UNIVERSAL CONSTRUCTIONS OF CRYSTALS:
THE CATEGORICAL MEANING OF THE SUGAWARA CONSTRUCTION

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Date: November 27, 2011.
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Notation and conventions

0.1. What we assume. These notes assume some familiarity with \(\infty\)-categories and derived algebraic geometry. Some basic facts are reviewed in [GL:DG], [GL:Stacks] and [GL:QCoh], and we refer the reader to loc.cit. for notation.

In the second half of the notes, we will use some familiarity with (DG) indschemes, and the reader is referred to [GL:IndSch]. The basic theory of crystals of quasi-coherent sheaves is written down in some detail in [GL:Crys].

0.2. Conventions. Whenever we say “category”, we actually mean \(\infty\)-category. To a certain extent, one can ignore the \(\infty\) aspect and pretend that we are working with usual categories. The same goes for schemes vs. derived schemes.

We fix a ground field \(k\), and all DG categories in this note are assumed \(k\)-linear. We let \(\text{Vect}_m\) denote the DG category of chain complexes over \(k\). By a scheme we shall mean a (DG) scheme over \(k\). We let \(\text{pt}\) denote the point-scheme \(\text{Spec}(k)\).

All DG categories are assumed cocomplete (contain infinite direct sums), unless specified otherwise. All functors between DG categories are assume continuous (commuting with colimits=cones and direct sums), also unless specified otherwise. For two DG categories \(D_1\) and \(D_2\) we denote the corresponding category of functors by \(\text{Funct}(D_1, D_2)_{\text{cont}}\).

The resulting 2-category denoted DG\text{Cat} has a natural symmetric monoidal structure given by tensor product. Its unit object is \(\text{Vect}\).

When we say “monoidal category” we mean a \(k\)-linear monoidal DG category, i.e., an algebra object in DG\text{Cat}. Similarly, for “symmetric monoidal”. For a monoidal category \(C\) we let \(C\text{-mod}\) denote the category left modules over it in DG\text{Cat}; these are by definition module categories over \(C\).

0.3. Notation. Here is an (incomplete) set of notational conventions:

For an \(\infty\)-category \(C\) and two objects \(c_1, c_2 \in C\) we shall write \(\text{Maps}_C(c_1, c_2)\) for the space of maps between them, considered as an \(\infty\)-groupoid.

If \(O\) is a monoidal category, we write \(\text{AssAlg}(O)\) for the category of associative algebras in \(O\). For \(A \in \text{AssAlg}(O)\), we let \(A\text{-mod}(O)\) denote the category of \(A\)-modules in \(O\).

If \(O\) is symmetric monoidal, and \(\mathcal{P}\) is a \(k\)-linear operad, we let \(\mathcal{P}\text{-Alg}(O)\) denote the category of \(\mathcal{P}\)-algebras in \(O\). For \(A \in \mathcal{P}\text{-Alg}(O)\), we let \(A\text{-mod}(O)\) denote the category of \(A\)-modules. Note, however, that for \(\mathcal{P} = \text{Ass}\), the two meanings of \(A\text{-mod}(O)\) are different: one means left modules, and the other means bi-modules. Hopefully, this will not lead to an ambiguity in the body of the text.

If \(D\) is a DG category equipped with a t-structure, we denote by \(D^\triangledown\) its heart.

1. Sheaves of categories

1.1. Presheaves.
1.1. We shall be working with the general notion of prestack.

\[ \text{PreStk} = \text{Funct}((\text{Sch}^{\text{aff}})^{\text{op}}, \infty\text{-Grpd}) \].

For \( S \in \text{Sch}^{\text{aff}} \) and \( Y \in \text{PreStk} \) we shall often write \( \text{Maps}(S, Y) \) for the value of \( Y \) on \( S \).

Yoneda embedding gives rise to a functor

\[ \text{Yon}_{\text{Sch}^{\text{aff}}} : \text{Sch}^{\text{aff}} \hookrightarrow \text{PreStk} \].

1.1.2. We shall say that a prestack is an fppf stack (or just a stack) if it satisfies fppf descent, see [GL:Stacks], Sect. 2.2.1.

The full subcategory of \( \text{PreStk} \) formed by stacks is denoted by \( \text{Stk} \). The inclusion

\[ \text{Stk} \hookrightarrow \text{PreStk} \]

admits a left adjoint, denoted \( L \), and referred to as “sheafification”:

\[ Y \mapsto L Y. \]

1.1.3. More generally, for a target category \( C \), we can consider the category

\[ \text{PreShv}(\text{Sch}^{\text{aff}}, C) := \text{Funct}((\text{Sch}^{\text{aff}})^{\text{op}}, C) \]

are refer to it as presheaves with values in \( C \).

For \( F \in \text{PreShv}(\text{Sch}^{\text{aff}}, C) \) we let \( F(S) \) denote the corresponding object of \( C \).

1.1.4. In most examples, \( C \) will be taken to be \( \infty\text{-Cat} \). In this case, for a morphism \( (S_1 \xrightarrow{\phi} S_2) \in \text{Sch}^{\text{aff}} \) and an object \( f(S_2) \in F(S_2) \) we shall denote by \( f(S_2)|_{S_1} \) the corresponding object of \( F(S_1) \). We shall call this operation “restriction” along \( \phi \).

1.1.5. We can consider the full subcategory

\[ \text{Shv}(\text{Sch}^{\text{aff}}, C) \subseteq \text{PreShv}(\text{Sch}^{\text{aff}}, C). \]

From now on, let us assume that \( C \) has colimits and colimits. In this case, the above inclusion has an adjoint, denoted also \( L \).

1.2. Extending presheaves to prestacks.

1.2.1. Note that any

\[ \mathcal{F} \in \text{PreShv}(\text{Sch}^{\text{aff}}, C) \]

admits a natural extension to a functor

\[ \text{PreStk}^{\text{op}} \to C. \]

By a slight abuse of notation we shall denote the extended functor also by \( \mathcal{F} \).

This procedure of extension can be characterized in several equivalent ways.

1.2.2. One way to say that is that it is the right Kan extension of \( \mathcal{F} \) along the functor

\[ (\text{Yon}_{\text{Sch}^{\text{aff}}})^{\text{op}} : (\text{Sch}^{\text{aff}})^{\text{op}} \hookrightarrow (\text{PreStk})^{\text{op}}. \]

I.e., it is the right adjoint to the restriction functor

\[ \text{Funct}((\text{PreStk})^{\text{op}}, C) \to \text{Funct}((\text{Sch}^{\text{aff}})^{\text{op}}, C). \]

1.2.3. Another way to characterize \( \mathcal{F} \) as a functor \( \text{PreStk}^{\text{op}} \to C \) is that this is a unique functor that restricts to the original \( \mathcal{F} \) on \( \text{Sch}^{\text{aff}} \), and which takes colimits in \( \text{PreStk} \) to limits in \( C \).
1.2.4. Finally, we can give an explicit expression of \( \mathcal{F}(\mathcal{Y}) \) for \( \mathcal{Y} \in \text{PreStk} \). Namely, 
\[
\mathcal{F}(\mathcal{Y}) \simeq \lim_{S \in (\text{Sch}^{\text{aff}}/\mathcal{Y})^{\text{op}}} \mathcal{F}(S).
\]

1.2.5. Note that one has to be careful with the notion of compatibility. I.e., for a pair of affine schemes \( S_1 \xrightarrow{\phi} S_2 \) and maps 
\[
y_i : S_i \to \mathcal{Y},
\]
the compatibility 
\[
y_1 \circ \phi \sim y_2
\]
is not a property, but an additional datum, since Maps(\( S_i, \mathcal{Y} \)) is a (\( \infty \)-) groupoid, and not a set. And similarly, with higher compositions. For an example of how this manifests itself see Sect. 1.5.4 below.

1.2.6. The following observation will play an important simplifying role:

**Lemma 1.2.7.** Assume that \( \mathcal{F} \in \text{PreShv}(\text{Sch}^{\text{aff}}, \textbf{C}) \) belongs to \( \text{Shv}(\text{Sch}^{\text{aff}}, \textbf{C}) \). Then for \( \mathcal{Y} \in \text{PreStk} \) the map 
\[
\mathcal{F}(L\mathcal{Y}) \to \mathcal{F}(\mathcal{Y}),
\]
corresponding to the canonical map \( \mathcal{Y} \to L\mathcal{Y} \), is an isomorphism.

This lemma implies that if \( \mathcal{F} \) happens to be a sheaf, we don’t have to bother with sheafifying our prestacks in order to evaluate \( \mathcal{F} \) on them.

1.3. **Examples presheaves.** Here are some examples of presheaves that we will consider in this lecture.

1.3.1. Let \( \textbf{C} = \text{Vect} \). Take 
\[
\mathcal{F}(S) = \Gamma(S, \mathcal{O}_S).
\]
This presheaf is a sheaf. Its value on \( \mathcal{Y} \in \text{PreStk} \) is \( \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \). Rather, this is the definition of what \( \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \) is.

1.3.2. Take \( \textbf{C} = \infty \)-\text{Cat} \. Note that for any \( \mathcal{F} \), an object \( f \in \mathcal{F}(\mathcal{Y}) \) is a compatible family of objects \( f(S) \in \mathcal{F}(S) \) for every affine scheme \( S \) mapping to \( \mathcal{Y} \).

Let us take \( \mathcal{F} \) to be one of the following:
(a) \( \mathcal{F} = \text{Qcoh}, \) i.e., \( \mathcal{F}(S) := \text{Qcoh}(S) \).
(b) \( \mathcal{F} = \text{AssAlg}(\text{Qcoh}), \) i.e., \( \mathcal{F}(S) := \text{AssAlg}(\text{Qcoh}(S)) \).
(c) \( \mathcal{F} = \text{LieAlg}(\text{Qcoh}), \) i.e., \( \mathcal{F}(S) := \text{LieAlg}(\text{Qcoh}(S)) \), or algebras with respect to any \( k \)-linear operad.

These presheaves are all sheaves.

1.3.3. For a prestack \( \mathcal{Y} \), we define the category \( \text{Qcoh}(\mathcal{Y}) \) as the value of the corresponding functor 
\[
\text{Qcoh} : (\text{PreStk})^{\text{op}} \to \infty \text{-Cat}
\]
on \( \mathcal{Y} \).

In fact, in this example, we could have considered a finer option for \( \textbf{C} \), namely we could have taken \( \textbf{C} := \infty \text{-DGCat}^{\text{SymMon}} \). This allows to define \( \text{Qcoh}(\mathcal{Y}) \) as a symmetric monoidal category.
1.3.4. We define the category \((\text{AssAlg}(\text{QCoh}))(\mathcal{Y})\) as the value of the corresponding functor \(\text{AssAlg}(\text{QCoh})\) on \(\mathcal{Y} \in \text{PreStk}\), and similarly for Lie algebras.

However, now there is something to prove:

**Lemma 1.3.5.** The category \((\text{AssAlg}(\text{QCoh}))(\mathcal{Y})\) identifies with \(\text{AssAlg}(\text{QCoh}(\mathcal{Y}))\), the latter being the category of associative algebras in \(\text{QCoh}(\mathcal{Y})\), regarded as a monoidal category.

A similar assertion takes place for Lie algebras, or algebras with respect to any \(k\)-linear operad (but for that we need to consider \(\text{QCoh}(\mathcal{Y})\) as a symmetric monoidal category).

1.4. **Sheaves of categories.** Now we will be getting into the realm of not completely documented mathematics. On the other hand, the definitions we are about to give can be reformulated so that we stay within the world of \((\infty, 1)\)-categories.

1.4.1. We take \(\mathcal{C} = \text{2-Cat}\), regarded as a \((\infty, 1)\)-category. We take \(\mathcal{F}\) to be the functor \(S \mapsto \text{QCoh}(S)\)-mod, i.e., to an affine scheme \(S\) we attach the 2-category of module categories over \(\text{QCoh}(S)\), considered as a monoidal category. We denote this functor by \(\text{ShvDGCat}\).

The following result is due to Toën:

**Theorem 1.4.2.** The functor \(\text{ShvDGCat}\) satisfies fppf descent.

1.4.3. We are now ready to give one of the main definitions of this talk:

**Definition 1.4.4.** For \(\mathcal{Y} \in \text{PreStk}\) we define \(\text{ShvDGCat}(\mathcal{Y})\) to be the value on \(\mathcal{Y}\) of the resulting functor

\[
\text{ShvDGCat} : (\text{PreStk})^{op} \to \text{2-Cat},
\]

i.e., to an affine scheme \(S\) we attach the 2-category of module categories over \(\text{QCoh}(S)\), considered as a monoidal category. We denote this functor by \(\text{ShvDGCat}\).

The following result is due to Toën:

**Theorem 1.4.2.** The functor \(\text{ShvDGCat}\) satisfies fppf descent.

1.4.5. A basic example of a sheaf of categories on \(\mathcal{Y}\) is \(\text{QCoh}_{\mathcal{Y}}\): for \(S \in \text{Sch}^{\text{aff}}\) we set \(\text{QCoh}_{\mathcal{Y}}(S) := \text{QCoh}(S)\).

