}\mathcal{D}$. In Section 3, we discuss how to recover

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$\mathfrak{F}_{\mathfrak{z}}$ with its connection from the infinitesimal symmetries of $\mathfrak{F}^0_{\mathfrak{z}}$ and we give a means to check the oper property. In Section 4 we formulate Lemma 4.1 which will allow us to construct the ${}^LB$-reduction and check the oper property. In Section 5 we prove Lemma 4.1.

2. Construction of the ${}^LG$-bundle

2.1. We want to construct an ${}^LG$-bundle $\mathfrak{F}^0_{\mathfrak{z}}$ on $\mathrm{Spec}(\mathfrak{z})$ via geometric Satake. Recall that an ${}^LG$-bundle is given by assigning to any (always finite-dimensional) representation V of ${}^LG$ a vector bundle $\mathfrak{F}^0_{\mathfrak{z}}(V)$ in a tensor-functorial way. Therefore, we concern ourselves now with the construction of $\mathfrak{z}$-modules.

2.2. Let Gr_G be the affine Grasmmannian. By a D_{crit}-module on Gr_G, we will mean a right twisted D-module on Gr_G (compactly supported) with twisting given by $\frac{1}{2}$ the determinant line bundle, i.e., $-\frac{1}{2}$ the Kac-Moody line bundle. The action of $\widehat{\mathfrak{g}}_{crit}$ on this (virtual) line bundle induces an action of $\widehat{\mathfrak{g}}_{crit}$ on $\Gamma(\mathrm{Gr}_G, M)$ for any D_{crit}-module M.

2.3. We will not need this section but include it for completeness.

The center $\mathfrak{Z}$ of the twisted enveloping algebra $U'(\widehat{\mathfrak{g}}_{crit})$ has a natural map to $\mathfrak{z}$ given its action on the vacuum module. In fact, this map can be shown to be surjective by realizing $\mathrm{Spf}(\mathfrak{Z})$ as the space of opers on the formal punctured disc. We claim that for any D_{crit}-module M, $\mathfrak{Z}$ acts through $\mathfrak{z}$. First, observe that this is true for the δ D_{crit}-module at the distinguished point e in Gr_G. Indeed, $\Gamma(\mathrm{Gr}_G, \delta_e)$ is the vacuum module $\mathbb{V}_{crit}$ and by definition of the map $\mathfrak{Z} \longrightarrow \mathfrak{z}$, $\mathfrak{Z}$ acts through the quotient $\mathfrak{z}$. Similarly, for any $g \in \mathrm{Gr}_G$, the $\widehat{\mathfrak{g}}_{crit}$-module $\Gamma(\mathrm{Gr}_G, \delta_g)$ is the vacuum module but with the choice of "maximal compact" the conjugate of $G(O)$ by a lift of g to $G(K)$ ($O = \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$). However, because $\mathfrak{Z}$ is fixed under conjugation, the same argument goes through. Because this is true for the δ D_{crit}-module at any point, the action of $\mathfrak{Z}$ on global sections of any D_{crit}-module is through $\mathfrak{z}$.

The format of this proof is convolution, to be expanded upon later in the seminar.

2.4. Let us collect a few facts about the geometric Satake equivalence that we will need. Let $\mathcal{H}_{sph}$ be the spherical Hecke category, i.e., the category of compactly supported $G(O)$-equivariant D-finitely generated right (untwisted) D-modules on Gr_G. Recall that geometric Satake gives an equivalence between $\mathcal{H}_{sph}$ and the category $\mathrm{Rep}\,{}^LG$ of finite dimensional representations of ${}^LG$.

Recall that the $G(O)$-orbits of Gr_G are indexed by the dominant coweights of G, where for such a coweight $\check{\lambda} : \mathbb{G}_m \longrightarrow G$ the corresponding orbit $\mathrm{Gr}^{\check{\lambda}}_G$ is the orbit containing $\check{\lambda}(t)$ where $t \in \mathbb{G}_m(K)$ is any uniformizer of $\mathcal{D}$. This description of the orbits does not depend on the choice of t.

Recall that the irreducible representations of ${}^LG$ are classified by the dominant coweights of G where the representation $V^{\check{\lambda}}$ is the representation of ${}^LG$ of highest weight $\check{\lambda}$.

Geometric Satake interchanges these two pictures as follows. The D-module on Gr_G which corresponds to $V^{\check{\lambda}}$ via the Satake equivalence is given as the intersection cohomology D-module of the orbit $\mathrm{Gr}_G^{\check{\lambda}}$, i.e., for $j^{\check{\lambda}} : \mathrm{Gr}_G^{\check{\lambda}} \hookrightarrow \mathrm{Gr}_G$, the D-module $j_{!*}(\mathscr{O}_{\mathrm{Gr}_G^{\check{\lambda}}})$.

2.5. Let $\mathcal{H}_{crit}$ be the category of $G(O)$-equivariant D-finitely generated D_{crit}-modules on Gr_G. As in the case of $\mathcal{H}_{sph}$, this is naturally a tensor category via the factorizaton structure on the determinant line bundle. According to [BD1] Section 4, there is a canonical identification of $\mathcal{H}_{crit}$ with $\mathcal{H}_{sph}$ as tensor categories. We fix such an identification for the rest of these notes. Note that this identification amounts to giving a square root $\mathscr{L}_{crit}$ of the determinant bundle.

By geometric Satake, this defines an equivalence of categories between $\mathcal{H}_{crit}$ and the category $\mathrm{Rep}\,{}^LG$. For $V \in \mathrm{Rep}\,{}^LG$, let $\mathcal{M}_{V,crit}$ be the corresponding D_{crit}-module on Gr_G.

2.6. We will need to appeal to the following theorem from [BD1]:

Theorem 2.1. *For any $V \in \mathrm{Rep}\,{}^LG$, the $\widehat{\mathfrak{g}}_{crit}$-module $R\Gamma(\mathrm{Gr}_G, \mathcal{M}_{V,crit})$ is a direct sum of copies of $\mathbb{V}_{crit}$ concentrated in cohomological degree 0.*

The formalism of groups acting on categories defines a monoidal action $\mathcal{H}_{crit}$ on the derived version of the category $\widehat{\mathfrak{g}}_{crit} - \mathrm{mod}^{G(O)}$.[1] The general convolution format implies that $R\Gamma(\mathrm{Gr}_G, \mathcal{M}_{V,crit}) \xrightarrow{\simeq} \mathcal{M}_{V,crit} * \mathbb{V}_{crit}$.

2.7. Now let us construct the ${}^LG$-bundle $\mathfrak{F}^0_{\mathfrak{z}}$ on $\mathrm{Spec}(\mathfrak{z})$. We consider $\mathfrak{F}^0_{\mathfrak{z}}$ as a tensor functor (i.e., monoidal and commuting with commutativity constraints) $\mathfrak{F}^0_{\mathfrak{z}} : \mathrm{Rep}\,{}^LG \longrightarrow \mathfrak{z} - mod$.

