MATH 131, PROBLEM SET 2. DUE: TUE., SEPT. 12

Problems marked with (*) are optional\(^1\).

1. Let \(X\) be a topological space, and \(Y \subset X\) be a subset. Define
   \[ Y := \{ x \in X \mid \text{for every open } x \in U \subset X, \ U \cap Y \neq \emptyset \}. \]
   (a) Show that \(Y\) is closed.
   (b) Show that \(Y\) equals the intersection of all closed subsets of \(X\) containing \(Y\).
   (c) Show that \(Y_1 \cup Y_2 = Y_1 \cup Y_2\).
   (d) Give an example (for \(X = \mathbb{R}\)) that (c) fails for intersections instead of unions.
   (e) Let \(X\) be a metric space. Show that
   \[ Y = \{ x \in X \mid \forall r > 0 \text{ we have } B(x, r) \cap Y \neq \emptyset \}. \]
   (f) Show that for \(X = \mathbb{R}\) and \(Y = (0, 1)\), we have \(Y = [0, 1]\).

2. Let \(X\) be a topological space and \(Y \subset X\). Define \(\overset{\circ}{Y} \subset Y\) to be the set
   \[ \{ y \in Y \mid \exists U \text{ open in } X \text{ such that } y \in U \subset Y \}. \]
   (a) Show that \(\overset{\circ}{Y}\) is open.
   (b) Show that \(\overset{\circ}{Y}\) equals the union of open subsets of \(X\) contained in \(Y\).
   (c) Show that \(X - \overset{\circ}{Y} = \overline{X} - \overline{Y}\).
   (d) Show that for \(X = \mathbb{R}\) and \(Y = [0, 1]\), we have \(\overset{\circ}{Y} = (0, 1)\).
   (e) Let \(X\) be a metric space. Show that
   \[ \overset{\circ}{Y} = \{ y \in Y \mid \exists r > 0 \text{ such that } B(y, r) \subset Y \}. \]
   (f) Show that in a metric space every open subset is a union of balls.

3. Let \(X\) be a topological space, and let \(Y \subset X\) be a subset. Recall the induced topology on \(Y\)
   (PSet 1, Problem 8). Show that for any topological space \(Z\), a map \(Z \to Y\) is continuous if and only if its composition with the tautological embedding \(Y \to X\) is a continuous map \(Z \to X\).

4. Let \(X\) be a topological space, and let \(\pi : X \to Y\) be a surjection of sets. Define a topology
   on \(Y\) so that for any topological space \(Z\), a map \(Y \to Z\) is continuous if and only if its precomposition with \(\pi\) is a continuous map \(X \to Z\).

5. Let \((X_1, \rho_1)\) and \((X_2, \rho_2)\) be metric spaces; consider the resulting topologies on \(X_1\) and \(X_2\), respectively. Show that the product topology on \(X_1 \times X_2\) is given by any of the following metrics:
   \[
   \rho((x_1, x_2), (y_1, y_2)) = \begin{cases} 
   \rho_1(x_1, y_1) + \rho_2(x_2, y_2), & \text{if } \rho_1(x_1, y_1)^2 + \rho_2(x_2, y_2)^2 < 1, \\
   \max(\rho_1(x_1, y_1), \rho_2(x_2, y_2)) & \text{otherwise.}
   \end{cases}
   \]

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\(^1\)Bonus points count directly towards the total score for the class.
(in particular, check that these are indeed metrics).

6. Let $X$ be a topological space, equipped with a basis $U_\alpha$. Let $Y \subset X$ be a subset. Show that in order to show that $Y$ is compact, it is sufficient to check that any cover of $Y$ consisting of open subsets from the basis contains a finite subcover.

7. Show that if $X_1$ and $X_2$ are compact, then $X_1 \times X_2$ (equipped with the product topology) is also compact.

8. Show that a subset $X \subset \mathbb{R}^n$ (where $\mathbb{R}^n$ is equipped with its usual topology) is compact if and only if it is closed and bounded (i.e., is contained in a ball of some radius for the usual metric on $\mathbb{R}^n$).

9. Show that the topological space $\mathbb{Q} \cap [0,1]$ equipped with the metric topology is not compact. (I.e., in the Heine-Borel theorem, it was important that we work with real numbers and not rational numbers.)

10. Show that if $X$ is a compact topological space and $f : X \to \mathbb{R}$ is a continuous function, then there exists a point $x \in X$ such that $f(x) \geq f(x')$ for all $x' \in X$ (in this case we say that $f$ attains its maximum value at $x$).