For an arbitrary \(D_{\mathcal{Y}} \in \text{ShvDGCat}(\mathcal{Y})\) we define the category of global sections

\[
\Gamma(\mathcal{Y}, D_{\mathcal{Y}}) := \text{Funct}_{\text{ShvDGCat}(\mathcal{Y})}(\text{QCoh}_{\mathcal{Y}}, D_{\mathcal{Y}})_{\text{cont}}.
\]

By construction, the category \(\Gamma(\mathcal{Y}, D_{\mathcal{Y}})\) is a module over \(\text{QCoh}(\mathcal{Y})\), considered as a monoidal category.

For example, for \(D_{\mathcal{Y}} = \text{QCoh}_{\mathcal{Y}}\), we have

\[
(1.1) \quad \Gamma(\mathcal{Y}, \text{QCoh}_{\mathcal{Y}}) \simeq \text{QCoh}(\mathcal{Y}).
\]
1.4.6. We can now ask the following question. On the one hand, for a prestack \( Y \) we can consider the 2-category \( \text{ShvDGCat}(Y) \). On the other hand, we can consider \( \text{QCoh}(Y) \) as a monoidal category, and consider the 2-category \( \text{QCoh}(Y)\text{-mod} \).

There is a pair of adjoint functors

\[
(1.2) \quad \text{QCoh}(Y)\text{-mod} \rightleftharpoons \text{ShvDGCat}(Y),
\]

where the right adjoint is the functor \( \Gamma(Y, D_Y) \), mentioned above. The left adjoint is “tensoring up”: given \( D(Y) \in \text{QCoh}(Y)\text{-mod} \), to \( S \in \text{Sch}^{\text{aff}}_Y \), we assign the \( \text{QCoh}(S) \)\text{-module category}

\[
\text{QCoh}(S) \otimes_{\text{QCoh}(Y)} D(Y).
\]

The above functors are mutually quasi-inverse equivalences for an affine scheme by definition. More generally, this is true for any scheme that is quasi-separated and quasi-compact. But it is not true for general prestacks.

We have, however, the following result:

**Theorem 1.4.7.** Let \( G \) be an affine algebraic group over a field of characteristic 0. Then for \( Y = \text{pt}/G \) (see Sect. 1.5), the functors (1.2) are equivalences.

This theorem formally implies that the assertion continues to hold for prestacks of the form \( X/G \), where \( G \) is as in the theorem, and \( X \) a quasi-separated and quasi-compact scheme.

1.4.8. Let \( A \) be a sheaf of associative algebras on \( Y \). (Note that by Lemma 1.3.5, the two ways to think about this notion are equivalent.) We attach to it a sheaf of categories on \( Y \), denoted \( A\text{-mod}_Y \) as follows:

To \( S \in \text{Sch}^{\text{aff}}_Y \) we attach the category \( A(S)\text{-mod}(\text{QCoh}(S)) \), where \( A(S) \) is the value on \( S \) of \( A \).

The object \( A\text{-mod}_Y \) comes equipped with a forgetful functor

\[
\text{oblv}^A : A\text{-mod}_Y \to \text{QCoh}_Y.
\]

We have:

**Lemma 1.4.9.** The functor \( A \mapsto A\text{-mod}_Y \) from \( \text{AssAlg}(\text{QCoh}(Y)) \) to the category of pairs \( (D_Y \in \text{ShvDGCat}(Y), F : D_Y \to \text{QCoh}_Y) \) is fully faithful.

As a consequence, we obtain:

**Corollary 1.4.10.** Let \( \phi : Y_1 \to Y_2 \) be a morphism of stacks, and let \( A_1 \) be an object of \( \text{AssAlg}(\text{QCoh}(Y_1)) \). Then a datum of its descent to an object \( A_2 \in \text{AssAlg}(\text{QCoh}(Y_2)) \) is equivalent to that of descent of the sheaf of categories \( A_1\text{-mod}_Y \), along with the forgetful functor \( \text{oblv}^{A_1} \).

**Remark 1.4.11.** It is essential that in Lemma 1.4.9 and Corollary 1.4.10 we use associative algebras, and not algebras for an arbitrary operad. However, these statements have an analog for Lie algebras. For that we need to replace the functor \( \text{ShvDGCat} \) by \( \text{ShvDGCat}^{\text{SymMon}} \), which attaches to \( S \in \text{Sch}^{\text{aff}} \) the 2-category of symmetric monoidal DG categories, tensored over \( \text{QCoh}(S) \). Then a sheaf of Lie algebras \( L \) gives rise to an object \( L\text{-mod}_Y \in \text{ShvDGCat}^{\text{SymMon}}(Y) \), and \( L \) can be recovered from \( L\text{-mod}_Y \) along with its forgetful functor to \( \text{QCoh}_Y \).
1.4.12. Let $A$ be as in Sect. 1.4.8, and let us regard it as an object of $\text{AssAlg}(\text{QCoh}(\mathcal{Y}))$ via Lemma 1.3.5. We have:

**Lemma 1.4.13.** The category $\Gamma(\mathcal{Y}, A\text{-mod}_Y)$ is canonically equivalent to $A\text{-mod}(\text{QCoh}(\mathcal{Y}))$.

This assertion is valid for algebras over an arbitrary $k$-linear operad.

1.5. **The classifying prestack of a group.** Here is a typical example of a prestack that we shall consider. Let $G$ be a group-scheme, or more generally, a group-object in $\text{PreStk}$.

1.5.1. Let $BG^\bullet$ be the corresponding simplicial object of $\text{PreStk}$, $BG^n = G^\times n$. We define $\text{pt}/G \in \text{PreStk}$ as the geometric realization of $BG^\bullet$, i.e.,

$$\text{colim}_{\Delta^{op}} BG^\bullet.$$ 

By definition, the $\infty$-groupoid $\text{Maps}(S, \text{pt}/G)$ has one object, whose group of automorphisms is $\text{Maps}(S, G)$.

**Remark 1.5.2.** Note that this is a naive version of $\text{pt}/G$, i.e., it is typically *not* a sheaf. Its sheafification $\mathcal{F}(\text{pt}/G)$ assigns to $S$ the groupoid of $\text{Maps}(S, G)$-torsors that are locally non-empty in the fppf topology. However, according to Sect. 1.2.7, this will not affect the value of those $\mathcal{F}$’s that are themselves sheaves. The reason we do not want to sheafify $\text{pt}/G$ is that this procedure does not appear very reasonable unless $G$ is a group-scheme, and the group-prestacks that we will consider will not be such.

1.5.3. More generally, for an object $\mathcal{Y} \in \text{PreStk}$ equipped with an action of $G$ we define a new object $\mathcal{Y}/G \in \text{PreStk}$, equipped with a forgetful map to $\text{pt}/G$.

1.5.4. Let $\mathcal{F}$ be a functor $(\text{Sch}^{\text{aff}})^{op} \to \mathbf{C}$.

**Definition 1.5.5.** We define the object of $\mathbf{C}$ of of weak $G$-modules in $\mathcal{F}(\text{pt})$ as

$$\text{G-mod}^{\text{weak}}(\mathcal{F}(\text{pt})) := \mathcal{F}(\text{pt}/G).$$

Of course, this depends not just on $\mathcal{F}(\text{pt})$, but on the entire datum of $\mathcal{F}$.

We have a tautological map in $\mathbf{C}$:

$$\text{G-mod}^{\text{weak}}(\mathcal{F}(\text{pt})) \to \mathcal{F}(\text{pt})$$

corresponding to $\text{pt} \to \text{pt}/G$.

1.5.6. Let us consider the case when $\mathbf{C} = \infty\text{-Cat}$.

We shall refer to $\text{G-mod}^{\text{weak}}(\mathcal{F}(\text{pt}))$ as the category of objects of $\mathcal{F}(\text{pt})$ endowed with a weak $G$-action.

I.e., a datum of action of action of $G$ on object $f \in \mathcal{F}(\text{pt})$ is by definition encoded by an object

$$f_{\text{pt}/G} \in \mathcal{F}(\text{pt}/G),$$

together with an isomorphism $(f_{\text{pt}/G})(\text{pt}) \simeq f$, where $\text{pt}$ is considered tautologically as an object of $\text{Sch}^{\text{aff}}_{/(\text{pt}/G)}$.

Informally, this datum of action can be described as follows. For every $S \in \text{Sch}^{\text{aff}}$ we consider

$$f(S) := f|_S \in \mathcal{F}(S).$$

Note that every $g \in \text{Maps}(S, G)$ defines an automorphism of the map $S \to \text{pt} \to \text{pt}/G$. 


Thus, every $g$ defines an automorphism of the object $f(S)$. So, we can say that a datum of weak action amounts to a compatible system of actions of the group $\text{Maps}(S,G)$ on $f(S)$ for every affine scheme $S$.

1.5.7. Take $\mathcal{F} = \text{QCoh}$. We have

$$\text{QCoh}(pt) \simeq \text{Vect}.$$ 

Thus, we obtain the notion of (weak) action of $G$ on a chain complex of vector spaces. I.e., we have the category

$$G\text{-mod}^{\text{weak}}(\text{Vect}) = \text{QCoh}(pt / G).$$

We shall also use the notation $\text{Rep}(G)$ for it.

When $G$ is a group-scheme, the category $\text{Rep}(G)$ has a natural $t$-structure. The corresponding abelian category $\text{Rep}(G)^{\otimes}$ identifies with the usual category of representations of $G$.

1.5.8. More generally, let us take $\mathcal{C} = \infty\text{-Cat}$, and for a DG category $D$ we consider a functor $\mathcal{F}$ equal to $\text{QCoh} \otimes D$. I.e., its value of $S \in \text{Sch}^{\text{aff}}$ is $\text{QCoh}(S) \otimes D$.

We thus obtain the notion of action of $G$ on an object of $D$: the category of such is

$$G\text{-mod}^{\text{weak}}(D) := (\text{QCoh} \otimes D)(pt / G),$$

i.e., the value of $\text{QCoh}(S) \otimes D$ on $pt / G \in \text{PreStk}$.

1.5.9. Taking $\mathcal{C} = \infty\text{-Cat}$ and $\mathcal{F} = \text{AssAlg}(\text{QCoh})$ we obtain the notion of action of $G$ on an associative algebra. And similarly, for algebras over any $k$-linear operad.

1.6. **Weak action of groups on categories.**

1.6.1. Let us now take $\mathcal{C} = 2\text{-Cat}$ and $\mathcal{F} = \text{ShvDGCat}$. By definition

$$\text{ShvDGCat}(pt) = \text{DGCat}.$$ 

According to our definition,

$$G\text{-mod}^{\text{weak}}(\text{DGCat}) := \text{ShvDGCat}(pt / G).$$

1.6.2. We shall think of objects of $G\text{-mod}^{\text{weak}}(\text{DGCat})$ as the following data: $D \in \text{DGCat}$, plus $D_{pt / G} \in \text{ShvDGCat}(pt / G)$, equipped with an identification

$$D_{pt / G}(pt) \simeq D.$$ 

We shall think of $D$ as above as endowed with a weak action of $G$. I.e., the data of a weak action on $D$ is encoded by an object $D_{pt / G} \in \text{ShvDGCat}(pt / G)$.

1.6.3. For $D_1, D_2$ two DG categories (weakly) acted on by $G$, we let

$$\text{Funct}_{G,\text{weak}}(D_1, D_2)_{\text{cont}}$$

be the category of morphisms between the corresponding objects

$$(D_i)_{pt / G} \in \text{ShvDGCat}(pt / G),$$

and refer to it as the category of weakly equivariant functors.

For a DG category $D$ with a weak action of $G$, we let $D^{G,\text{weak}}$ be the category of $G$-equivariant functors $\text{Vect} \rightarrow D$, where $\text{Vect}$ is considered as a DG category with the trivial weak $G$-action, i.e., the corresponding object in $\text{ShvDGCat}(pt / G)$ is $\text{QCoh}_{pt / G}$.

By definition,

$$D^{G,\text{weak}} = \Gamma(pt / G, D_{pt / G}).$$
We shall refer to \( D^{G,\text{weak}} \) as the category of (weakly) \( G \)-equivariant objects in \( D \).

According to (1.1), we have:

**Lemma 1.6.4.** We have a natural equivalence \( \text{Vect}^{G,\text{weak}} \simeq \text{Rep}(G) \).

1.6.5. Note that for \( D \) equipped with a weak action of \( G \), the category \( D^{G,\text{weak}} \) is equipped with an action of the monoidal category \( \text{Vect}^{G,\text{weak}} \simeq \text{Rep}(G) \).

We can reformulate Theorem 1.4.7 as follows:

**Theorem 1.6.6.** Let \( G \) be an affine algebraic group over a field of characteristic 0. Then the functor \( \mathbf{D} \mapsto \mathbf{D}^{G,\text{weak}} \)

\[
\text{G-mod}^{\text{weak}}(\mathbf{D}\text{Cat}) \to \text{Rep}(G)\text{-mod}
\]

is an equivalence.

**Remark 1.6.7.** The inverse functor to one in the theorem is given by

\[
\mathbf{D} \mapsto \text{Vect} \otimes \text{Rep}(G) \mathbf{D},
\]

where \( \text{Vect} \) is naturally endowed with a structure of commutation between the trivial action of \( \text{Rep}(G) \) and the trivial weak \( G \)-action.

