1. Show that the following subset of $\mathbb{R}^2$ is connected but not path-connected:

$\{(x, \sin(\frac{1}{x})), \ x > 0\} \cup \{(0, y), \ y \in [-1, 1]\}$.

2. Let $X$ be a topological space and let

$X = \bigcup_{a \in A} X_a$

be its decomposition into connected (resp., path-connected) components.

(a) Show that if $f : Y \to X$ is a continuous map and $Y$ is connected (resp., path-connected), then $f$ factors through a map $f_a : Y \to X_a$ for some $a \in A$.

(b) Show that if $X$ is written as a topological disjoint $X_1 \sqcup X_2$, then there exists a decomposition $A = A_1 \sqcup A_2$ so that $X_i = \bigcup_{a \in A_i} X_a$.

(c) Show that if $X$ is written as a topological disjoint union

$X = \bigcup_{b \in B} Y_b,$

where each $Y_b$ is connected (resp., path-connected), then there is a natural bijection $\phi : A \to B$ such that under this bijection $X_a = Y_{\phi(a)}$.

3. Let $X$ be a topological space. Show that the (natural) map

$\pi : ([0, 1] \times X) \sqcup ([1, 2] \times X) \to [0, 2] \times X$

has the property that a subset $U \subset [0, 2] \times X$ is open if and only if

$\pi^{-1}(U) \subset ([0, 1] \times X) \sqcup ([1, 2] \times X)$

is open. Use this to show that homotopy is an equivalence relation.

4. Let $f : X_1 \to X_2$ be a continuous map between topological spaces.

(a) Show that the following conditions are equivalent: (i) $f$ is a homeomorphism; (ii) composition $f$ induces a bijection $C(Y, X_1) \to C(Y, X_2)$ for any $Y$; (iii) precomposition with $f$ induces a bijection $C(X_2, Y) \to C(X_1, Y)$ for any $Y$.

(a') Show that the same is true if you replace the word “topological space” by “group”, the word “homeomorphism” by “isomorphism” and $C(X, Y)$ by the set of homomorphisms.

(b) Show that the following conditions are equivalent: (i) $f$ is a homotopy equivalence; (ii) composition $f$ induces a bijection $[Y, X_1] \to [Y, X_2]$ for any $Y$; (iii) precomposition with $f$ induces a bijection $[X_2, Y] \to [X_1, Y]$ for any $Y$.

(c) Show that if $f : X \to Y$ is a homotopy equivalence, then the induced map $\pi_0(X) \to \pi_0(Y)$ is a bijection.

5. Let $X$ be the subset of $\mathbb{R}^2 = \mathbb{C}$ equal to

$\{z, \ |z - 1| = 1\} \cup \{z, \ |z + 1| = 1\},$

i.e., a figure 8 comprised of two circles of radius 1 touching each other at one point. Construct a homotopy inverse to the embedding of $X$ into $\mathbb{C} \setminus \{1, -1\}$.

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6. Let $X$ be a topological space with a marked point $x \in X$. For two elements $\gamma_1$ and $\gamma_2$ of $\Omega(X, x)$ we define their concatenation by the formula

$$\gamma_1 \circ \gamma_2(s) = \begin{cases} 
\gamma_2(2s), & s \leq \frac{1}{2}, \\
\gamma_1(2s - 1), & s \geq \frac{1}{2}.
\end{cases}$$

(a) Show that $\gamma_1 \circ \gamma_2$ is a well-defined element of $\Omega(X, x)$.

(b) Show that if $\gamma'_1 \sim \gamma''_1$ and $\gamma'_2 \sim \gamma''_2$, then $\gamma'_1 \circ \gamma'_2 \simeq \gamma''_1 \circ \gamma''_2$. Here $\gamma' \sim \gamma''$ means that $\gamma'$ and $\gamma''$ belong to the same path component of $\Omega(X, x)$.

7. Let $X$ be a topological space with a marked point $x \in X$, and $\gamma$ be an element in $\Omega(X, x)$. Show that

$$\Gamma(t, s) = \begin{cases} 
\gamma(\frac{s}{1-t}), & s \leq 1 - \frac{t}{2}, \\
x, & s \geq 1 - \frac{t}{2}
\end{cases}$$

defines a continuous path in $\Omega(X, x)$ from $\gamma$ to the concatenation $\gamma_{\text{unit}} \circ \gamma$, where $\gamma_{\text{unit}}$ is the constant path with value $x$.

8 (optional), bonus 2pts. A topological space is said to be contractible if it is homotopy equivalent to the point. A topological space $X$ is said to be locally contractible if every point $x \in X$ has a contractible open neighborhood. Show that if $X$ is locally contractible, then $C([0, 1], X)$ is locally path-connected.