Define $\mathfrak{F}^0_{\mathfrak{z}}(V) = \mathrm{Hom}_{\widehat{\mathfrak{g}}_{crit}}(\mathbb{V}_{crit}, \Gamma(\mathrm{Gr}_G, \mathcal{M}_{V,crit})) = (\mathcal{M}_{V,crit} * \mathbb{V}_{crit})^{G(O)}$. Theorem 2.1 implies that $\mathfrak{F}^0_{\mathfrak{z}}(V) \otimes_{\mathfrak{z}} \mathbb{V}_{crit}$ considered as a $\widehat{\mathfrak{g}}_{crit}$-module via the action on $\mathbb{V}_{crit}$ is isomorphic to $\mathcal{M}_{V,crit} * \mathbb{V}_{crit}$. Therefore, we have:

$$\mathcal{M}_{V_1 \otimes V_2} * \mathbb{V}_{crit} \simeq (\mathcal{M}_{V_1} * \mathcal{M}_{V_2}) * \mathbb{V}_{crit} \simeq \mathcal{M}_{V_1} * (\mathfrak{F}^0_{\mathfrak{z}}(V_2) \otimes_{\mathfrak{z}} \mathbb{V}_{crit}) \simeq \mathfrak{F}^0_{\mathfrak{z}}(V_2) \otimes_{\mathfrak{z}} (\mathcal{M}_{V_1} * \mathbb{V}_{crit})$$

Here the last equality is true because convolving with a direct sum of copies of the vacuum module is exact and then by functoriality. The last term is isomorphic to $\mathfrak{F}^0_{\mathfrak{z}}(V_1) \otimes_{\mathfrak{z}} \mathfrak{F}^0_{\mathfrak{z}}(V_2) \otimes_{\mathfrak{z}} \mathbb{V}_{crit}$. Therefore, applying $\mathrm{Hom}_{\widehat{\mathfrak{g}}_{crit}}(\mathbb{V}_{crit}, -)$ we get:

$$\mathfrak{F}^0_{\mathfrak{z}}(V_1 \otimes V_2) \xrightarrow{\simeq} \mathfrak{F}^0_{\mathfrak{z}}(V_1) \otimes_{\mathfrak{z}} \mathfrak{F}^0_{\mathfrak{z}}(V_2)$$

[1] Let us give a moral argument why it acts, at least in the untwisted setting. Given a group ind-scheme acting on a category in an appropriate sense and given a subgroup, the corresponding Hecke category acts on the "invariants" of the action, i.e., the corresponding equivariant category.

as desired. It's direct to check the compatibility with associativity constraints, so our functor $\mathfrak{F}_{\mathfrak{z}}^0$ is monoidal.

2.8. Next, let us indicate the proof that this monoidal functor is compatible with the commutativity constraint. Our argument uses chiral algebras, which will be formally introduced later in the seminar. The reference for chiral algebras is [BD2].

Why do we need to appeal to chiral algebras? Recall that the definition of the commutativity constraint on the affine Grassmannian, which is central to this theorem, is constructed most naturally by considering the factorization affine Grassmannian and looking at what happens as two different points collide. To compare our monoidal functor with this, we need a version of the Kac-Moody algebra which lives not only at a single point on the curve, but which interpolates these different algebras just as the global affine Grassmannian interpolates the affine Grassmannians living at each point. Chiral algebras are designed to do this.

2.9. Let us collect a few facts about chiral algebras.

First, a chiral algebra is a D-module on a curve X equipped with extra structure. A D_X-algebra is equally well labelled as a commutative chiral algebra.

For a chiral algebra A, there are associated D_{X^i}-modules $A^{(i)}$ forming the factorization algebra structure of A, where $A^{(1)} = A$ and i is any non-negative integer. For A commutative, $A^{(i)}$ is a D_{X^i}-algebra.

For a chiral algebra A, there is a notion of chiral module over A which is in particular a D-module on X. More generally, there is a category $\mathcal{C}_A^{(i)}$ of modules of A on X^i for all i which are D_{X^i}-modules. Then $\mathcal{C}_A^{(i)}$ has $A^{(i)}$ as a distinguished object. As i varies, the $\mathcal{C}_A^{(i)}$ form a chiral category which we sometimes abbreviate, however unfairly, $\mathcal{C}_A$.

2.10. There is a Kac-Moody chiral algebra $\mathcal{A}_{\mathfrak{g},crit}$ such that modules over this chiral algebra supported at a closed point $x \in X$ are equivalent to modules over the corresponding Kac-Moody algebra. An important fact about this chiral algebra is that the fiber of $\mathcal{A}_{\mathfrak{g},crit}$ over such a point x is the vacuum module for the corresponding Kac-Moody algebra.

The center of this chiral algebra is denoted $\mathfrak{z}_X$. One has $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{g},crit}}(\mathcal{A}_{\mathfrak{g},crit}, \mathcal{A}_{\mathfrak{g},crit}) = \mathfrak{z}_X$. By the way, this proves that endomorphisms of the vacuum module is a commutative algebra.

2.11. Recall that we have group factorization schemes $G(K)_X$ and $G(O)_X$ such that $G(K)_X/G(O)_X$ is the factorization affine Grassmannian. This defines the chiral critical Hecke category $\mathcal{H}_{crit,X}$. Because $G(K)_X$ acts on the chiral category $\mathcal{C}_{\mathcal{A}_{\mathfrak{g},crit}}$, the chiral critical Hecke category acts on $\mathcal{C}_{\mathcal{A}_{\mathfrak{g},crit}}^{G(O)_X}$. Because the objects V of $\mathcal{H}_{crit}$ are Aut-equivariant (see Section 3.1 and Section 4.1), we have induced objects $M_{V,X}$ of $\mathcal{H}_{crit,X}^{(1)}$.

2.12. For $M \in \mathcal{H}_X^{(i)}$, define $\mathfrak{F}^0_{\mathfrak{z}_X}(M)$ to be $\mathrm{Hom}_{\mathcal{C}^{(i)}_{\mathcal{A}_{\mathfrak{g},crit}}}(\mathcal{A}^{(i)}_{\mathfrak{g},crit}, M * \mathcal{A}^{(i)}_{\mathfrak{g},crit})$. Note that this is a module over $\mathfrak{z}_X^{(i)}$. A generalization of Theorem 2.1 says that for $V \in \mathcal{H}_X^{(i)}$, we have:

$$\mathfrak{F}^0_{\mathfrak{z}_X}(M) \otimes_{\mathfrak{z}_X^{(i)}} \mathcal{A}^{(i)}_{\mathfrak{g},crit} \xrightarrow{\simeq} M * \mathcal{A}^{(i)}_{\mathfrak{g},crit}$$

Consideration of the convolution picture implies that for $i = 2$, we have:

$$\mathfrak{F}^{0,(2)}_{\mathfrak{z}_X}(j_{!*}(M_{V_1,X} \boxtimes M_{V_2,X})) \xrightarrow{\simeq} (\mathfrak{F}^0_{\mathfrak{z}_X}(M_{V_1,X}) \boxtimes \mathfrak{F}^0_{\mathfrak{z}_X}(M_{V_2,X})) \otimes_{\mathfrak{z}\boxtimes\mathfrak{z}} \mathfrak{z}^{(2)}$$

Applying Δ^* and restricting to a point on our curve, we see that our functor $\mathfrak{F}^0_{\mathfrak{z}}$ is compatible with commutativity constraints because both arise from switching coordinates on X^2 and then pulling back.