1.6.8. Let \( A \) be an associative algebra. From Corollary 1.4.10, we obtain:

**Corollary 1.6.9.** The datum of action of \( G \) on \( A \) is equivalent to that of an action of \( G \) on the category \( A\text{-mod} \) together with the data of \( G \)-equivariance on the forgetful functor \( A\text{-mod} \to \text{Vect} \).

This corollary has a variant for Lie algebras along the lines of Remark 1.4.11.

From Lemma 1.4.13, we obtain:

**Corollary 1.6.10.** The category \( (A\text{-mod})^G \) identifies canonically with \( A_{\text{pt}}/G\text{-mod}(\text{Rep}(G)) \), where \( A_{\text{pt}}/G \) is regarded as an object of \( \text{AssAlg}(\text{Rep}(G)) \).

This corollary is valid for algebras over any \( k \)-linear operad.

2. **Crystals**

2.1. **The notion of crystal.**

2.1.1. If \( Y \) is a prestack we define the new prestack \( Y_{\text{dR}} \) by

\[
\text{Maps}(S, Y_{\text{dR}}) = \text{Maps}(S_{\text{red}}, Y), \quad S \in \text{Sch}^{\text{aff}}
\]

where \( S_{\text{red}} \) denotes the underlying reduced scheme.

We have a tautological projection \( Y \to Y_{\text{dR}} \).

Note that even we start with \( Y \) being a scheme \( X \), the resulting prestack \( X_{\text{dR}} \) is far from being a scheme.

2.1.2. Here is another key definition for this talk.

**Definition 2.1.3.** For \( F \in \text{PreShv}(\text{Sch}^{\text{aff}}, \mathbf{C}) \) we define the object of \( \mathbf{C} \) of crystals in \( F(Y) \), as \( F(Y_{\text{dR}}) \).

We shall also use the notation \( F_{\text{crys}}(Y) := F(Y_{\text{dR}}) \).

2.1.4. For example, consider the case \( \mathbf{C} = \text{Vect} \) and \( F(S) = \Gamma(S, \mathcal{O}_S) \). Let \( X \) be a scheme of finite type. It follows from Sect. 2.1.6 that \( F_{\text{crys}}(X) \) identifies with de Rham cohomology of \( X \).
2.1.5. Consider the case when $C = \infty\text{-Cat}$. In this case we shall refer to $\mathcal{F}_{\text{crys}}(\mathcal{Y})$ as the category of crystals in $\mathcal{F}(\mathcal{Y})$.

In other words, we shall say that $f_{\mathcal{Y}} \in \mathcal{F}(\mathcal{Y})$ is a crystal if we are given an object $f_{\mathcal{Y}_{\text{dR}}} \in \mathcal{F}(\mathcal{Y}_{\text{dR}})$ and an identification

\[ f_{\mathcal{Y}} \cong (f_{\mathcal{Y}_{\text{dR}}})|_{\mathcal{Y}}. \]

So, we shall often think of crystals in $\mathcal{F}(\mathcal{Y})$ as objects $f \in \mathcal{F}(\mathcal{Y})$ equipped with a data of descent to $\mathcal{Y}_{\text{dR}}$.

2.1.6. In the example of $\mathcal{F} = \text{QCoh}$, the category of crystals in $\text{QCoh}(\mathcal{Y})$, i.e., $\text{QCoh}(\mathcal{Y}_{\text{dR}})$, is by definition the category of (left) $\text{D}$-modules on $\mathcal{Y}$.

In the case when $\mathcal{Y}$ is a smooth scheme $X$ of finite type, one easily shows that the category $\text{QCoh}_{\text{crys}}(X)$ indeed identifies with (the DG category whose homotopy category is) the usual derived category of $\text{D}$-modules on $X$.

2.1.7. Similarly, if we take Examples (b) and (c) from Sect. 1.3.2, we obtain the notions of crystal of associative algebras and Lie algebras on $\mathcal{Y}$.

Using Lemma 1.3.5, we identify the categories

\[ \text{AssAlg}_{\text{crys}}(\mathcal{Y}) \text{ and } \text{LieAlg}_{\text{crys}}(\mathcal{Y}) \]

with

\[ \text{AssAlg}(\text{QCoh}_{\text{crys}}(\mathcal{Y})) \text{ and } \text{LieAlg}(\text{QCoh}_{\text{crys}}(\mathcal{Y})), \]

respectively.

2.2. **Crystals of categories.** The next main definition is:

**Definition 2.2.1.** The 2-category of crystals of categories on $\mathcal{Y}$ is by definition

\[ \text{ShvDGCat}_{\text{crys}}(\mathcal{Y}) := \text{ShvDGCat}(\mathcal{Y}_{\text{dR}}). \]

2.2.2. I.e., informally, the data of a crystal of categories on $\mathcal{Y}$ assigns to every $S \in \text{Sch}^{\text{aff}}$ together with a map

\[ y' : S_{\text{red}} \to \mathcal{Y}, \]

a DG category $\mathbf{D}(S)$ equipped with an action of $\text{QCoh}(S)$. For two points $(S_i, y'_i)$, a morphism $\phi : S_1 \to S_2$ and an isomorphism

\[ y'_1 \circ \phi \sim y'_2, \]

we must be given an identification

\[ \text{QCoh}(S_1) \otimes_{\text{QCoh}(S_2)} \mathbf{D}(S_2) \simeq \mathbf{D}(S_1), \]

together with a homotopy-coherent system of compatibilities for $n$-fold compositions.

2.2.3. A basic example of a crystal of categories is $\text{QCoh}_{\mathcal{Y}}$. I.e., the sheaf of categories that we denoted $\text{QCoh}_{\mathcal{Y}}$ has a natural structure of crystal. The corresponding sheaf of categories on $\mathcal{Y}_{\text{dR}}$ is just $\text{QCoh}_{\mathcal{Y}_{\text{dR}}}$.
2.2.4. Let $D_y$ be a sheaf of categories on $Y$ with a structure of crystal. By definition, we have a functor
\[ \Gamma(Y_{\mathrm{dR}}, D_{y_{\mathrm{an}}}) \to \Gamma(Y, D_y). \]

**Definition 2.2.5.** We define the category crystalline objects in $\Gamma(Y, D_y)$, denoted $\Gamma_{\mathrm{crys}}(Y, D_y)$, as $\Gamma(Y_{\mathrm{dR}}, D_{y_{\mathrm{an}}})$.

Tautologically,
\[ \Gamma_{\mathrm{crys}}(Y, D_y) = \quad = \text{Funct}_{\mathrm{ShvDGCat}_{\mathrm{crys}}}(\mathrm{QCoh}_Y, D_{y_{\mathrm{an}}}), \]

For example, for $D_y = \text{QCoh}_Y$, the category of crystalline objects in $\Gamma(Y, \text{QCoh}_Y) \simeq \text{QCoh}(Y)$ is $\text{QCoh}_{\mathrm{crys}}(Y)$.

2.2.6. Let $Y$ be a prestack, and let $A$ be a sheaf of associative algebras on $Y$. From Corollary 1.4.10 we obtain:

**Corollary 2.2.7.** The following data are equivalent:
(a) A structure on $A$ of crystal of associative algebras;
(b) A structure on $A_{\mathrm{mod}_{Y}}$ of a crystal of categories and a structure on $\text{ obl }^A$ of functor between crystals of categories.

This corollary has a variant for Lie algebras along the lines of Remark 1.4.11.

From Lemma 1.4.13, we obtain:

**Corollary 2.2.8.** Under the equivalence of Corollary 2.2.7, the category of crystalline objects in $\Gamma(Y, A_{\mathrm{mod}_{Y}})$ identifies with $A_{\mathrm{mod}}(\text{QCoh}_{\mathrm{crys}}(Y))$, where $A$ is regarded as an object of $\text{AssAlg}(\text{QCoh}_{\mathrm{crys}}(Y))$.

This corollary is valid for algebras over any $k$-linear operad.

2.3. Strong actions.

2.3.1. The following observation is formal:

**Lemma 2.3.2.** For a group-object $G \in \text{PreStk}$, the prestacks
\[ \text{pt}/(G_{\mathrm{dR}}) \text{ and (pt}/G)_{\mathrm{dR}} \]
are canonically isomorphic.

2.3.3. Let $\mathcal{F}$ be a functor $(\text{Sch}^{\text{aff}})^{\text{op}} \to C$.

**Definition 2.3.4.** We define the object of $C$ of of strong $G$-modules in $\mathcal{F}(\text{pt})$ as
\[ G_{\text{mod}}^{\text{strong}}(\mathcal{F}(\text{pt})) := \mathcal{F}((\text{pt}/G)_{\text{dR}}). \]

Equivalently, $G_{\text{mod}}^{\text{strong}}(\mathcal{F}(\text{pt}))$ is the object of $C$ of crystals in $G_{\text{mod}}^{\text{weak}}(\mathcal{F}(\text{pt}))$.

We have natural maps on $C$:
\[ G_{\text{mod}}^{\text{strong}}(\mathcal{F}(\text{pt})) \to G_{\text{mod}}^{\text{weak}}(\mathcal{F}(\text{pt})) \to \mathcal{F}(\text{pt}), \]
corresponding to
\[ \text{pt} \to \text{pt}/G \to (\text{pt}/G)_{\text{dR}}. \]
2.3.5. For example, it follows from Sect. 2.3.7 that for \( C = \text{Vect} \) and \( \mathcal{F}(S) = \Gamma(S, \mathcal{O}_S) \), the object \( G\text{-mod}^{\text{strong}}(k) \) is the \( G \)-equivariant cohomology of the point.

2.3.6. Consider the case when \( C = \infty\text{-Cat} \). We shall refer to \( G\text{-mod}^{\text{strong}}(\mathcal{F}(pt)) \) as the category of objects of \( \mathcal{F}(pt) \) endowed with a strong \( G \)-action.

I.e., a strong action of \( G \) on \( f \in \mathcal{F}(pt) \) is a datum of an object \( f_{(pt/G)_{\text{dR}}} \in \mathcal{F}((pt/G)_{\text{dR}}) \) together with an isomorphism

\[
f_{(pt/G)_{\text{dR}}}(pt) \simeq f.
\]

Equivalently, we can rephrase it as follows: a data of a strong action of \( G \) on \( f \in \mathcal{F}(pt) \) is that of a weak action, plus a structure of crystal on the corresponding object \( f_{pt/G} \in \mathcal{F}(pt/G) \).

2.3.7. Consider the example \( \mathcal{F} = \text{QCoh} \), i.e., \( \mathcal{F}(pt) = \text{Vect} \). By definition, the corresponding category \( G\text{-mod}^{\text{strong}}(\text{Vect}) \) is \( \text{QCoh}_{\text{crys}}(pt/G) \). When \( G \) is a group-scheme of finite type, the latter is (the DG category underlying) the usual \( G \)-equivariant derived category of the point.

2.4. **Strong actions of groups on categories.** Let us take \( D = 2\text{-Cat} \) and \( \mathcal{F} = \text{ShvDGCat} \). We thus obtain the notion of strong action of \( G \) on a category:

\[ G\text{-mod}^{\text{strong}}(\text{DGCat}) := \text{ShvDGCat}((pt/G)_{\text{dR}}). \]

Thus, we shall think of a DG category with a strong action of \( G \) as \( D \in \text{DGCat} \) and \( D_{(pt/G)_{\text{dR}}} \), equipped with an identification

\[ D_{(pt/G)_{\text{dR}}}(pt) \simeq D. \]

Note that we also have the intermediate object

\[ D_{pt/G} := D_{(pt/G)_{\text{dR}}}|_{pt/G}. \]

2.4.1. For two DG categories \( D_1 \) and \( D_2 \) equipped with strong actions of \( G \), we obtain the corresponding category

\[ \text{Funct}_{G,\text{strong}}(D_1, D_2)_{\text{cont}}. \]

For a DG category \( D \), equipped with a strong action of \( G \), we define

\[ D^{G,\text{strong}} := \text{Funct}_{G,\text{strong}}(\text{Vect}, D)_{\text{cont}}. \]

Here \( \text{Vect} \) is regarded naturally as a category with a strong action of \( G \), where the corresponding object in \( \text{ShvDGCat}((pt/G)_{\text{dR}}) \) is \( \text{QCoh}_{(pt/G)_{\text{dR}}} \).

2.4.2. Equivalently,

\[ D^{G,\text{strong}} = \Gamma_{\text{crys}}(pt/G, D_{pt/G}) := \Gamma((pt/G)_{\text{dR}}, D_{(pt/G)_{\text{dR}}}). \]

Note that we have natural forgetful functors

\[ (2.1) \quad D^{G,\text{weak}} \to D^{G,\text{strong}} \to D. \]

**Lemma 2.4.3.** Suppose that \( G \) is a pro-unipotent group scheme. Then the composed functor \( D^{G,\text{strong}} \to D \) is fully faithful.
2.4.4. Assume that \( D \) is a t-structure. In this case, the categories 
\[ D^{G,\text{weak}} \] and 
\[ D^{G,\text{strong}} \]
have t-structures, and the forgetful functors (2.1) are t-exact. We have:

**Lemma 2.4.5.** Suppose that \( G \) is a connected group-scheme. Then the functor 
\[ (D^{G,\text{strong}})^{\triangleright} \to D^{\triangleright} \]
is fully faithful.

