Problems marked with (*) are optional. 

1. Let $X$ be a topological space that is 1st countable. Show that for another topological space $Y$, a function $f : X \to Y$ is continuous if and only if for every convergent sequence $\{x_n\} \to x$, the sequence $\{f(x_n)\}$ converges to $y$.

2. Let $X$ be a topological space. A subset $Y \subset X$ is said to be dense if $\overline{Y} = X$. We shall say that $X$ is separable if it contains a countable dense subset.
   (a) Show that if $X$ is 2nd countable, then it is separable.
   (b) Show that if $X$ is separable and the topology on $X$ is metrizable (i.e., there exists a metric that induces the given topology), then it is 2nd countable.

3. Let $X$ be a topological space. For a subset $Y \subset X$ we shall say that $x \in X$ is a limit point of $Y$ if for every open $x \in U$ the intersection $(U - \{x\}) \cap Y$ is non-empty.
   (a) Show that the closure of $Y$ is the union of $Y$ and the set of its limit points.
   (b) Suppose that $X$ is 1st countable. Then $x$ is a limit point of $Y$ if and only if there exists a sequence $\{y_n\}$ with $x \neq y_n \in Y$ converging to $x$.
   (c) Show that $Y$ has no limit points if and only if it is closed and the induced topology on $Y$ is discrete.
   (d) Let $X$ be compact. Show that any infinite subset of $X$ has a limit point.

4. Let $X$ be a topological space and let $\{x_n\}$ be a sequence of distinct points in $X$; let $Y$ be the subset of $X$ comprised of elements of this sequence.
   (a) Show that the limit of any subsequence of $\{x_n\}$ is a limit point of $Y$.
   (b) Assume that $X$ is 1st countable and T1. Show that any limit point of $Y$ can be obtained as the limit of a subsequence of $\{x_n\}$.
   (c) Deduce that if $X$ is compact and also 1st countable and T1, then any sequence contains a convergent subsequence.

5. Consider the set $X = \mathbb{N} \cup \infty$ (natural numbers and one more point, called $\infty$). Define a topology on it by taking declaring the open subsets to be of the form $S$ for any subset $S \subset \mathbb{N}$ and $(\mathbb{N} - T) \cup \infty$, where $T \subset \mathbb{N}$ is a finite subset.
   (a) Show that this is indeed a topology.
   (b) Show that $X$ is Hausdorff.
   (c) Show that $X$ is 2nd countable.
   (d) Show that $X$ is compact.
   (e) Show that for a topological space $Y$, a map $X \to Y$ given by $n \mapsto y_n$, $\infty \mapsto y$ is continuous if and only if the sequence $\{y_n\}$ converges to $y$.

6. Show that a subset of a totally bounded metric space is totally bounded.

7. Let $X$ be a metric space.

---

Date: September 17, 2017 (the document may be modified).

1Bonus points count directly towards the total score for the class.
(a) Let \( i : X \to \overline{X} \) be a metric-preserving map. Let \( f : X \to Y \) be a map such that there exists a constant \( C \) with
\[
\rho_Y(f(x_1), f(x_2)) \leq C \cdot \rho_X(x_1, x_2).
\]
Assume that the image of \( i \) is dense in \( \overline{X} \) and that \( Y \) is complete. Show that there exists a uniquely defined continuous map \( \tilde{f} : X \to Y \) such that \( \tilde{f} = \tilde{f} \circ i \).

(b) We say that \( i \) realizes \( \overline{X} \) as a completion of \( X \) if \( X \) is itself complete. Show that if \( i_1 : X \to \overline{X}_1 \) and \( i_2 : X \to \overline{X}_2 \) are two completions, then there exists an isometry \( f : \overline{X}_1 \simeq \overline{X}_2 \), compatible with the embeddings \( i_1 \) and \( i_2 \).

(c) Show that if \( X \) is totally bounded, and \( \overline{X} \) is a completion of \( X \), then \( \overline{X} \) is compact.

(d*) Show that any metric space admits a completion.

8. Define \( \ell_\infty \subset \mathbb{R}^N := \prod_{n} \mathbb{R} \) as the subset of those vectors \( a := (a_i, i \in \mathbb{N}) \) for which there exists a constant \( C \) such that \( |a_i| \leq C \).

(a) Show that \( \rho(a, b) := \sup_n |a_i - b_i| \) defines a metric on \( \ell_\infty \).

(b) Show that \( \ell_\infty \) is complete.

(c) Show directly that the subset \( X \subset \ell_\infty \) consisting of \( a \) with \( |a_i| \leq \frac{1}{2^n} \) is compact.

9. Define \( \ell_1 \subset \mathbb{R}^N \) as the subset of those vectors \( a \) for which there exists a constant \( C \) such that for all \( n \),
\[
\sum_{i=1,\ldots,n} |a_i| \leq C.
\]

(a) Show that
\[
\rho(a, b) := \sum_{i=1,\ldots,\infty} |a_i - b_i| := \sup_n \sum_{i=1,\ldots,n} |a_i - b_i|
\]
defines a metric on \( \ell_1 \).

(b) Show that the (obvious) embedding \( \mathbb{R}^N_f \hookrightarrow \ell_1 \) realizes \( \ell_1 \) as the completion of \( \mathbb{R}^N_f \) with respect to the restricted metric.

(c) Show that the subset \( X \subset \ell_1 \) consisting of \( a \) with \( |a_i| \leq \frac{1}{2^n} \) is compact.

10. We retain the notations from the previous two problems.

(a) Show that the identity map on \( \mathbb{R}^N_f \) extends uniquely to a continuous map \( \ell_1 \to \ell_\infty \).

(b) Show that the topology on \( \mathbb{R}^N_f \), induced by the embedding \( \mathbb{R}^N_f \hookrightarrow \ell_1 \) is strictly stronger (contains more open subsets) than that induced by \( \mathbb{R}^N_f \hookrightarrow \ell_\infty \).

(c) Show that the two topologies on \( X \) induced by \( X \hookrightarrow \ell_\infty \) and \( X \hookrightarrow \ell_1 \), respectively, coincide.