3. Automorphisms of the Formal Disc

3.1. We fix a coordinate t of the formal disc $\mathcal{D}$.

Let[2] Aut be the group scheme of automorphisms of $\mathcal{D}$ which preserve the point 0. This is a group scheme (of infinite type) whose points are power series without constant term and with invertible t-coefficient with group law the composition of power series, i.e., for a $\mathbb{C}$-algebra A:

$$\mathrm{Aut}(A) = \{\sum_{i>0} a_i t^i | a_i \in A, a_1 \in A^\times\}$$

There is a canonical homomorphism $\mathrm{Aut} \longrightarrow \mathbb{G}_m$ given by considering how any uniformizer is scaled under the action of Aut. We refer to this as the "standard character." Any uniformizer defines a splitting of this character.

Let Aut^+ be the group scheme of all automorphisms of $\mathcal{D}$. This is a group ind-scheme whose points are power series with nilpotent constant term and invertible t-coefficient and group law the composition of power series, i.e., for a $\mathbb{C}$-algebra A, we have:

$$\mathrm{Aut}^+(A) = \{\sum_{i\geq 0} a_i t^i | a_i \in A, a_1 \in A^\times, a_0^N = 0 \text{ for } N >> 0\}$$

By this description, we see that Aut^+ acts on $\mathcal{D}^\times$ as well.

There is a natural embedding of Aut into Aut^+ which realizes Aut as the reduced part of Aut^+. Therefore, giving an action of Aut^+ is equivalent[3] to giving an action of the Harish-Chandra pair $(\mathrm{Lie}(\mathrm{Aut}^+), \mathrm{Aut})$. The quotient $\mathrm{Aut}^+ / \mathrm{Aut}$ is canonically isomorphic to $\mathcal{D}$ in an Aut^+-equivariant way.

[2]Our notation differs from that of [BD1]. Our Aut is their Aut^0 and our Aut^+ is their Aut. Probably their notation is better because of the compatibility with the notation Der, c.f. Section 3.3.

[3]This is a generalization of a familiar fact from formal groups: to give an action of a formal group is the same thing as giving an action of its Lie algebra.

3.2. Note the crystalline nature of the action of Aut^+ on $\mathcal{D}$: this action identifies 0 with infinitesimally close points.

3.3. The Lie algebra $\mathrm{Lie}(\mathrm{Aut}^+)$ is denoted Der. The action of Der on $\mathcal{D}$ realizes Der as the Lie algebra of derivations on the disc, i.e., expressions $a\partial_t$ with $a \in \mathbb{C}[[t]]$ and with the usual Lie bracket of vector fields. The Lie algebra of Aut is given by $t \cdot \mathrm{Der}$ the vector fields vanishing at the origin.

Let $L_i = -t^{i+1}\partial_t \in \mathrm{Der}$ for $i \geq -1$. Observe that $[L_0, L_n] = -nL_n$ and that the span of L_{-1}, L_0 and L_1 forms a copy of sl_2 inside of Der.

3.4. Let $K_1 \subset K_2$ be affine group ind-schemes with K_1 a group scheme ($\mathrm{Aut} \subset \mathrm{Aut}^+$ for our purposes) such that K_2/K_1 a formally smooth ind-scheme of ind-finite type and let S be a scheme over $\mathbb{C}$ equipped with an action of K_2. Let $i : S \longrightarrow S\widehat{\times}K_2/K_1$ be the embedding given by the identity in K_2.

By a quasi-coherent sheaf on a ind-scheme, we mean in the $*$-sense, i.e., a compatible family of sheaves on some realization of the ind-scheme with respect to the $*$-pull-back.

Proposition 3.1. *The functor i^* is an equivalence between coherent sheaves on $S\widehat{\times}K_2/K_1$ equivariant with respect to the diagonal action of K_2 and K_1-equivariant coherent sheaves on S. This functor lifts to an equivalence between K_2-equivariant coherent sheaves on $S\widehat{\times}K_2/K_1$ with (weakly)*[4] *equivariant connection along K_2/K_1 and coherent sheaves on S equivariant with respect to the action of the Harish-Chandra pair $(\mathfrak{k}_2, K_1)$.*

Proof. Consider the map $S\widehat{\times}K_2 \longrightarrow S$ which at the level of points is $(s, k) \mapsto k^{-1} \cdot s$. This is equivariant with respect to the K_1-action on $S\widehat{\times}K_2$ via its right action on the second coordinate and the natural action on S, and K_2-equivariant for the diagonal action on the first term and the trivial action on the second. Therefore, this induces an isomorphism $S\widehat{\times}(K_2/K_1) \longrightarrow (S\widehat{\times}K_2)/K_1$ in a K_2-equivariant way. This gives the result.

Let us try to imitate the formalism of the argument above for the second part. To do this, we need the de Rham space[5] of a scheme Z, which we recall is defined by $DR(Z)(R) = \varprojlim Z(R/I)$ where I ranges over all nilpotent ideals of R (for R Noetherian, this is just $Z(R_{red})$). A coherent sheaf over $DR(Z)$ is equivalent to a D-module on Z. Note that for a group K with formal completion at the identity $\widehat{K}$, we have $DR(K) = K/\widehat{K}$.

[4]Note that it does not make sense to speak of strongly equivariant bundles with connection over the first coordinate when a connected group acts non-trivially on the second coordinate.

[5]"Space" means merely a functor from the category of commutative rings to pro-sets.

Now we can imitate the proof above. We have the following equality, where all quotients are understood in the stack sense:

$$K_2\backslash S\widehat{\times}DR(K_2/K_1) \xrightarrow{\simeq} K_2\backslash S\widehat{\times}DR(K_2)/DR(K_1) \xrightarrow{\simeq} K_2\backslash(S\widehat{\times}(\widehat{K_2}\backslash K_2))/DR(K_1)$$

But the last expression is isomorphic to $DR(K_1)\backslash(\widehat{K_2}\backslash S)$ as before. Coherent sheaves on $K_2\backslash S\widehat{\times}DR(K_2/K_1)$ of the above isomorphisms are (weakly, of course) K_2-equivariant coherent sheaves on $S\widehat{\times}K_2/K_1$ with connection along the second coordinate, while coherent sheaves on $DR(K_1)\backslash(\widehat{K_2}\backslash S)$ are $\widehat{K_2}$-equivariant coherent sheaves which are strongly K_1-equivariant, i.e., they have a $\mathfrak{k}_2$-action and are strongly K_1-equivariant, i.e., they have an action of the Harish-Chandra pair $(\mathfrak{k}_2, K_1)$. □

3.5. Note that the Proposition 3.1 implies similar equivalences with torsors for some affine algebraic group Γ replacing coherent sheaves. Therefore, because Aut^+ equivariance is equivalent to an action of the Harish-Chandra pair $(\mathrm{Der}, \mathrm{Aut})$, we obtain the following corollary, which is [BD1] 3.5.3.:

Corollary 3.2. *Let* Aut^+ *act on* S. *The functor* i^* *is an equivalence between* Aut^+-*equivariant* Γ-*bundles on* $S\widehat{\times}\mathcal{D}$ *and* Aut-*equivariant* Γ-*bundles on* S. *The functor* i^* *is an equivalence between* Aut^+-*equivariant* Γ-*bundles on* $S\widehat{\times}\mathcal{D}$ *with connection along* $\mathcal{D}$ *and* Aut^+-*equivariant* Γ-*bundles on* S.