2.5. **Digression: normal subgroups.**

2.5.1. Let \( \phi : G_1 \to G_2 \) be a homomorphsim of group-objects in \( \infty\text{-Grpd} \). A data of “normal subgroup” for \( \phi \) is that of action of \( G_2 \) on \( G_1 \) by automorphisms, which is compatible with the adjoint action of \( G_1 \) on itself via \( \phi \).

If \( \phi : G_1 \to G_2 \) is a normal subgroup data, we can form a quotient \( G_1/G_2 \), which is a group-object in \( \infty\text{-Grpd} \).

2.5.2. If \( G_2 \) acts on an object \( c \) of some \( \infty\text{-category} \( C \), then there is a natural notion of compatible trivialization of the induced \( G_1 \)-action.

Namely, \( c \) gives rise to a functor 
\[ \tilde{c} : \infty\text{-Grpd}/(pt/G_2) \to C, \]
and the action of \( G_2 \) on \( G_2 \) gives rise to a functor \( \tilde{G}_1 \) from \( \infty\text{-Grpd}/(pt/G_2) \) to the category of group-objects in \( \infty\text{-Grpd} \).

We have a natural action of \( \tilde{G}_1 \) on \( \tilde{c} \). By definition, a compatible trivialization of the induced \( G_1 \)-action on \( c \) is a trivialization of the \( \tilde{G}_1 \)-action on \( \tilde{c} \).

**Lemma 2.5.3.** A data of compatible trivialization of the induced \( G_1 \)-action on \( c \) is equivalent to the data of factoring the \( G_2 \)-action on \( c \) through the quotient \( G_2/G_1 \).

2.5.4. The above definitions make sense when we replace the category of group-objects in \( \infty\text{-Grpd} \) by the category of functors from any other source category to it. We shall take the source category to be \( (\text{Sch}^{\text{aff}})^{\text{op}} \), and so the above category of functors identifies with that of group-objects in \( \text{PreStk} \). Thus, we obtain the notion of normal subgroup for a homomorphism \( G_1 \to G_2 \) of group-objects of \( \text{PreStk} \).

2.6. **Strong actions via formal completions.**

2.6.1. Recall (see [GL:IndSch], Defn. 6.1.2) that for a prestack \( Y \) and a prestack \( Z \) mapping to it, we define the formal completion \( y^\wedge_2 \) as 
\[ y^\wedge_2 \simeq y \times_{y_{\text{dr}}} Z_{\text{dR}}. \]

I.e., for \( S \in \text{Sch}^{\text{aff}} \), an \( S \)-point of \( y^\wedge_2 \) is a pair \((y, z')\), where \( y \) is an \( S \)-point of \( Y \) and \( z' \) the lift of the induces \( S_{\text{red}} \)-point \( y' \) of \( Y \), to \( Z \).

2.6.2. For a group-object \( G \) in \( \text{PreStk} \) we thus obtain another group-object \( G^\wedge_{\{1\}} \), where \( \{1\} \) refers to the unit map \( pt \to G \).

It follows from the definitions that \( G^\wedge_{\{1\}} \) is naturally a normal subgroup of \( G \).

**Lemma 2.6.3.** We have a canonical isomorphism \( G_{\text{dR}} \simeq G/G^\wedge_{\{1\}} \).
2.6.4. It follows from Lemma 2.5.3 that the data of strong action of a group on a category $\mathbf{D}$ can be reformulated as that of a weak action, plus a compatible trivialization of the induced $G_{(1)}^\wedge$-action.

2.7. Lie algebras. In this subsection the fact that we work with derived affine schemes, rather than ordinary affine schemes, becomes essential.

2.7.1. Let $\text{FormGrp}_{\text{tilt}}$ be a full subcategory of the category of group-objects in $\text{PreStk}$ consisting of objects $G$ that satisfy the following three conditions:

- $G$ is locally almost of finite type, see [GL:Stacks], Sect. 1.3;
- For $S$ reduced, $\text{Maps}(S, G) = \{1\}$;
- $S$ admits deformation theory, see [GL:IndSch], Sect. 4.8 (drop both connectivity and coconnectivity).

We have the following assertion:

**Proposition 2.7.2.** Over a field of characteristic 0, the category $\text{FormGrp}_{\text{tilt}}$ is canonically equivalent to that of Lie algebras.

**Proof.** (sketch)

Given a Lie algebra $L$ we define the corresponding prestack $G = \text{Exp}(L)$ as follows:

$$\text{Maps}(\text{Spec}(R), \text{Exp}(L)) := L \otimes A^{nil},$$

where $A^{nil} \subset A$ is the ideal of nilpotent elements.

The Baker-Campbell-Hausdorff formula defines a group structure on expressions $\text{Exp}(l)$ for $l \in L \otimes A^{nil}$. The nilpotence condition implies that the corresponding series has finitely many terms.

More conceptually, we can rephrase this as follows. Let $U(L)^+$ be the augmentation ideal in the universal enveloping algebra $U(L)$. Consider the linear dual $(U(L)^+)^*$ as a pro-object in the category of commutative algebras, $U(L)^* = \lim_n (U(L)^+_n)^*$, where $U(L)_n$ is the $n$th term of the PBW filtration, and the algebra structure arises from the cocommutative coalgebra structure on $U(L)$. Then

$$\text{Maps}(\text{Spec}(R), \text{Exp}(L)) = \colim_n \text{Maps}_{\text{ComAlg}}((U(L)^+_n)^*, A^{nilp}).$$

Now, the algebra structure on $U(L)$ gives the above expression a structure of group.

□

The same proof also shows:

**Corollary 2.7.3.** The categories $L$-mod and $\text{Rep}(\text{Exp}(L))$ are naturally equivalent in a way respecting the forgetful functor to $\text{Vect}$.

2.7.4. Let us return to Sect. 2.6.4. We can assign to $G_{(1)}^\wedge$, viewed as a normal subgroup of $G$ a sheaf of Lie algebras

$$\mathfrak{g}_{pt/G} \in \text{LieAlg}(\text{QCoh}(pt/G)) \simeq \text{LieAlg}(\text{Rep}(G)).$$

By Sect. 1.5.9, the object $\mathfrak{g}_{pt/G}$ encodes an action of $G$ on $\mathfrak{g} := \text{Lie}(G)$; by definition, this is the adjoint action.
From Sect. 2.5.2 we obtain that if $D$ is a DG category endowed with a strong action of $G$, the identity functor on $D^{G,\text{weak}}$ lifts canonically to a functor
\begin{equation}
D^{G,\text{weak}} \to \mathfrak{g}/G\text{-mod}(D^{G,\text{weak}}).
\end{equation}
(In the above formula, we use the following notation: of $L$ is a Lie algebra in a symmetric monoidal category $O$ and a $O$-module category $M$, we denote by $L\text{-mod}(M)$ the category of $L$-modules in $M$.)

**Lemma 2.7.5.** The category $D^{G,\text{strong}}$ identifies with the equalizer of
\begin{equation}
D^{G,\text{weak}} \rightrightarrows \mathfrak{g}/G\text{-mod}(D^{G,\text{weak}}),
\end{equation}
where the first arrow is the functor of (2.2), and the second arrow is the functor of zero action.

Thus, for an object $d \in D^{G,\text{weak}}$ the action of $\mathfrak{g}/G$ on it, given by the functor (2.2) measures the obstruction for $d$ to be strongly equivariant.

2.7.6. Consider the category $\text{Vect}^{G,\text{strong}}$. (Its homotopy category is by definition the $G$-equivariant derived category of the point.) The category $\text{Vect}^{G,\text{strong}}$ has a natural monoidal structure by virtue of being $QCoh((pt/G)_{\text{dR}})$.

For every $D$ with a strong action of $G$, the category $D^{G,\text{strong}}$ is a module over $\text{Vect}^{G,\text{strong}}$.

However, we do not have an analog of Theorem 1.4.7 in this case. In particular, the assignment
\begin{equation}
D \mapsto D^{G,\text{strong}}
\end{equation}
may send a non-zero category to zero.

2.7.7. Harish-Chandra datum. Let $G$ be a group-object of $\text{PreStk}$, which is locally almost of finite type, and which admits deformation theory. E.g., we can take $G$ to be a group DG scheme and group DG indscheme locally almost of finite type.

Let $A$ be an associative algebra equipped with a (weak) action of $G$. As was explained above, the category $A\text{-mod}$ is then acted on (weakly) by $G$. We wish to ask what does it take to upgrade this action to a strong one. The answer is provided by the following proposition:

**Proposition 2.7.8.** A datum of lifting the weak $G$-action on $A\text{-mod}$ to a strong one is equivalent to the datum of a homomorphism of associative algebras $\phi : U(g) \to A$ endowed with the following compatibilities:
(a) $\phi$ is equivariant with respect to the $G$-action on $A$ and the adjoint action on $U(g)$;
(b) The derivative of the $G$-action on $A$ equals the $g$-action obtained from $\phi$ by taking commutators.
(c) A homotopy between (b) and the image under $\phi$ of the corresponding structure on $U(g)$.

We shall refer to the datum of a map $\phi : U(L) \to A$ as a Harish-Chandra datum for the $G$-action on $A$. I.e., a Harish-Chandra datum is equivalent to making a weak action strong.

2.7.9. Suppose that the algebra $A$ is connective (concentrated in non-positive cohomological degrees). Then the category $A\text{-mod}$ has a t-structure, characterized by the property that the forgetful functor to $\text{Vect}$ is t-exact. Assume that $G$ is a group DG scheme almost of finite type (e.g., an algebraic group of finite type over $k$).

We can use Proposition 2.7.8 to characterize the essential image of the forgetful functor
\begin{equation}
(A\text{-mod}^{G,\text{strong}})^{\nabla} \to A\text{-mod}^{\nabla},
\end{equation}
which, according to Lemma 2.4.5, is fully faithful.
Let $V$ be an object of $A\text{-mod}$. Then the data of $\phi$ from Proposition 2.7.8 defines on the vector space underlying $V$ a structure of $g$-module.

**Lemma 2.7.10.** The object $V$ lies in the essential image of $\left( A\text{-mod}^{G,\text{strong}} \right) \triangleright$ if and only if the above action of $g$ on $V$ comes from an action of $G$.

2.7.11. Let us spell out the content of Lemma 2.7.5 in concrete terms for $D = A\text{-mod}$, where a strong action is described via Proposition 2.7.8. Let $M$ be an object of $A\text{-mod}^{G,\text{weak}}$. Consider the forgetful functor

$$F : A\text{-mod} \to \text{Vect}.$$  

This functor is naturally $G$-equivariant. In particular, it gives rise to a functor

$$A\text{-mod}^{G,\text{weak}} \to \text{Vect}^{G,\text{weak}} = \text{Rep}(G) \to g\text{-mod}.$$  

On the other hand, the datum of $\phi$ defines another structure of $g$-module on $F(M)$. It is easy to see that the discrepancy between these two actions is a datum of action of the Lie algebra object $g \in \text{Rep}(G)$ on $M \in A\text{-mod}^{G,\text{weak}}$ with respect to the tautological monoidal action of $\text{Rep}(G)$ on $A\text{-mod}^{G,\text{weak}}$ by tensor products.

Lemma 2.7.5 says that the data of lifting of $M$ to a strongly equivariant object is equivalent to that of trivialization of the latter action.

In particular, we have:

**Lemma 2.7.12.** Assume that an object $M \in A\text{-mod}^{G,\text{weak}}$ comes from an object of $M \in A\text{-mod}^{G,\text{weak}}$. Then the resulting two actions of $g$ on $F(M)$ are canonically homotopic.

2.7.13. Unfortunately, the actual situation that we will be dealing with does not fall into the paradigm of correspondence between Lie algebras and group-objects of PreStk, described by Proposition 2.7.2.

Namely, the group-objects that we shall encounter, such as the loop group $G((t))$, are not locally of finite type. Hence, the objects that serve as their “Lie algebras” are not quite Lie algebras, but topological Lie algebras. Most importantly, what changes is the notion of module, as the latter are required to be discrete and the action continuous.

For the subclass of group-objects of PreStk that will be of interest for us, the relevant Lie-type objects are Tate Lie algebras. However, for the purposes of this lecture, we shall pretend that they are just ordinary Lie algebras.

We should remark that the analog of Proposition 2.7.2 for Tate Lie algebras has not yet been established, or even formulated in a reasonable generality.

2.7.14. **An aside.** The equivalence of Proposition 2.7.2 is closely related to the following more general result of Lurie.

Namely, let $\text{FormMod}_{\text{laft}}$ denote the full subcategory of PreStk consisting of objects $Y$ equipped with a distinguished point $\{1\}$ and that satisfy the same three conditions as in the case $\text{FormGrp}_{\text{laft}}$ (but we are not putting a group structure on $Y$).

We have:

**Theorem 2.7.15.** There exists a canonical equivalence between $\text{FormMod}_{\text{laft}}$ and the category of Lie algebras.

The compatibility between Theorem 2.7.15 and Proposition 2.7.15 is expressed by the following observation:
Lemma 2.7.16. There is an equivalence of categories $\text{FormMod}_{\text{left}} \to \text{FormGrp}_{\text{left}}$ given by $\mathcal{Y} \mapsto \text{pt} \times_{\mathcal{Y}} \text{pt}$.