Remark 3.3. Roman Travkin has suggested the following proof of the second statement. First, forgetting the symmetries, pull-back is an equivalence between coherent sheaves on S and coherent sheaves on $S\widehat{\mathcal{D}}$ with connection along the second coordinate. Indeed, this is immediate from the crystalline perspective. This functor is Aut^+-equivariant and therefore induces an equivalence between Aut^+-equivariant objects. A similar proof works in the general setting of Proposition 3.1.

3.6. Suppose that $S = \mathrm{Op}_{^LG}(\mathcal{D})$. Then Corollary 3.2 says that the Aut-equivariant ^LG-bundle $\mathfrak{F}^0_{Op}$ on $\mathrm{Op}_{^LG}(\mathcal{D})$ obtained by pull-back of the tautological bundle on $\mathrm{Op}_{^LG}(\mathcal{D})\widehat{\times}\mathcal{D}$ admits a natural Aut^+-equivariant structure which incorporates the connection of the tautological bundle. Furthermore, $\mathfrak{F}^0_{\mathrm{Op}}$ has a Aut-equivariant reduction $\mathfrak{F}^0_{\mathrm{Op},^LB}$ to ^LB. These structures recover the ^LG-bundle with connection and reduction to ^LB on $\mathrm{Op}_{^LG}(\mathcal{D})\widehat{\times}\mathcal{D}$ entirely!

3.7. Given a scheme $S = \mathrm{Spec}(A)$ with an action of Aut^+ and an Aut^+-equivariant ^LG-bundle $\mathfrak{F}^0$ with Aut-equivariant reduction $\mathfrak{F}^0_{^LB}$ of $\mathfrak{F}^0$ to ^LB, we would like to characterize when the induced bundle with connection on $S\widehat{\times}\mathcal{D}$ comes from a map $S \longrightarrow \mathrm{Op}_{^LG}(\mathcal{D})$.

3.8. We will do this in a convenient way under assumptions of the action of Der on A which are satisfied when $A = \mathfrak{z}$ (see Section 4.2). Note that L_0 (see Section 3.3) acts on A diagonalizably with integer eigenvalues. We assume that these eigenvalues are all non-negative with the eigenvalue 0 occurring with multiplicity one and the eigenvalue 1 occurring with multiplicity 0. Note that the unit 1 of A must be the unique up to scaling eigenvector with eigenvalue 0.

In this case, $\mathrm{Spec}(A)$ has a distinguished closed point $* \in \mathrm{Spec}(A)$ defined by the maximal ideal consisting of the span of the eigenvectors with positive eigenvalue. This maximal ideal is preserved by $sl_2 \subset \mathrm{Der}$ (see Section 3.3) because of the assumptions on the action.

One way to check that the eigenvalues are non-negative is to see that the action of Aut on A extends to an action of the algebraic semigroup of all endomorphisms of the disc as an ind-scheme. In this case, the action of $\mathbb{G}_m \subset \mathrm{Aut}$ extends to an action of the semigroup $\mathbb{A}^1$ where 0 in $\mathbb{A}^1$ corresponds to the composition $\mathcal{D} \longrightarrow 0 \longrightarrow \mathcal{D}$. However, if the action extends to $\mathbb{A}^1$, then the only characters of $\mathbb{G}_m$ which can appear look as $z \mapsto z^n$ for $n \geq 0$ as desired. One should note that this is not true for $\mathfrak{z}$, but we will use this in Section 5.

3.9. The following is [BD1] 3.5.8.

Proposition 3.4. *Let A be an algebra with an action of* Aut^+ *satisfying the conditions of Section 3.8 and equipped with an* Aut^+ *equivariant ${}^LG$-bundle $\mathfrak{F}^0$ with a* Aut-*equivariant reduction $\mathfrak{F}^0_{{}^LB}$ to ${}^LB$. Then this bundle comes from a map* $\mathrm{Spec}(A) \longrightarrow \mathrm{Op}_{{}^LG}(\mathcal{D})$ *if and only if the induced* Aut-*equivariant ${}^LH$-bundle is obtained by pull-back from $\mathbb{C}$ of the* Aut-*equivariant bundle $\rho(\omega_{\mathcal{D}}/t\omega_{\mathcal{D}})$.*

The proof will occupy Sections 3.10-3.13.

3.10. First, let us show the necessity of this condition on ${}^LH$. Recall the following general property of opers.

For an oper on X, the ${}^LH$-bundle induced by the ${}^LB$-bundle is canonically described as follows. Take $\rho : \mathbb{G}_m \longrightarrow^L H$ (which exists because ${}^LG$ is adjoint) and push forward the line bundle ω_X to get a ${}^LH$-bundle $\rho(\omega_X)$. More generally, for a Y-family of opers on X (Y a D_X-scheme), the induced ${}^LH$-bundle on Y is the pull-back of $\rho(\omega_X)$ to Y along the structure map $Y \longrightarrow X$.

Therefore, the tautological ${}^LB$-bundle $\mathfrak{F}_{\mathrm{Op},{}^LB}$ on $\mathrm{Op}_{{}^LG}(\mathcal{D})\widehat{\times}\mathcal{D}$ induces the ${}^LH$-bundle which is the pull-back along the second coordinate of $\rho(\omega_{\mathcal{D}})$. Pulling back along $\mathrm{Op}_{{}^LG}(\mathcal{D}) \to \mathrm{Op}_{{}^LG}(\mathcal{D})\widehat{\times}\mathcal{D}$ implies necessity.

3.11. Now we aim to show sufficiency. We have the bundle $\mathfrak{F}$ on $\mathrm{Spec}(A)\widehat{\times}\mathcal{D}$ with connection ∇ over $\mathcal{D}$ corresponding to $\mathfrak{F}^0$ via Corollary 3.2 and with reduction to $\mathfrak{F}_{{}^LB}$. We need to show that the connection satisfies the oper properties, i.e., that in the notation of Section 1.2 $c(\nabla) \in ({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-1}_{\mathfrak{F}_{{}^LB}} \otimes \omega_{\mathcal{D}}$ and that the non-degeneracy condition on the projection of $c(\nabla)$ to $({}^L\mathfrak{g}/{}^L\mathfrak{b})^{\alpha}_{\mathfrak{F}_{{}^LB}}$ for all simple coroots α. Let us show this first condition.

Let $({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-k}$ be the $-k$th associated graded piece of the natural ${}^L\mathfrak{b}$-module filtration on ${}^L\mathfrak{g}/{}^L\mathfrak{b}$ as in [BD1] 3.1.1 so that ${}^L\mathfrak{b}$ acts on $({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-k}$ through ${}^L\mathfrak{h}$ and the characters are sums of k negative simple roots. Then $({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-k}_{\mathfrak{F}_{{}^L B}}$ is explicitly computed as an Aut^+-equivariant bundle to be $\pi_2^*\omega_{\mathscr{D}}^{\otimes -k} \otimes ({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-k}$ as follows. The action of ${}^L B$ on $({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-k}$ is through ${}^L H$ and the induced action of $\mathbb{G}_m$ through ρ is given as the diagonal action through the $-k$-power of the standard character. Therefore, by the condition on the induced ${}^L H$-bundle we get this computation.

But $c(\nabla)$ is an Aut-invariant section of $({}^L\mathfrak{g}/{}^L\mathfrak{b})_{\mathfrak{F}_{{}^L B}} \otimes \pi_2^*\omega_{\mathcal{D}}$ and since $({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-k}_{\mathfrak{F}_{{}^L B}} \otimes \pi_2^*\omega_{\mathcal{D}} \xrightarrow{\simeq} \pi_2^*\omega_{\mathscr{D}}^{\otimes -k+1} \otimes ({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-k}$ we see that $c(\nabla)$ lies in $({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-1}_{\mathfrak{F}_{{}^L B}} \otimes \pi_2^*\omega_{\mathcal{D}}$ as desired.

3.12. To proceed we will need the following observation. Trivialize (however non-canonically) the bundle $\mathfrak{F}^0$ over $* \in \mathrm{Spec}(A)$. Because $sl_2 \subset \mathrm{Der}$ acts on $\mathfrak{F}^0$ and preserves the point $*$, sl_2 acts on this fiber. The trivialization of $\mathfrak{F}^0$ defines an embedding $sl_2 \hookrightarrow^L \mathfrak{g}$. A different choice of trivialization of $\mathfrak{F}_0$ would conjugate this embedding by an element of ${}^L G$ so we have an embedding of sl_2 into $\mathfrak{g}$ canonically defined modulo conjugacy. We claim that this is the principal embedding.

Because $\mathfrak{F}^0_{{}^L B}$ is preserved by Aut, L_0 and L_1 map to ${}^L\mathfrak{b}$. On $\mathcal{D}$, L_0 acts on the fiber of $\omega_{\mathcal{D}}$ as multiplication by -1 and therefore acts on the induced ${}^L H$-bundle by $-\rho$ after trivializing $\omega_{\mathcal{D}}$ by dt and using the induced trivialization on our ${}^L H$-bundle. This implies that the conjugacy class of L_0 is the same is $-\rho$, which implies that our embedding is principal.

3.13. Now we can prove the non-degeneracy condition. As in Section 3.11, $c(\nabla)$ lies in the Aut-invariant part of $({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-1}_{\mathfrak{F}_{{}^L B}} \otimes \pi_2^*\omega_{\mathcal{D}}$, and this bundle is isomorphic to the constant vector bundle for the space $({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-1}$. Therefore, its Aut^+-invariants are isomorphic to the vector space $({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-1} \xrightarrow{\simeq} \oplus_{\check{\alpha}}\mathbb{C}$. Therefore, to check the non-degeneracy, it suffices to do this at a single point of $\mathrm{Spec}(A)\widehat{\times}\mathcal{D}$.

Consider the restriction of our data to $* \times \mathcal{D}$. We have seen in Section 3.12 that the embedding of $sl_2 \subset \mathrm{Der}$ into ${}^L\mathfrak{g}$ induced by taking a trivialization of the pull-back of $\mathfrak{F}$ is a principal embedding. In particular, L_{-1} maps to a principal nilpotent element f. Therefore, since $-L_{-1} = \partial_t$ we see that $c(\nabla) \in ({}^L\mathfrak{g}/{}^L\mathfrak{b})^{-1}_{\mathfrak{F}_{{}^L B}} \otimes \omega_{\mathcal{D}}$ must be $-f dt$ which implies the desired result.

4. Construction of the ${}^L B$-bundle

4.1. Note that Aut^+ acts on Gr_G and in fact, the semi-direct product of Aut^+ with $G(O)$ acts on Gr_G. The orbits of Gr_G are preserved under the action of Aut because they are of the form $G(O) \cdot \check{\lambda}(t)$ which reduces us to the case $G = \mathbb{G}_m$ where this is clear. The square root $\mathscr{L}_{crit}$ of the determinant bundle is a Aut^+-equivariant square root, i.e., it is Aut^+-equivariant and the natural isomorphism $\mathscr{L}_{crit}^{\otimes 2} \longrightarrow \mathscr{L}_{det}$ is Aut^+-equivariant. Therefore, by the explicit description of the category $\mathcal{H}_{crit}$, we see that every object of this is Aut^+-equivariant. By construction of the functor $\mathfrak{F}^0_{\mathfrak{z}}$, the ${}^L G$-bundle on $\mathrm{Spec}(\mathfrak{z})$ constructed in Section 2 is Aut^+-equivariant.

According to Section 3.4, we should check that $\mathfrak{z}$ satisfies the conditions of Section 3.8 and construct an Aut-equivariant reduction of this bundle to ${}^LB$ which induces the correct Aut-equivariant ${}^LH$-bundle.

4.2. Let us show that $\mathfrak{z}$ satisfies the conditions of Section 3.8. The associated graded of $\mathfrak{z}$ with respect to its natural Aut^+-equivariant filtration is[6] $(\mathrm{Sym}(\mathfrak{g} \otimes K/O))^{G(O)}$. However, on this space L_0 clearly has non-negative eigenvalues and a unique 0-eigenspace. Furthermore, if it had any eigenvectors with eigenvalue 1, we see that these would have to live in $\mathfrak{g} \otimes K/O$ and in particular in $\mathfrak{g} \otimes 1/t$ for a uniformizer t. However, there are no G-invariant elements in $\mathfrak{g}$ besides 0 because $\mathfrak{g}$ is semi-simple.

4.3. Fix a uniformizer t of $\mathcal{D}$, which in particular defines an operator $L_0 = -\partial_t \in$ Der. Let $IC_{crit,\check{\lambda}}$ be the twisted intersection cohomology D_{crit}-module on Gr_G corresponding to the locally closed subset $\mathrm{Gr}_G^{\check{\lambda}}$.

The ${}^LB$-reduction comes from the following lemma, which will be proved by explicit calculations in Section 5.

Lemma 4.1. *The lowest eigenvalue of L_0 acting on $\Gamma(\mathrm{Gr}_G, \mathrm{IC}_{crit,\check{\lambda}})$ is $-\check{\lambda}(\rho)$. This eigenspace is one-dimensional over $\mathbb{C}$.*

We assume this lemma for the remainder of this section.

4.4. Let $\mathcal{L}_{\check{\lambda}} \subset \Gamma(\mathrm{Gr}_G, \mathrm{IC}_{crit,\check{la}})$ be the eigenspace described by the lemma. First, we claim that this line is independent of the choice of uniformizer t. It suffices to show that this line is fixed by the kernel $\mathrm{Ker}(\mathrm{Aut} \longrightarrow \mathbb{G}_m)$ of the standard character because then Aut acts on such invariants through $\mathbb{G}_m$ and the operator L_0 becomes canonical.

But this is clear: because $[L_0, L_i] = -iL_i$, L_0 acts on $L_i\mathcal{L}_{\check{\lambda}}$ with eigenvalue $-\check{\lambda}(\rho)-i$ so this line must be 0 by Lemma 4.1.

4.5. Next, we prove that $\mathcal{L}_{\check{\lambda}}$ lies in $\Gamma(\mathrm{Gr}_G, \mathrm{IC}_{crit,\check{\lambda}})^{G(O)} = \Gamma(\mathrm{Gr}_G, \mathrm{IC}_{crit,\check{\lambda}})^{\mathfrak{g}(O)}$.