Remark 2.7.17. We can think of the above assignment $\mathcal{Y} \mapsto \text{pt} \times_{\mathcal{Y}} \text{pt}$ as associating to a prestack the group of automorphisms of its distinguished point. Naively, the functor in the opposite direction would send $G$ to $\text{pt}/G$. However, the resulting prestack $\text{pt}/G$ will in general fail to have deformation theory. So, we need to perform an additional manipulation to it to make it satisfy this condition.

3. Universal constructions

3.1. The groups of automorphisms.

3.1.1. We fix an integer $n$, to be the dimension of the variety on which we will be performing our universal constructions.

Consider the topological ring $\hat{\mathcal{O}} = k[[t_1, ..., t_n]]$. We let $\mathfrak{m}$ denote its maximal ideal, (i.e. the ideal generated by the $t_m$'s).

We let $\text{Aut}(\hat{\mathcal{O}})$ be the group-object in $\text{PreStk}$ that assigns to $S = \text{Spec}(R)$ the group of $R$-linear continuous automorphisms of the ring $R[[t_1, ..., t_n]]$.

One can show that $\text{Aut}(\hat{\mathcal{O}})$ is in fact a group ind-scheme (but not locally of finite type).

We let $\circ \text{Aut}(\hat{\mathcal{O}})$ denote a group sub-object of $\text{Aut}(\hat{\mathcal{O}})$ that assigns to $S = \text{Spec}(R)$ the subgroup of those automorphisms of $R[[t_1, ..., t_n]]$ that preserve the evaluation map $R[[t_1, ..., t_n]] \to R$.

It is easy to see that $\circ \text{Aut}(\hat{\mathcal{O}})$ is in fact a group-scheme over $k$, and that the inclusion $\circ \text{Aut}(\hat{\mathcal{O}}) \hookrightarrow \text{Aut}(\hat{\mathcal{O}})$ induces an isomorphism when evaluated on reduced affine schemes. I.e., the formal completion of $\circ \text{Aut}(\hat{\mathcal{O}})$ in $\text{Aut}(\hat{\mathcal{O}})$ equals $\text{Aut}(\hat{\mathcal{O}})$.

3.1.2. More generally, for an integer $i$ we let $\circ \text{Aut}(\hat{\mathcal{O}})_i$ denote the group subscheme of $\circ \text{Aut}(\hat{\mathcal{O}})$ whose points are those automorphisms of $R[[t_1, ..., t_n]]$ that preserve the projection $R[t_1, ..., t_n] \to R \otimes k[t_1, ..., t_n]/\mathfrak{m}^i$.

I.e., $\circ \text{Aut}(\hat{\mathcal{O}})_0 = \circ \text{Aut}(\hat{\mathcal{O}})$, and it is easy to see that for $i \geq 1$, the group-scheme $\circ \text{Aut}(\hat{\mathcal{O}})_i$ is pro-unipotent.

We let $\text{Aut}(\hat{\mathcal{O}})_i$ denote the formal completion of $\circ \text{Aut}(\hat{\mathcal{O}})_i$ in $\text{Aut}(\hat{\mathcal{O}})$.

3.1.3. Let $\text{Der}(\hat{\mathcal{O}})$ denote the Lie algebra of $\text{Aut}(\hat{\mathcal{O}})$.

Caution: The Lie algebra $\text{Der}(\hat{\mathcal{O}})$ carries a natural topology, which we must take into account, but which we shall ignore, see Sect. 2.7.13.

The Lie algebra $\text{Der}(\hat{\mathcal{O}})$ is freely spanned by the elements $f \cdot \partial_{t_m}, f \in \hat{\mathcal{O}}$.

We let $\circ \text{Der}(\hat{\mathcal{O}})$ (resp., $\circ \text{Der}(\hat{\mathcal{O}})_i$) denote the Lie algebra of $\circ \text{Aut}(\hat{\mathcal{O}})$ (resp., $\circ \text{Aut}(\hat{\mathcal{O}})_i$). It is freely spanned by the elements $f \cdot \partial_{t_m}$ with $f \in \mathfrak{m}$ (resp., $f \in \mathfrak{m}^i$).
3.2. **The space of jets of coordinates.** Let $X$ be a smooth affine scheme of dimension $n$. Since we have not sheafified our classifying stacks $pt/G$, we will have to assume that the tangent sheaf of $X$ admits a trivialization. However, Lemma 1.2.7 allows to get rid of this assumption, as well as of the fact that $X$ needs to be affine.

3.2.1. We let $\text{Coord}(X)$ denote the scheme that attaches to $S = \text{Spec}(R)$ the data of $(x, \alpha)$, where $x$ is a map $S \to X$, and $\alpha$ is an isomorphism between the formal schemes

$$(S \times X)_{\Gamma_x} \simeq \text{Spf}(R[[t_1, \ldots, t_n]]),$$

which makes the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{id} & \text{Spec}(R) \\
\downarrow & & \downarrow \\
(S \times X)_{\Gamma_x} & \xrightarrow{\alpha} & \text{Spf}(R[[t_1, \ldots, t_n]]) \\
\downarrow & & \downarrow \\
S & \xrightarrow{id} & \text{Spec}(R)
\end{array}
$$

commute. In the above formula $\Gamma_x$ is the graph of the map $x$:

$$\Gamma_x \simeq S \to S \times X.$$

It is easy to see that $\text{Coord}(X)$ is an $\text{Aut}(\hat{\O})$-torsor, locally trivial in Zariski topology. However, our assumption on $X$ implies that this torsor is in fact globally trivial.

Thus, we obtain a map

$$\pi_X : X \to pt / \text{Aut}(\hat{\O}).$$

3.2.2. Similarly, let $X_i$ be the scheme that attaches to $S = \text{Spec}(R)$ the data of $(x, \alpha_i)$, where $x$ is a map $S \to X$, and $\alpha_i$ is an isomorphism between the $i$-th infinitesimal neighborhood of $\Gamma_S$ in $S \times X$ and

$$\text{Spec}(R \otimes k[[t_1, \ldots, t_n]]/m_i),$$

which makes the corresponding diagram commute.

The scheme $\text{Coord}(X)$ is a (trivial) $\text{Aut}(\hat{\O})$-torsor over $X_i$. Thus, we obtain a map

$$\pi_{X_i} : X_i \to pt / \text{Aut}(\hat{\O})_i.$$

3.2.3. Let $\mathcal{F}$ be as in Sect. 1.1.3. We obtain that pullback defines maps

$$\mathcal{F}(pt / \text{Aut}(\hat{\O})) \to \mathcal{F}(X)$$

and

$$\mathcal{F}(pt / \text{Aut}(\hat{\O})_i) \to \mathcal{F}(X_i).$$

3.2.4. Suppose now that $\mathcal{F}$ takes values in $\infty$-$\text{Cat}$. For $f_{pt / \text{Aut}(\hat{\O})} \in \mathcal{F}(pt / \text{Aut}(\hat{\O}))$, we let $f_X$ denote the resulting object of $\mathcal{F}(X)$, and similarly for $X_i$.

We shall apply this construction for $\mathcal{F}$ in the examples of Sect. 1.3.2. Thus, we obtain a functor

$$\text{Rep}(\text{Aut}(\hat{\O})) \to \text{QCoh}(X),$$

and also

$$\text{AssAlg}(\text{Rep}(\text{Aut}(\hat{\O}))) \to \text{AssAlg}(\text{QCoh}(X))$$

(or more generally, for algebras over an arbitrary operad). And similarly for $X_i$.
3.2.5. Applying the above construction to $F = \text{ShvDGCat}$, we obtain that a DG category $D$ acted on (weakly) by $\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})}$ gives rise to a sheaf of categories $D_X$ on $X$. The same goes for (weakly) $\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})}$-equivariant functors.

For example, if $A$ is an associative algebra acted on by $\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})}$, then the sheaf of categories $(A\text{-mod})_X$ corresponding to $A$-mod, regarded as a category acted on weakly by $\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})}$, identifies with $(A_X\text{-mod})_X$. In particular, $\Gamma(X, (A\text{-mod})_X) \simeq A_X\text{-mod}(\text{QCoh}(X))$.

3.3. Moving the point. A key observation is that we have a canonical action of the entire $\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})}$ on $\text{Coord}(X)$. It is defined as follows:

3.3.1. For $S = \text{Spec}(R)$, an $S$-point $(x, \alpha)$ of $\text{Coord}(X)$, and an $S$-point $\gamma$ of $\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})}$ we construct a new $S$-point $\gamma \cdot x$ of $\text{Coord}(X)$ as follows:

The $S$-point $\gamma \cdot x$ equals the map

$$S \to \text{Spf}(R[t_1, \ldots, t_n]) \xrightarrow{\gamma} \text{Spf}(R[t_1, \ldots, t_n]) \xrightarrow{\alpha} (S \times X)_{\gamma \cdot x} \to S \times X \to X.$$  

In particular, the points $x$ and $\gamma \cdot x$ are infinitesimally close (i.e., they agree on $S_{\text{red}}$). This implies that we have a canonical isomorphism

$$(3.1) \quad (S \times X)_{\gamma \cdot x} \simeq (S \times X)^\wedge_{\gamma \cdot x},$$  

and we obtain the datum of $\gamma \cdot \alpha$ by composing the identification (3.1) with $\alpha$.

3.3.2. The action of $\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})}$ on $\text{Coord}(X)$ gives rise to the maps

$$(3.2) \quad \text{Coord}(X)/\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})} \to X_{\text{dR}} \quad \text{and} \quad \text{Coord}(X)/\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})}_i \to (X_i)_{\text{dR}}.$$  

Lemma 3.3.3. The maps $(3.2)$ are isomorphisms.

From Lemma 3.3.3 we obtain that we have canonical maps

$$\pi_{X,\text{dR}} : X_{\text{dR}} \to \text{pt}/\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})} \quad \text{and} \quad \pi_{X_i,\text{dR}} : (X_i)_{\text{dR}} \to \text{pt}/\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})}_i.$$  

Moreover, it is easy to see that the following commutative diagrams are Cartesian:

$$\begin{array}{ccc}
X & \longrightarrow & X_{\text{dR}} \\
\pi_X \downarrow & & \downarrow \pi_{X,\text{dR}} \\
\text{pt}/\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})} & \longrightarrow & \text{pt}/\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})}
\end{array}$$  

and

$$\begin{array}{ccc}
X_i & \longrightarrow & (X_i)_{\text{dR}} \\
\pi_{X_i} \downarrow & & \downarrow \pi_{X_i,\text{dR}} \\
\text{pt}/\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})}_i & \longrightarrow & \text{pt}/\overset{\circ}{\text{Aut}(\hat{\mathcal{O}})}_i.
\end{array}$$
3.3.4. The morphisms $\pi_{X, dr}$ and $\pi_{X, i, dr}$ allow to give universal constructions of crystals. Namely, let $\mathcal{F}$ be as in Sect. 1.1.3. We obtain that pullback defines the maps in $\mathcal{C}$:

$$\mathcal{F}(pt / Aut(\hat{\mathcal{O}})) \to \mathcal{F}_{crys}(X)$$

and

$$\mathcal{F}(pt / Aut(\hat{\mathcal{O}})_i) \to \mathcal{F}_{crys}(X_i).$$

In particular, the constructions reviewed in Sects. 3.2.4 and 3.2.5 are applicable, once we replace $\circ Aut(\hat{\mathcal{O}})$ by $Aut(\hat{\mathcal{O}})$, and the categories of loc.cit. by crystalline objects in them.

For example, if $A$ is an associative algebra acted on by $Aut(\hat{\mathcal{O}})$, we obtain that the sheaf of categories $(A_{-mod})_{X, dr}$, equal to the pullback of $A_{-mod}$, when the latter is viewed as a category acted in weakly by $Aut(\hat{\mathcal{O}})$, identifies with $(A_X)_{-mod}$ as a crystal of categories.

In particular, $\Gamma_{crys}(X, (A_{-mod})_X) \simeq A_{-mod}(\text{QCoh}_{crys}(X))$, where $A_X$ is viewed as an object of $\text{AssAlg}(\text{QCoh}_{crys}(X))$.

4. Chiral modules as a crystal

All schemes considered in this section are assumed separated and quasi-compact. In fact, with no restriction of generality, one can consider only affine schemes.

4.1. Modules over a chiral algebra as a sheaf of categories.

4.1.1. Let $X$ be a scheme of finite type, and let $A$ be a chiral algebra on it. Recall the category of $A$-chiral $\mathcal{O}$-modules on $X$, denoted $A_{-mod}(\text{QCoh}(X))$.

In fact, we can define the category $A_{-mod}(\text{QCoh}(S))$ for any scheme $S$ mapping to $X$. Namely, for $S \to X$ we define $A_{-mod}(\text{QCoh}(S))$ to have as objects $M \in \text{QCoh}(S)$ equipped with an action map

$$j_* j^*(A \boxtimes M) \to \Gamma_{*, dr}(M),$$

that satisfies certain axioms.