An argument similar to the one above goes through. Namely, L_0 acts on $\mathfrak{g}(O)$ with non-positive eigenvalues and with $\mathrm{Ker}(\mathfrak{g}(O) \to \mathfrak{g})$ having strictly negative eigenvalues. Therefore, by the argument above, this kernel acts by 0 on this line. Because L_0 acts with the eigenvalue 0 on $\mathfrak{g} \subset \mathfrak{g}(O)$, $\mathfrak{g}$ preserves the eigenspaces of L_0 and in particular acts on on $\mathcal{L}_{\check{\lambda}}$ because this eigenspace is 1-dimensional. However, $\mathfrak{g}$ has no characters because it is semisimple, so $\mathfrak{g}$ acts on $\mathcal{L}_{\check{\lambda}}$ trivially. Since the embedding of $\mathfrak{g} \hookrightarrow \mathfrak{g}(O)$ induces an isomorphism with the quotient $\mathfrak{g}$ of $\mathfrak{g}(O)$, this implies that $\mathfrak{g}(O)$ acts on $\mathcal{L}_{\check{\lambda}}$ and trivially so.

[6]Actually, all we need is that there is an Aut^+-equivariant (or even just Aut-equivariant) embedding, which is easier to show than the "is" statement.

4.6. With these two observations, let us construct the ${}^LB$ reduction. We will do this using the Plucker relations. Recall that the Plucker relations say that to construct a reduction of a ${}^LG$-torsor $\mathfrak{F}$ to ${}^LB$, it is enough to give the following data. For each dominant coweight $\check{\lambda}$ of G, we are required to specify a line bundle $\mathscr{L}^{\check{\lambda}} \subset \mathfrak{F}(V^{\check{\lambda}})$ with vector bundle quotient, where $V^{\check{\lambda}}$ is as in Section 2.4. This line bundle corresponds to the natural ${}^LB$-submodule of $V^{\check{\lambda}}$ given by the highest weight line. We require that we have isomorphisms $\mathscr{L}^{\check{\lambda}+\check{\lambda}'} \xrightarrow{\simeq} \mathscr{L}^{\check{\lambda}} \otimes \mathscr{L}^{\check{\lambda}'}$ making the following diagram commute:

$$\begin{array}{ccccc} \mathscr{L}^{\check{\lambda}+\check{\lambda}'} & & \longrightarrow & & \mathscr{L}^{\check{\lambda}} \otimes \mathscr{L}^{\check{\lambda}'} \\ \downarrow & & & & \downarrow \\ \mathfrak{F}(V^{\check{\lambda}+\lambda'}) & \longrightarrow & \mathfrak{F}(V^{\check{\lambda}} \otimes V^{\check{\lambda}'}) & \longrightarrow & \mathfrak{F}(V^{\check{\lambda}}) \otimes \mathfrak{F}(V^{\check{\lambda}'}) \end{array}$$

Here we are taking the natural map $V^{\check{\lambda}+\lambda'} \hookrightarrow V^{\check{\lambda}} \otimes V^{\check{\lambda}'}$.

4.7. We have constructed $\mathbb{C}$-lines $\mathcal{L}_{\check{\lambda}}$ inside of $\mathfrak{F}^0_{\mathfrak{z}}(V^{\check{\lambda}})$ by the definition of $\mathfrak{F}^0_{\mathfrak{z}}$ and the compatibility between the geometric Satake and highest weight representations spelled out in Section 2.4. Take the $\mathfrak{z}$-modules generated by them, which are free submodules by Theorem 2.1 and because $\mathfrak{z}$ is a polynomial algebra. Furthermore, comparing the L_0 eigenvalues and using Lemma 4.1, we see that these lines satisfy the Plucker relations automatically. These line bundles are evidently Aut-equivariant and therefore we have an Aut-equivariant reduction to ${}^LB$.

Let us show that the quotients of these vector bundles by these line bundles are projective $\mathfrak{z}$-modules. Because $\mathfrak{z}$ is graded after a choice of uniformizer by the map $\mathbb{G}_m \subset$ Aut and is in non-positive[7] degrees with $\mathbb{C}$ as the degree 0 part, it suffices to show that $\mathrm{Tor}_1(\mathbb{C}, \mathfrak{F}^0_{\mathfrak{z}}(V^{\check{\lambda}})/\mathcal{L}_{\check{\lambda}}) = 0$ where $\mathbb{C}$ is realized as the degree 0 quotient $\mathfrak{z} \twoheadrightarrow \mathbb{C}$. To show this Tor vanishes, we need to show that $\mathcal{L}_{\check{\lambda}} \hookrightarrow \mathfrak{F}_{\mathfrak{z}}(V^{\check{\lambda}})$ remains an injection after tensoring with $\mathbb{C}$. But this is the inclusion of $\mathcal{L}_{\check{\lambda}}$ into $\mathfrak{F}_{\mathfrak{z}}(V^{\check{\lambda}})/\mathfrak{z}_{<0} \cdot \mathfrak{F}_{\mathfrak{z}}(V^{\check{\lambda}})$ which is non-zero because $\mathcal{L}_{\check{\lambda}}$ is the highest graded line.

4.8. Finally, let us check that the ${}^LH$ reduction is what we expect. Indeed, for a coweight $\check{\lambda}$ of G, the line bundle induced from our ${}^LH$-bundle is the trivial line bundle on $\mathrm{Spec}(\mathfrak{z})$ with Aut-action the $\check{\lambda}(\rho)$-th power of the standard character because of Lemma 4.1. Because of Proposition 3.4, this completes the construction of the oper on $\mathrm{Spec}(\mathfrak{z})\widehat{\times}\mathcal{D}$.

5. Proof of Lemma 4.1

5.1. We fix the following notational convention. For a map $f : Z \longrightarrow Z'$ of schemes, we will use f_*^{sh} to denote the sheaf-theoretic push-forward of a sheaf, f_*^D to denote the D-module push-forward, and similarly for the pull-back functors f^*_{sh} and f^*_D.

[7] Note that the $\mathbb{G}_m$ grading is opposite to the L_0 eigenvalue.

5.2. Let $U_{\check{\lambda}}$ be $\mathrm{Gr}_G \setminus \left(\overline{\mathrm{Gr}^{\check{\lambda}}_G} \setminus \mathrm{Gr}^{\check{\lambda}}_G\right)$. That is, $U_{\check{\lambda}}$ the complement in the affine Grassmannian of the boundary of the orbit $\mathrm{Gr}^{\check{\lambda}}_G$. Let $j : U_{\check{\lambda}} \hookrightarrow \mathrm{Gr}^{\check{\lambda}}_G$ be the corresponding open embedding and let $i : \mathrm{Gr}^{\check{\lambda}}_G \hookrightarrow U_{\check{\lambda}}$ be the corresponding closed embedding.

We denote by $\mathscr{L}^{\check{\lambda}}_{crit}$ the restriction of $\mathscr{L}_{crit}$ to $\mathrm{Gr}^{\check{\lambda}}_G$ and $\mathscr{L}^{\check{\lambda},r}_{crit}$ the corresponding right D_{crit}-module $\mathscr{L}^{\check{\lambda}}_{crit} \otimes \omega_{\mathrm{Gr}^{\check{\lambda}}_G}$ on $\mathrm{Gr}^{\check{\lambda}}_G$.

5.3. Note that $j^D_{!*}\mathscr{L}^{\check{\lambda},r}_{crit} \xrightarrow{\simeq} R^0 j^D_* \mathscr{L}^{\check{\lambda},r}_{crit}$. Indeed, because $\mathscr{L}^{\check{\lambda},r}_{crit}$ is a simple D_{crit}-module on $\mathrm{Gr}^{\check{\lambda}}_G$, the natural inclusion realizes $j^D_{!*}\mathscr{L}^{\check{\lambda},r}_{crit}$ as the unique simple submodule of $R^0 j^D_* \mathscr{L}^{\check{\lambda},r}_{crit}$ and the semisimplicity of $\mathcal{H}_{crit}$ then implies the result.

Because j is an open embedding, the composition of the functor $R^0 j^D_*$ with the forgetful functor from D_{crit}-modules to sheaves is equal to $R^0 j^{sh}_*$. Therefore, $\Gamma(\mathrm{Gr}_G, \mathrm{IC}_{crit,\check{\lambda}}) = \Gamma(U_{\check{\lambda}}, i^D_* \mathscr{L}^{\check{\lambda},r}_{crit})$.

5.4. We claim that $\mathscr{L}^{\check{\lambda},r}_{crit} = \mathscr{L}^{\check{\lambda}}_{crit} \otimes \omega_{\mathrm{Gr}^{\check{\lambda}}_G}$ is a trivial $G(O)$-equivariant line bundle on $\mathrm{Gr}^{\check{\lambda}}_G$.

We will demonstrate this using the following general setup: let Z be a scheme with an action of an algebraic group K and equipped with a K-equivariant line bundle $\mathscr{L}$. To see that this line bundle is trivial as an equivariant, one only needs to check that at a fixed point $z \in Z(\mathbb{C})$ the action of $\mathrm{Stab}_z(K)$ on $\mathscr{L}_z$ is through the trivial character.

Fix a Cartan H of G so that we can make sense of the point $\check{\lambda}(t)$. We will compute the character of $\mathrm{Stab}_{\check{\lambda}(t)}(G(O))$ on the fibers of $\omega_{\mathrm{Gr}^{\check{\lambda}}_G}$ and on $\mathscr{L}_{crit}$, or equivalently, $\mathscr{L}^{\otimes 2}_{crit}$ the determinant line bundle. We largely follow [BD1] Sections 8 and 9.

5.5. The following observation will be of repeated use for us. Let α be a root of G for the Cartan subgroup H and fix a non-zero root vector $y_\alpha \in \mathfrak{g}$ for each root α. Then $\mathrm{Ad}_{\check{\lambda}(t)}(y_\alpha) = y_\alpha t^{\check{\lambda}(\alpha)} \in \mathfrak{g}((t))$. Similarly, $\mathrm{Ad}_{\check{\lambda}(t)}(y_\alpha t^i) = y_\alpha t^{\check{\lambda}(\alpha)} t^i$

5.6. Let $G \hookrightarrow G(O)$ be the natural embedding via constant jets and let $G(O)$ be the natural splitting given by evaluation of a jet at 0. We wish to compare the stabilizers of $\check{\lambda}(t)$ in $G(O)$ and in G (where G acts via $G \hookrightarrow G(O)$). Let us denote the former group by $S_{G(O)}$ and the later by S_G so that there is an induced embedding $S_G \hookrightarrow S_{G(O)}$ with splitting $S_{G(O)} \longrightarrow S_G$. Explicitly, these stabilizers are the groups of points g in $G(O)$ or G such that $\check{\lambda}(t)^{-1} g \check{\lambda}(t) \in G(O)$.

We will show that H maps onto the maximal toric quotient of $S_{G(O)}$, which will in turn allow us to compute the character.

5.7. Observe that S_G contains the group B^-. Indeed, clearly H is contained in G because $H(K)$ is commutative. To see that $N^- \subset S_G$, note that N^- is generated by $g_\alpha = \exp(y_\alpha)$ where α is a negative root. Then by Section 5.5, we have $\check\lambda(t)^{-1}g_\alpha\check\lambda(t) = \exp(y_\alpha t^{-\check\lambda(\alpha)})$ and because α is negative $-\check\lambda(\alpha) \geq 0$ so that $y_\alpha t^{-\check\lambda(\alpha)} \in \mathfrak{n}^-(O) \subset \mathfrak{g}(O)$ as desired.

A fortiori, S_G is parabolic and contains H. Therefore, H maps onto the maximal toric quotient of S_G.

5.8. We claim that the kernel of the map $S_{G(O)} \longrightarrow S_G$ is pro-unipotent. This would then complete the proof that H maps onto the maximal toric quotient of $S_{G(O)}$.

More generally, the kernel of $G(O) \longrightarrow G$ is pro-unipotent. Indeed, it suffices to check this claim when $G = GL_n$ where this kernel is the space of matrices of the form $1 + tM_n(\mathbb{C}[[t]])$.

5.9. Now let us compute the relevant characters of H. Observe that the tangent space of $\mathrm{Gr}_G^{\check\lambda}$ at $\check\lambda(t)$ is:

$$\frac{\mathfrak{g}(O)}{\mathfrak{g}(O) \cap \mathrm{Ad}_{\check\lambda(t)}\,\mathfrak{g}(O)}$$

Decompose $\mathfrak{g}(O)$ as $\mathfrak{n}^-(O) \oplus \mathfrak{h}(O) \oplus \mathfrak{n}(O)$. By Section 5.5, $\mathrm{Ad}_{\check\lambda(t)}$ is the identity on $\mathfrak{h}(O)$, expands $\mathfrak{n}^-(O)$ inside of $\mathfrak{n}^-(K)$, and contracts $\mathfrak{n}^(O)$ into itself. Therefore, we have the natural isomorphism:

$$\frac{\mathfrak{n}(O)}{\mathrm{Ad}_{\check\lambda(t)}\,\mathfrak{n}(O)} \xrightarrow{\simeq} \frac{\mathfrak{g}(O)}{\mathfrak{g}(O) \cap \mathrm{Ad}_{\check\lambda(t)}\,\mathfrak{g}(O)}$$

By Section 5.5, the character α of H appears with dimension $\check\lambda(\alpha)$. Indeed, this space has basis $y_\alpha, y_\alpha t, \ldots, y_\alpha t^{\check\lambda(\alpha)-1}$.

Since the fiber of $\omega_{\mathrm{Gr}_G^{\check\lambda}}$ at $\check\lambda(t)$ is the determinant of the dual of this space, the character of H acting on there is $-\sum_{\alpha>0}\check\lambda(\alpha)\cdot\alpha$.

5.10. Now let us compute the character of H acting on the fiber of the determinant line bundle at $\check\lambda(t)$. To compute this, note that the fiber is the determinant of the vector space:

$$\frac{\mathfrak{g}(O)}{\mathfrak{g}(O) \cap \mathrm{Ad}_{\check\lambda(t)}\,\mathfrak{g}(O)} \otimes \left(\frac{\mathrm{Ad}_{\check\lambda(t)}\,\mathfrak{g}(O)}{\mathfrak{g}(O) \cap \mathrm{Ad}_{\check\lambda(t)}\,\mathfrak{g}(O)}\right)^* \xrightarrow{\simeq}$$
$$\frac{\mathfrak{g}(O)}{\mathfrak{g}(O) \cap \mathrm{Ad}_{\check\lambda(t)}\,\mathfrak{g}(O)} \otimes \left(\frac{\mathfrak{g}(O)}{\mathrm{Ad}_{\check\lambda(t)^{-1}}\,\mathfrak{g}(O) \cap \mathfrak{g}(O)}\right)^*$$

Here the isomorphism comes from conjugating by $\check\lambda(t)$ and is thus H-equivariant.