4.1.2. Before we can proceed further, let us explain the meaning of the terms in (4.1). For a scheme $S$ and another scheme $S'$ mapping to it, we define the category of relative crystals on $S'$, denoted $\text{QCoh}_{crys}/S(S')$ as

$$\text{QCoh}_{crys}/S(S' \times S \text{dR} \times S).$$

The map in (4.1) takes place in the category $\text{QCoh}_{crys}/S(X \times S)$, where $X \times S$ is regarded as a scheme over $S$ via the projection on the second factor.

It is easy to see that for $S' = X \times S$, regarded as a scheme over $S$ as above, the category $\text{QCoh}_{crys}/S(X \times S)$ identifies with

$$\text{QCoh}_{crys}(X) \otimes \text{QCoh}(S).$$

In particular, for $A \in \text{QCoh}_{crys}(X)$ and $M \in \text{QCoh}(S)$, the expression $A \boxtimes M$ makes sense as an object of $\text{QCoh}_{crys}/S(X \times S)$

We let $\Gamma$ denote the graph map $S \to S \times X$, and $j$ the embedding of its complement. The functors $j^*$ and $j_*$ is the pair of adjoint functors that relate the category $\text{QCoh}_{crys}/S(X \times S)$ with $\text{QCoh}_{crys}/S(X \times S - \Gamma)$.

We can consider the map $\Gamma$ as a morphism of schemes over $S$. Let

$$\Gamma^*: \text{QCoh}_{crys}/S(X \times S) \to \text{QCoh}_{crys}/S(S) \simeq \text{QCoh}(S)$$

denote the pullback functor. The functor $\Gamma_{*, dr}$ is the left adjoint of $\Gamma^*$.
When formulating the axioms on the map (4.1), one deals with the categories
\[ \text{QCoh}_{\text{crys}}(X^n \times S) \]
for various \( n \), and the objects in it with a similar meaning.

4.1.3. Let \( \phi : S_1 \to S_2 \) be a morphism of schemes over \( X \). Tensoring up defines a functor
\[ A\text{-mod}(\text{QCoh}(S_2)) \to A\text{-mod}(\text{QCoh}(S_1)), \]
and in fact a functor
\[ (S_2) \otimes_{\text{QCoh}(S_1)} A\text{-mod}(\text{QCoh}(S_2)) \to A\text{-mod}(\text{QCoh}(S_1)). \]

**Lemma 4.1.4.** The functor (4.2) is an equivalence.

4.1.5. Thus, we obtain that the assignment
\[ (S \in \text{Sch}^{\text{aff}}_X) \mapsto A\text{-mod}(\text{QCoh}(S)) \]
forms a sheaf of categories over \( X \), denoted \( A\text{-mod}_X \), such that
\[ A\text{-mod}(\text{QCoh}(X)) \simeq \Gamma(X, A\text{-mod}_X). \]

4.1.6. By a similar token, for \( S \to X \) one defines the category
\[ A\text{-mod}(\text{QCoh}_{\text{crys}}(S)) \simeq A\text{-mod}(\text{D-mod}(S)) \]
of \( A \)-modules in crystals/D-modules on \( S \).

4.2. Modules over a chiral algebra as a crystal of categories.

4.2.1. We now claim that the sheaf of categories \( A\text{-mod}_X \), introduced above, is in fact a crystal of categories. I.e., we claim that the category \( A\text{-mod}(\text{QCoh}(S)) \) is defined for \( S \) mapping to \( X_{\text{dR}} \). In other words, we claim that the expression (4.1) (and similar expressions for formulating the axioms), are defined for \( S \) mapping just to \( X_{\text{dR}} \) rather than to \( X \).

4.2.2. First, we note that the open subset
\[ X \times S - \Gamma \subset X \times S \]
only depends on the map \( S_{\text{red}} \to X \). So, the functors \( j^* \) and \( j_* \) are still well-defined.

The meaning of \( \Gamma_{\text{dR},*} \) is similar: it is the left adjoint of the pullback functor
\[ \Gamma^* : \text{QCoh}_{\text{crys}}(X \times S) \to \text{QCoh}(S), \]
corresponding to the graph map
\[ S \to X_{\text{dR}} \times S. \]
4.2.3. As in Lemma 4.1.4, we obtain:

**Lemma 4.2.4.** For a map $S_1 \to S_2 \in \text{Sch}_X$, the resulting functor

$$\text{QCoh}(S_1) \otimes_{\text{QCoh}(S_2)} \mathcal{A}\text{-mod}(\text{QCoh}(S_2)) \to \mathcal{A}\text{-mod}(\text{QCoh}(S_1))$$

is an equivalence.

So, the assignment

$$(S \in \text{Sch}^{\text{aff}}_{X_{dR}}) \mapsto \mathcal{A}\text{-mod}(\text{QCoh}(S))$$

indeed forms a crystal of categories on $X$.

The following easily results from the definitions:

**Corollary 4.2.5.** The category $\Gamma_{\text{crys}}(X, \mathcal{A}\text{-mod}_X)$ of crystalline objects in

$$\Gamma(X, \mathcal{A}\text{-mod}_X) \simeq \mathcal{A}\text{-mod}(\text{QCoh}(X))$$

identifies canonically with the category $\mathcal{A}\text{-mod}(\text{QCoh}_{\text{crys}}(X))$ of $\mathcal{A}$-chiral $\mathcal{D}$-modules.

And more generally:

**Corollary 4.2.6.** For any $S \to X$, the category $\Gamma_{\text{crys}}(S, \mathcal{A}\text{-mod}_X|_S)$ of crystalline objects in

$$\mathcal{A}\text{-mod}(S) \simeq \Gamma(S, \mathcal{A}\text{-mod}_X|_S)$$

identifies canonically with the category $\mathcal{A}\text{-mod}(\text{QCoh}_{\text{crys}}(S)) \simeq \mathcal{A}\text{-mod}(\mathcal{D}\text{-mod}(S))$ of $\mathcal{A}$-chiral $\mathcal{D}$-modules on $S$.

4.3. **Lie-* algebras.**

4.3.1. Let now $L$ be a Lie-* algebra on $X$. There are four categories that can be attached to it:

$L\text{-mod}^*(\text{QCoh}(X)), \ L\text{-mod}^{ch}(\text{QCoh}(X)), \ L\text{-mod}^*(\text{QCoh}_{\text{crys}}(X)), \ L\text{-mod}^{ch}(\text{QCoh}_{\text{crys}}(X))$.

By repeating the constructions of Sects. 4.1 and 4.2, we upgrade the categories

$L\text{-mod}^*(\text{QCoh}(X))$ and $L\text{-mod}^{ch}(\text{QCoh}(X))$

two crystals of categories over $X$, denoted $L\text{-mod}_X^*$ and $L\text{-mod}_X^{ch}$, respectively, such that

$L\text{-mod}^*(\text{QCoh}(X)) \simeq \Gamma(X, L\text{-mod}_X^*), \ L\text{-mod}^{ch}(\text{QCoh}(X)) \simeq \Gamma(X, L\text{-mod}_X^{ch})$

and

$L\text{-mod}^*(\text{QCoh}_{\text{crys}}(X)) \simeq \Gamma_{\text{crys}}(X, L\text{-mod}_X^*), \ L\text{-mod}^{ch}(\text{QCoh}_{\text{crys}}(X)) \simeq \Gamma_{\text{crys}}(X, L\text{-mod}_X^{ch})$.

4.3.2. Recall now that to a Lie-* algebra $L$ on $X$ one can attach its universal enveloping chiral algebra, denoted $U^{cl}(L)$.

**Proposition 4.3.3.** The canonical map $L \to U^{cl}(L)$ induces an equivalence of crystals of categories

$$U^{cl}(L)\text{-mod}_X \to L\text{-mod}_X^{ch}.$$

We shall prove this proposition by constructing the inverse functor. Recall that the construction of $U^{cl}(L)$ was given in terms of the corresponding factorization algebra, i.e., for each non-empty finite set $I$ we constructed a crystal

$$(U^{cl}(L))(I) \in \text{QCoh}_{\text{crys}}(X^I),$$

along with the factorization isomorphisms. In order to construct the inverse functor on modules, we will have to interpret the category of modules over a chiral algebra also in factorization terms.
4.3.4. Digression: factorization description of chiral modules. Thus, let $\mathcal{A}$ be a chiral algebra, and let $\mathcal{A}(I)$ be the corresponding crystal on $X^I$ for every non-empty finite set $I$. Let $S$ be a scheme mapping to $X_{dR}$. We can describe the category

$$\mathcal{A}\text{-mod}(\text{QCoh}(S))$$

in terms of $\mathcal{A}(I)$’s as follows.

An object of $\mathcal{A}\text{-mod}(\text{QCoh}(S))$ is an assignment for every finite (possibly empty) set $I$ of an object $M(I) \in \text{QCoh}_{\text{crys}}/S(X^I \times S)$, together with the following system of identifications.

Let $I_1 \sqcup (I_2 \sqcup *)$ be a partition of the finite set $I \sqcup *$. (To simplify the notation, we are only considering two-element partitions, whereas one should really consider arbitrary partitions.)

Let

$$(X^I \times S_{\text{red}})_{I_1,I_2} \subset X^I \times S_{\text{red}}$$

be the following open subset: we require that the points $\{x_j, j \in I_1\}$ and $\{x_j, j \in I_2\} \cup \{\phi(s)\}$ be disjoint. Here $\phi$ denotes the map $S_{\text{red}} \to X$.

Let $(X^I \times S)_{I_1,I_2}$ be the corresponding open subset of $X^I \times S$; let $j$ denote its embedding. The first type of factorization isomorphism that we require is

$$j^*(M(I)) \simeq j^*(\mathcal{A}(I_1) \boxtimes M(I_2)).$$

Let now $I \to J$ be a surjection of finite sets. Let $\Delta_{I,J} : X^I \to X^J$ be the corresponding diagonal morphism. The second type of factorization isomorphism is

$$\Delta_{I,J}^*(M(I)) \simeq M(J).$$

Finally, consider the graph map

$$\Gamma : S \to X^I_{dR} \times S.$$

The third type of factorization isomorphism is between $\Gamma^*(M(I))$ and $M(*) \in \text{QCoh}(S)$.

In fact, the second and the third factorization isomorphisms need to be combined into one, where one considers surjections of finite sets $(I \sqcup *) \to (J \sqcup *)$ that send $*$ to $*$.

The above factorizations isomorphism need to satisfy the natural compatibility conditions.

4.3.5. Let us return to the chiral universal enveloping algebra of a Lie-* algebra. Recall that for a finite set $I$, the crystal $(U^\text{cd}(L))(I)$ was constructed as follows. Consider the diagram

$$
\begin{array}{ccc}
X \times X^I & \xrightarrow{p_1} & X \\
\downarrow p_2 & & \\
X^I & & \\
\end{array}
$$

Let $j$ be the open embedding of the locus in $X \times X^I$, where the first coordinate is required to be disjoint from the coordinates along $i \in I$.

We consider two Lie algebras in $\text{QCoh}_{\text{crys}}(X^I)$:

$$\mathcal{L}(I,X) := (p_2)_{dR,*} \circ p_1^*(L) \quad \text{and} \quad \mathcal{L}(I,X) := (p_2 \circ j)_{dR,*} \circ (j \circ p_1)^*(L).$$

The two acquire a Lie algebra structure from the Lie-* algebra structure on $L$. 
By adjunction, we have a natural map
\[ \mathcal{L}(I, X) \to \tilde{\mathcal{L}}(I, X) \]
and we have
\[ (U^d(L))(I) := \text{Ind}_{\mathcal{L}(I, X)}(O_{X^I}). \]

It is easy to see that although the Lie algebras \( \mathcal{L}(I, X) \) and \( \tilde{\mathcal{L}}(I, X) \) depend globally on \( X \) (they change if we shrink \( X \) to an open subset), the resulting crystal \( (U^d(L))(I) \) does not change.

4.3.6. Let now \( M \) be a chiral \( L \)-module on \( S \), and we wish to construct the objects
\[ M(I) \in \text{QCoh}_{\text{crys}}(X^I \times S), \]
with the appropriate factorization isomorphisms.

Consider the diagram
\[ (X \times X^I \times S)_0 \xrightarrow{p_1} X \]
\[ \downarrow \quad \quad \downarrow \]
\[ p_2 \quad \quad \quad \quad \quad X^I \times S, \]
where \( (X \times X^I \times S)_0 \subset X \times X^I \times S \) is the open subset where the first coordinate is required to be disjoint from \( \phi(s) \), where \( \phi : S_{\text{red}} \to X \). Let \( j \) be the open embedding of the locus in \( (X \times X^I \times S)_0 \), where the coordinates along \( i \in I \) also also required to be disjoint from the first one.

We consider
\[ \mathcal{L}(I, X, S) := (p_{2,3})_{dR,*} \circ p_1^*(L) \quad \text{and} \quad \tilde{\mathcal{L}}(I, X, S) := (p_{2,3} \circ j)_{dR,*} \circ (j \circ p_1)^*(L), \]
both being Lie algebras in \( \text{QCoh}_{\text{crys}}(X^I \times S) \).