As in Section 5.9, the character of H acting on the determinant of this vector space is:

$$\sum_{\alpha>0}\check\lambda(\alpha)\cdot\alpha - \sum_{\alpha>0} -\check\lambda(\alpha) = 2\sum_{\alpha>0}\check\lambda(\alpha)\cdot\alpha$$

Since this is twice the character of H acting on the fiber of $\mathscr{L}_{crit}$, comparing with the computation from Section 5.9 we see that we have completed the plan outlined in Section 5.4 and therefore see that $\mathscr{L}_{crit}^{\check{\lambda}} \otimes \omega_{\mathrm{Gr}_G^{\check{\lambda}}}$ is a trivial $G(O)$-equivariant line bundle.

5.11. Because Aut acts on $\mathscr{L}_{crit}^{\check{\lambda}} \otimes \omega_{\mathrm{Gr}_G^{\check{\lambda}}}$ in a way compatible with its action on $G(O)$, Lie(Aut) acts on the line $\Gamma(\mathrm{Gr}_G^{\check{\lambda}}, \mathscr{L}_{crit}^{\check{\lambda}} \otimes \omega_{\mathrm{Gr}_G^{\check{\lambda}}})^{G(O)}$. Let us compute the eigenvalue of L_0 on this line.

Because $L_0 y_\alpha t^i = -i y_\alpha t^i$, the decomposition from Section 5.9 implies that L_0 acts on the fiber of $\omega_{\mathrm{Gr}_G^{\check{\lambda}}}$ as multiplication by:

$$-\sum_{\alpha>0} \sum_{i=0}^{\check{\lambda}(\alpha)-1} -i = \frac{1}{2} \sum_{\alpha>0} \check{\lambda}(\alpha)(\check{\lambda}(\alpha)-1)$$

By the same computation, L_0 acts on the fiber of the determinant line bundle by the sum of $\frac{-1}{2}\sum_{\alpha>0} \check{\lambda}(\alpha)(\check{\lambda}(\alpha)-1)$ and minus (because of duality) the determinant of the action on $\left(\frac{\mathrm{Ad}_{\check{\lambda}(t)} \mathfrak{g}(O)}{\mathfrak{g}(O) \cap \mathrm{Ad}_{\check{\lambda}(t)} \mathfrak{g}(O)}\right)$. As in the computation in Section 5.9, the decomposition of this space induced by $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ induces an isomorphism of it with:

$$\frac{\mathrm{Ad}_{\check{\lambda}(t)} \mathfrak{n}^-(O)}{\mathfrak{g}(O) \cap \mathrm{Ad}_{\check{\lambda}(t)} \mathfrak{n}^-(O)}$$

This has a basis given by $y_\alpha t^{-i}$ where α ranges over negative roots and where i ranges from 1 to $-\check{\lambda}(\alpha)$. Therefore, L_0 acts on this space as multiplication by (note it doesn't matter whether α runs over negative or positive roots in the sum):

$$\sum_{\alpha>0} \sum_{i=1}^{\check{\lambda}(\alpha)} i = \frac{1}{2} \sum_{\alpha>0} \check{\lambda}(\alpha)(\check{\lambda}(\alpha)+1)$$

Therefore, L_0 acts on the determinant line bundle as $\frac{-1}{2}\sum_{\alpha>0} \check{\lambda}(\alpha)(\check{\lambda}(\alpha)-1) - \frac{1}{2}\check{\lambda}(\alpha)(\check{\lambda}(\alpha)+1) = -\sum_{\alpha>0} \check{\lambda}(\alpha)^2$ and therefore on the fiber of $\mathscr{L}_{crit}$ as $-\frac{1}{2}\sum_{\alpha>0} \check{\lambda}(\alpha)^2$. Finally, adding this to the computation from the canonical sheaf on our orbit, we see that L_0 acts on our line bundle as multiplication by:

$$\frac{1}{2} \sum_{\alpha>0} \check{\lambda}(\alpha)(\check{\lambda}(\alpha)-1) - \frac{1}{2} \sum_{\alpha>0} \check{\lambda}(\alpha)^2 = \frac{-1}{2} \sum_{\alpha>0} \check{\lambda}(\alpha) = -\check{\lambda}(\rho)$$

5.12. Let us put everything together towards the proof of Lemma 4.1. We have seen that $\Gamma(\mathrm{Gr}_G, \mathrm{IC}_{crit,\check{\lambda}}) = \Gamma(U_{\check{\lambda}}, i_* \mathscr{L}_{crit}^{\check{\lambda}})$. On the other hand, clearly $\Gamma(\mathrm{Gr}_G^{\check{\lambda}}, \mathscr{L}crit \otimes \omega_{\mathrm{Gr}_G^{\check{\lambda}}})$ embeds into this space. We have seen that the corresponding line bundle is trivial as a $G(O)$-equivariant line bundle, so there is a unique distinguished $G(O)$-equivariant line in these global sections. Furthermore, this line is fixed by L_0 and to compute the eigenvalue of L_0 on this line we only need to compute how L_0 scales the fiber of our line bundle at a point fixed by L_0. We have seen that L_0 has eigenvalue $-\check{\lambda}(\rho)$ on this fiber.

Therefore, to complete the proof of Lemma 4.1, we need to see that this is the lowest eigenvalue of L_0 acting on $\Gamma(U_{\check{\lambda}}, i_* \mathscr{L}^{\check{\lambda}}_{crit})$ and that its eigenspace has dimension 1. It's clear that every other section of this module differs from our given section as multiplication by a function or by a vector field normal to $\mathrm{Gr}^{\check{\lambda}}_G$. Therefore, we need to compute the action of Der on the normal bundle to $\mathrm{Gr}^{\check{\lambda}}_G$ $U_{\check{\lambda}}$ and on functions on $\mathrm{Gr}^{\check{\lambda}}_G$ and show that L_0 has non-negative eigenvalues with the only zero eigenvalue being constant functions.

5.13. We follow [BD1] Section 9. Let us show that L_0 acts on $\mathscr{O}_{\mathrm{Gr}^{\check{\lambda}}_G}$ with non-negative eigenvalues. Indeed, the action of Aut extends on here extends to an action of the algebraic semigroup End of all endomorphisms of the disc by construction of the action of Aut. Therefore, as in Section 3.8, the eigenvalues must be non-negative.

One can prove this alternatively as follows while deducing the 1-dimensionality of the 0-eigenspace. We have seen that $\mathrm{Gr}^{\check{\lambda}}_G$ maps to a (partial) flag variety and that $\mathrm{Ker}(G(O) \longrightarrow G)$ acts transitively on the fibers. Since L_0 acts on functions on this kernel with non-negative eigenvalues and its only 0-eigenvalue is constant functions, and because the only global functions on the base are constants, we see that L_0 must act with non-negative eigenvalues and unique 0-eigenspace the constant functions.

Finally, note that we can explicitly compute the sheaf of normal vectors. Indeed, it admits a Aut-equivariant surjection from:

$$\mathscr{O}_{\mathrm{Gr}^{\check{\lambda}}_G} \otimes \frac{\mathfrak{g}(K)}{\mathfrak{g}(O)}$$

given by the action of $G(K)$ on Gr_G and sections of this sheaf obviously have strictly positive eigenvalues by the above and by explicit computation on $\mathfrak{g}(K)/\mathfrak{g}(O)$.

This completes the proof of Lemma 4.1.

References

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