As before, we have a homomorphism \( \mathcal{L}(I, X, S) \to \tilde{\mathcal{L}}(I, X, S) \), and the structure on \( M \in \text{QCoh}(S) \) of Lie-* module over \( L \) defines a structure of \( \mathcal{L}(I, X, S) \)-module on
\[ O_{X^I} \boxtimes M \in \text{QCoh}_{\text{crys}}(X^I \times S). \]

Finally, we define the sought-for crystals
\[ M(I) := \text{Ind}_{\mathcal{L}(I, X, S)}(O_{X^I} \boxtimes M). \]

4.4. Crystals of Lie algebras associated to a Lie-* algebra. In this subsection, the bug alluded to in Sect. 2.7.13 will come to bear heavily.

4.4.1. We shall assume that \( X \) is a smooth curve. Let \( L \) be a Lie-* algebra on \( X \).

Recall that the category of crystals on a scheme \( X \) of finite type can be realized either as the category of left D-modules, or as the category of right D-modules. These two realizations give the category \( \text{QCoh}_{\text{crys}}(X) \) two different t-structures, with the one corresponding to the “right” realization much better behaved. (An account of this material will shortly become available in [GL:Crys].)

We shall assume that the crystal underlying \( L \) lies in the heart of the t-structure of the “right” realization, and that it is coherent as a D-module.
We are going to associate to $L$ two crystals of (topological) Lie algebras on $S$. This will be a variant of the construction of the Lie algebras $\mathcal{L}(I, X, S)$ and $\overset{\circ}{\mathcal{L}}(I, X, S)$ in Sect. 4.3.6.

4.4.2. Let $(S, x) \in \text{Sch}^{\text{aff}}/X_{\text{dR}}$. Consider the formal completion $\hat{\Gamma}_x$ of $S_{\text{red}} \hookrightarrow X \times S$. We let $D_x$ denote the scheme obtained from $\hat{\Gamma}_x$ by taking the colimit. I.e., if we write $\hat{\Gamma}_x \simeq \underset{\alpha}{\text{colim}} \ S_{\alpha} \in \text{PreStk}$, then

$$D_x = \text{colim} \ S_{\alpha} \in \text{Sch}^{\text{aff}}.$$ 

Equivalently, if $\hat{\Gamma}_x = \text{Sph}(R)$, where $R$ is a topological commutative algebra, then $D_x = \text{Spec}(R)$. Note that $D_x$ is not of finite type over $k$.

The scheme $D_x$ contains $S_{\text{red}}$ as a closed subset. We let $\overset{\circ}{D}_x$ be the complementary open; let $j$ denote the corresponding open embedding. Since $X$ is a curve, the morphism $j$ is affine. Let $p$ denote the projection $D_x \rightarrow S$.

Informally, we can think about $D_x$ (resp., $\overset{\circ}{D}_x$) as the family of formal discs (resp., formal punctured discs) parameterized by $S$. Again, we emphasize that we only needed $x$ to be defined as a map $S_{\text{red}} \rightarrow X$ in order to construct $D_x$ and $\overset{\circ}{D}_x$.

4.4.3. Consider the restrictions $(L \boxtimes \mathcal{O}_S)|_{D_x} \in \text{QCoh}_{\text{crys}}/S(D_x)$ and $j_* \circ j^* \left((L \boxtimes \mathcal{O}_S)|_x \right)_{D_x}$.

We define sheaves of Lie algebras $\mathcal{L}(S)$ and $\overset{\circ}{\mathcal{L}}(S)$ on $S$ by

$$\mathcal{L}(S) := p_{*, \text{dR}} \left((L \boxtimes \mathcal{O}_S)|_{D_x} \right) \quad \text{and} \quad \overset{\circ}{\mathcal{L}}(S) := p_{*, \text{dR}} \left(j_* \circ j^* \left((L \boxtimes \mathcal{O}_S)|_x \right)_{D_x} \right).$$

4.4.4. The functor $p_{*, \text{dR}} : \text{QCoh}_{\text{crys}}/S(D_x) \rightarrow \text{QCoh}(S)$, used above, does not a priori make sense.

In general, the functor $g_{*, \text{dR}} : \text{QCoh}_{\text{crys}}/T$ for $g : T \rightarrow S$ is defined when $T$ is of finite type over $S$, which is not the case for $D_x$. However, one could give an ad hoc definition for the above morphism $p$.

However, we do not actually want to define $p_{*, \text{dR}}$ in this framework, because we do not want to obtain a mere object of $\text{QCoh}(S)$, but a sheaf of Tate vector spaces.

It is important to keep in mind that the functor from the category of sheaves of Tate vector spaces to $\text{QCoh}$ does not commute with pullbacks (we need to take pullbacks taking the topology into account). So, for $\phi : S_1 \rightarrow S_2$, we do not have an isomorphism between

$$\mathcal{L}(S_1) \quad \text{and} \quad \phi^*(\mathcal{L}(S_2)),$$

where $\phi^*$ is understood as the usual pullback $\text{QCoh}(S_2) \rightarrow \text{QCoh}(S_1)$.

We will not dwell on this point in this talk, simultaneously assuming that $p_{*, \text{dR}}$ is defined, and disregarding the topology on the resulting object of $\text{QCoh}(S)$. 

4.4.5. The above construction admits a generalization along the lines of Sect. 4.3.6. Namely for a finite set $I$ (possibly empty), one can produce sheaves of (topological) Lie algebras

$$\mathcal{L}(I, S)$$

and

$$\mathcal{\hat{L}}(I, S)$$

in $\text{Qcoh}^{\text{crys}}(X^I \times S)$. We recover $\mathcal{L}(S)$ and $\mathcal{\hat{L}}(S)$ for $I = \emptyset$.

The sheaves $\mathcal{L}(I, S)$ and $\mathcal{\hat{L}}(I, S)$ are local analogs of $\mathcal{L}(I, X, S)$ and $\mathcal{\hat{L}}(I, X, S)$, respectively. We have natural maps

$$\mathcal{L}(I, X, S) \to \mathcal{L}(I, S)$$

and

$$\mathcal{\hat{L}}(I, X, S) \to \mathcal{\hat{L}}(I, S),$$

and the former can be obtained from the latter by a certain completion procedure.

4.4.6. We let $\mathcal{L}_X$ (resp., $\mathcal{\hat{L}}_X$) denote the resulting crystals of (topological) Lie algebras on $X$. We have:

Lemma 4.4.7. The crystals of categories

$$L\text{-mod}_X^\bullet$$

and

$$L\text{-mod}_X^{\text{ch}}$$

identify, respectively with

$$\mathcal{L}_X\text{-mod}_X$$

and

$$\mathcal{\hat{L}}_X\text{-mod}_X.$$

Again, when defining the crystals of categories $\mathcal{L}_X\text{-mod}_X$ and $\mathcal{\hat{L}}_X\text{-mod}_X$, one needs to take the topology on these Lie algebras into account.

5. Kac-Moody, KL and Sugawara

5.1. Statement of the main result.

5.1.1. Let $\mathfrak{g}$ be a reductive Lie algebra, and $\kappa$ a level. We consider the category

$$\hat{\mathfrak{g}}_\kappa\text{-mod}.$$

Again, the Lie algebra $\hat{\mathfrak{g}}_\kappa$ carries a natural topology, and this topology is used in the definition of $\hat{\mathfrak{g}}_\kappa\text{-mod}$.

5.1.2. We consider the group scheme $G[[t]] = G(\hat{\mathfrak{O}})$ and the group indscheme $G(\langle t \rangle)$. By Proposition 2.7.8, extended to the non-finite type case, the category $\hat{\mathfrak{g}}_\kappa\text{-mod}$ carries a canonical strong action of $G[[t]]$.

If $\kappa = 0$, it also carries a strong action of $G(\langle t \rangle)$; for general $\kappa$, we have a twisted strong action of $G(\langle t \rangle)$, but we will not need it here.

By definition, the category KL is defined as

$$\hat{\mathfrak{g}}_\kappa\text{-mod}^{G[[t]]\text{-strong}}.$$

5.1.3. Let $L_{\mathfrak{g}, \kappa}$ be the Lie-* algebra associated to $\mathfrak{g}$ and $\kappa$. Let $A_{\mathfrak{g}, \kappa}$ denote the corresponding $(\kappa$-twisted) universal enveloping chiral algebra.

The main practical goal of this talk is to explain the following:

Theorem 5.1.4. For any smooth curve $X$, there exists a canonically defined functor

$$\text{KL} \to A_{\mathfrak{g}, \kappa}\text{-mod}(\text{D-mod}(X_1)).$$

Note that $X_1$ is the space of 1-jets of coordinates on $X$. Since $X$ is a curve, it identifies with the punctured tangent bundle to $X$. 
5.2. The Sugawara construction.

5.2.1. Consider the group scheme $\text{Aut}(\hat{O})$ for $n = 1$. Note that there is a natural weak action of $\text{Aut}(\hat{O})$ on the categories

$$\hat{\mathfrak{g}}_{\kappa}\text{-mod and KL}$$

that comes from the action of $\text{Aut}(\hat{O})$ on $\hat{\mathfrak{g}}_{\kappa}$ and $G[t]$ by automorphisms; see Sect. 1.6.8. The forgetful functor

$$\text{(5.1)} \quad \text{KL} \to \hat{\mathfrak{g}}_{\kappa}\text{-mod}$$

is naturally weakly $\text{Aut}(\hat{O})$-equivariant.

The Sugawara construction will be encoded for us in the following statement:

**Theorem 5.2.2.** The weak actions of $\text{Aut}(\hat{O})$ on $\hat{\mathfrak{g}}_{\kappa}\text{-mod}$ and KL admit canonical extensions to strong actions. The structure of weak equivariance on the functor (5.1) admits a canonical extension to a structure of strong equivariance with respect to the above strong actions.

In the next section we shall explain how the datum of strong action incarnates in terms of Kac-Moody modules.

From now on, we shall assume Theorem 5.2.2, plus one additional property that this construction satisfies (see Sect. 5.2.3 below), and deduce Theorem 5.1.4 from it.

5.2.3. The strong action of $\text{Aut}(\hat{O})$ on KL restricts to a strong action of $\text{Aut}(\hat{O})_1$. Recall the forgetful functor

$$\text{KL}^{\text{Aut}(\hat{O})_1, \text{strong}} \to \text{KL},$$

and recall that according to Lemma 2.4.3, the latter functor is fully faithful.

We shall use the following property of the Sugawara construction:

**Lemma 5.2.4.** The functor $\text{KL}^{\text{Aut}(\hat{O})_1, \text{strong}} \to \text{KL}$ is an equivalence.

This lemma can actually be proved without knowing the explicit formula for Sugawara:

**Proof.** By fully-faithfulness, it suffices to show that the essential image of the functor in question generates KL. We shall take as generators the objects from the abelian category $\mathcal{O}_{\kappa}$.

According to Lemma 2.7.10, again applied in the non-finite type situation, it suffices to show that the action of $\text{Der}(\hat{O})_1$ on the vector space underlying an object $V \in \mathcal{O}_{\kappa}$, comes from an action of $\text{Aut}(\hat{O})_1$. As the group in question is pro-unipotent, we need to show that the action of $\text{Der}(\hat{O})_1$ on $V$ is locally nilpotent.

Recall that every object $V \in \mathcal{O}_{\kappa}$ admits a decomposition into generalized eigenspaces with respect to $L_0$. The compatibility of point (a) in Proposition 2.7.8 says that the action of $\text{Der}(\hat{O})_1$ strictly raises the grading. Since the grading on a given module is bounded from above, the assertion follows. 

□
5.2.5. From Theorem 5.2.2 and Lemma 5.2.4 we deduce the following corollary, which is all we will need for Theorem 5.1.4.

**Corollary 5.2.6.** The forgetful functor \( KL \to \hat{g}_\kappa\text{-mod} \) canonically extends to a functor

\[
KL \to \left( \hat{g}_\kappa\text{-mod} \right)^{\text{Aut}(\hat{\Omega})_1,\text{weak}}.
\]

We will actually prove more (and this will be used in the next talk): we will construct a functor

(5.2)

\[
KL \to \left( \hat{g}_\kappa\text{-mod} \right)^{\text{Aut}(\hat{\Omega})_1,\text{strong}}.
\]

The functor in Corollary 5.2.6 is then obtained by composing with the forgetful functor

\[
\left( \hat{g}_\kappa\text{-mod} \right)^{\text{Aut}(\hat{\Omega})_1,\text{strong}} \to \left( \hat{g}_\kappa\text{-mod} \right)^{\text{Aut}(\hat{\Omega})_1,\text{weak}}.
\]

**Proof.** Note that since the map

\[
\text{Aut}(\hat{\Omega})_1 \to \text{Aut}(\hat{\Omega})_1
\]

induces an isomorphism when evaluated on reduced affine schemes, and so

\[
(pt / \text{Aut}(\hat{\Omega})_1)_{\text{dR}} \to (pt / \text{Aut}(\hat{\Omega})_1)_{\text{dR}}
\]

is an isomorphism, for any DG category \( D \) equipped with a strong action of \( \text{Aut}(\hat{\Omega})_1 \), the restriction functor

\[
D^{\text{Aut}(\hat{\Omega})_1,\text{strong}} \to D^{\text{Aut}(\hat{\Omega})_1,\text{strong}}
\]

is an equivalence.

Hence, from Lemma 5.2.4, we obtain an equivalence

\[
KL \simeq KL^{\text{Aut}(\hat{\Omega})_1,\text{strong}}.
\]

The sought-for functor (5.2) is then obtained as a composition

\[
KL \simeq KL^{\text{Aut}(\hat{\Omega})_1,\text{strong}} \to \left( \hat{g}_\kappa\text{-mod} \right)^{\text{Aut}(\hat{\Omega})_1,\text{strong}},
\]

where the second arrow comes from the forgetful functor \( KL \to \hat{g}_\kappa\text{-mod} \), which is strongly \( \text{Aut}(\hat{\Omega}) \)-equivariant (by Theorem 5.2.2), and hence also strongly \( \text{Aut}(\hat{\Omega})_1 \)-equivariant. \( \square \)

5.3. **Proof of Theorem 5.1.4.**

5.3.1. Combining Sect. 3.3.4 with Corollaries 4.2.6 and 5.2.6, we obtain that in order to prove Theorem 5.1.4 it suffices to establish the following:

**Proposition 5.3.2.** The crystals of categories \( (\hat{g}_\kappa\text{-mod})_X \) and \( A_{\hat{g}_\kappa\text{-mod}}X \) are canonically equivalent.
5.3.3. Consider the crystal $L_{g,\kappa,tw}$-mod$_X^{ch}$, where the subscript "tw" refers to the fact that we are considering the category of chiral $L_{g,\kappa}$-modules on which the center $\omega \subset L_{g,\kappa}$ acts as identity. I.e., we have a pull-back square of crystals of categories over $X$:

\[
\begin{array}{ccc}
L_{g,\kappa,tw}$-mod$_X^{ch} & \longrightarrow & L_{g,\kappa}$-mod$_X^{ch} \\
\downarrow & & \downarrow \\
\text{Qcoh}_X & \longrightarrow & (\omega)$-mod$_X^{ch}.
\end{array}
\]

From Proposition 4.3.3 (generalized to the twisted case), we have an equivalence of crystals of categories:

$A_{g,\kappa}$-mod$_X \simeq L_{g,\kappa,tw}$-mod$_X^{ch}$.

5.3.4. Let $(\overset{\circ}{g}_{g,\kappa})_X$ denote the crystal of (topological) Lie algebras on $X$ associated to the Lie-* algebra $L_{g,\kappa}$ by the procedure of Sect. 4.4.

Let also $\overset{\circ}{\omega}_X$ denote the crystal of Lie algebras associated to $\omega$, regarded as a Lie-* algebra. Note that residue defines a canonical map

$\overset{\circ}{\omega}_X \to O_X$.

Let us denote by $(\overset{\circ}{g}_{g,\kappa,tw})_X$ the pushout

$(\overset{\circ}{g}_{g,\kappa})_X \sqcup \overset{\circ}{\omega}_X \to O_X$.

It follows from Lemma 4.4.7 that the crystal of categories $L_{g,\kappa,tw}$-mod$_X^{ch}$ identifies canonically with

$(\overset{\circ}{g}_{g,\kappa,tw})_X$-mod.

5.3.5. Combining, this with Sect. 3.3.4, we obtain that Proposition 5.3.2 follows from the next assertion:

Lemma 5.3.6. The crystals of topological Lie algebras $(\overset{\circ}{g}_{g,\kappa,tw})_X$ and $(\overset{\circ}{g}_{g})_X$ on $X$ are canonically isomorphic.

5.3.7. Proof of Lemma 5.3.6. We will prove the lemma by providing another description of the corresponding crystal of Lie algebras. To simplify the notation, we will assume that $\kappa = 0$, i.e., we will be dealing with $L_g = g \otimes D_X$, and $\overset{\circ}{g} = g((t))$.

On the same occasion we will describe the corresponding crystal $(\overset{\circ}{g})_X$, corresponding to the non-punctured situation. It will be identified with $(g[[t]])_X$.

For $x : S \to X_{\text{dR}}$ recall the schemes $D_x$ and $\overset{\circ}{D}_x \overset{j}{\to} D_x$. We claim that

$(\overset{\circ}{g})_X \simeq p_* (g \otimes O_X \times S |_{D_x})$ and $(\overset{\circ}{g})_X \simeq p_* (j_* \circ j^* (g \otimes O_X \times S |_{D_x}))$.

Indeed, this follows from the fact that the de Rham push-forward of a crystal induced from a quasi-coherent sheaf is canonically isomorphic to the usual push-forward of that quasi-coherent sheaf.

The identifications with $(\overset{\circ}{g}_{g})_X$ and $(g[[t]])_X$ follow from the canonical $\text{Aut}(\overset{\circ}{g})$-equivariant isomorphism

$\text{Coord}(X) \times \overset{\circ}{\Gamma}_x \simeq \text{Coord}(X) \times \text{Spf}(R[[t]])$,

where $S = \text{Spec}(R)$. 
6. The Sugawara identity on the curve

The goal of this section is to explain how the axioms of strong action translate into a certain identity on sections of the D-module on \( X \) underlying the chiral algebra \( \mathcal{A}_{g,k} \). This identity is stated in (6.5) below.

6.1. The action of infinitesimal symmetries. In this and the next subsection we shall take \( \mathcal{C} \) of Sect. 1.1.3 to be DGCat and \( \mathcal{F} \) to be QCoh. However, the same discussion applies to \( \mathcal{C} = 2\text{-Cat} \) and \( \mathcal{F} = \text{ShvDGCat} \).

6.1.1. By Sect. 3.2.4, we have a canonically defined functor

\[
\text{Rep}(\hat{\mathcal{O}}_i) \to \text{QCoh}(X_i), \ V \mapsto V_{X_i}.
\]

Consider the composed functor

\[
(6.1) \quad \text{Rep}(\hat{\mathcal{O}}_i) \to \text{QCoh}(X_i) \to \text{Rep}(\text{Aut}(X)) \to \text{Vect}.
\]

6.1.2. Let \( \text{Aut}(X) \in \text{PreStk} \) denote the group-object of automorphisms of \( X \). This is in fact a group indscheme, locally of finite type. Let \( \text{Der}(X) \) denote its Lie algebra.

We claim:

**Lemma 6.1.3.** The functor (6.1) canonically factors as

\[
\text{Rep}(\hat{\mathcal{O}}_i) \to \text{Rep}(\text{Aut}(X)) \to \text{Vect}.
\]

**Proof.** This follows from the fact that \( \text{Aut}(X) \) acts canonically on \( X_i \) and the \( \hat{\mathcal{O}}_i \)-torsor \( \text{Coord}(X) \), so that the map

\[
X_i \to \text{pt} / \hat{\mathcal{O}}_i
\]

is \( \text{Aut}(X) \)-equivariant.

6.1.4. In what follows, for \( V \in \text{Rep}(\hat{\mathcal{O}}_i) \), we shall regard \( V_{X_i} \) as an object of \( \text{Der}(X) \)-mod. We shall refer to it as the “Lie action”. I.e., we obtain a canonically defined functor

\[
(6.2) \quad \text{Rep}(\hat{\mathcal{O}}_i) \to \text{Der}(X)\text{-mod}.
\]

For example, for \( V \in \text{Rep}(\hat{\mathcal{O}}_i) \) given by the standard character of \( \mathbb{G}_m \) and the canonical projection \( \hat{\mathcal{O}}_i \to \mathbb{G}_m \), the resulting \( V_X \) identifies with \( \omega_X \), and the resulting action of \( \text{Der}(X) \) on \( \Gamma(X,\omega_X) \) is the usual action by Lie derivatives.

The same applies to any of the sheaves on \( X \) (such as \( \text{Diff}_X \)) attached naturally to \( X \).

6.2. Action on crystals.

6.2.1. Let \( \mathcal{M} \) be a left D-module on \( X \), i.e., a crystal. It is well-known that \( \Gamma(X,\mathcal{M}) \) carries a canonical action of \( \text{Der}(X) \). Indeed, it comes by comparing the pullbacks of \( \mathcal{M} \) under the two maps

\[
\text{Aut}(X)^{\hat{\mathcal{O}}_i}_{(1)} \times X \Rightarrow X.
\]
6.2.2. Let us now start with an object \( V \in \text{Rep}(\text{Aut}(\hat{O}_i)) \), and consider the corresponding object \( V_{X_i} \in \text{QCoh}(X_i) \).

On the one hand, we obtain that \( \Gamma(X_i, V_{X_i}) \) carries a Lie action of \( \text{Der}(X) \) as in Sect. 6.1.4. On the other hand, it carries an action of \( \text{Der}(X) \) due to the fact that \( V_{X_i} \) has a crystal structure by Sect. 3.3.4.

It is **not** true that these two actions coincide. Our current goal is to describe the discrepancy between the two.

6.2.3. Consider the formal group \( (\text{Aut}(\hat{O}_i))_{\{1\}} \) equipped with the adjoint action of \( \text{Aut}(\hat{O}_i) \). By Sect. 3.3.4, it gives rise to a crystal of formal groups on \( X_i \); we denote it by \( ((\text{Aut}(\hat{O}_i))_{\{1\}})_{X_i} \).

Consider the group-object

\[
\text{Sect}_{\text{crys}} \left( X_i, ((\text{Aut}(\hat{O}_i))_{\{1\}})_{X_i} \right)
\]

of PreStk of crystalline sections of \( ((\text{Aut}(\hat{O}_i))_{\{1\}})_{X_i} \). I.e., for \( S \in \text{Sch}^{aff} \), we let the group of its \( S \)-points be

\[
\text{Maps}_{\text{crys}/S} \left( S \times X_i, ((\text{Aut}(\hat{O}_i))_{\{1\}})_{X_i} \right).
\]

**Lemma 6.2.4.** **There exists a canonically defined map**

\[
\text{Aut}(X)_{\{1\}} \to \text{Sect}_{\text{crys}} \left( X_i, ((\text{Aut}(\hat{O}_i))_{\{1\}})_{X_i} \right).
\]

6.2.5. Let now \( M \) be a crystalline object of \( ((\text{Aut}(\hat{O}_i))_{\{1\}})_{X_i}-\text{mod}(\text{QCoh}(X_i)) \).

From Lemma 6.2.4 we obtain:

**Corollary 6.2.6.** **The vector space** \( \Gamma(X_i, M) \) **carries a canonical action of** \( \text{Aut}(X)_{\{1\}} \), **which commutes with the** \( \text{Der}(X) \)-**action of Sect. 6.2.1.**

We call the resulting action of \( \text{Der}(X) \) on \( \Gamma(X_i, M) \) “the inner action”.

We obtain that for \( M \) as above, the vector space \( \Gamma(X_i, M) \) canonically upgrades to an object of the category

\[
(\text{Der}(X) \oplus \text{Der}(X))-\text{mod},
\]

where the first copy of \( \text{Der}(X) \) acts via the crystalline structure, and the second copy by the inner action.

6.2.7. Let \( V \) be again an object of \( \text{Rep}(\text{Aut}(\hat{O}_i)) \). Note that the functor

\[
V \mapsto V_{X_i} \in \text{QCoh}_{\text{crys}}(X_i)
\]

of Sect. 3.3.4 canonically upgrades to a functor

\[
\text{Rep}(\text{Aut}(\hat{O}_i)) \to \left( (\text{Rep}(\text{Aut}(\hat{O}_i))_{X,-\text{mod}(\text{QCoh}(X_i))) \right)_{\text{crys}},
\]

and hence, to a functor

\[
\text{Rep}(\text{Aut}(\hat{O}_i)) \to \left( ((\text{Aut}(\hat{O}_i))_{X,-\text{mod}(\text{QCoh}(X_i))) \right)_{\text{crys}}.
\]
Hence, from Sect. 6.2.5, we obtain that the composed functor
\[ \text{Rep}(\text{Aut}(\hat{\mathcal{O}})_i) \to \text{QCoh}_{\text{crys}}(X_i) \to \text{QCoh}(X_i) \xrightarrow{\Gamma(X,-)} \text{Vect} \]
canonically factors through a functor
\[ (6.3) \quad \text{Rep}(\text{Aut}(\hat{\mathcal{O}})_i) \to (\text{Der}(X) \oplus \text{Der}(X))\text{-mod}. \]

6.2.8. We are finally ready to formulate the main result of this subsection:

**Theorem 6.2.9.** The following diagram of functors canonically commutes:
\[
\begin{array}{ccc}
\text{Rep}(\text{Aut}(\hat{\mathcal{O}})_i) & \xrightarrow{(6.3)} & (\text{Der}(X) \oplus \text{Der}(X))\text{-mod} \\
\downarrow & & \downarrow \\
\text{Rep}(\text{Aut}(\hat{\mathcal{O}})_i) & \xrightarrow{(6.2)} & \text{Der}(X)\text{-mod},
\end{array}
\]
where the right vertical arrows corresponds to the diagonal homomorphism
\[ \text{Der}(X) \to \text{Der}(X) \oplus \text{Der}(X). \]

The proof is a straightforward verification.

6.2.10. Thus, Theorem 6.2.9 answers the question posed in Sect. 6.2.2. Namely, the discrepancy between the Lie action and the one coming from the crystal structure is given by the inner action Corollary 6.2.6.

6.3. **Throwing in a category.**

6.3.1. Let now \( D \) be a category equipped with a weak action of \( \text{Aut}(\hat{\mathcal{O}})_i \). Assume that we are given a \( \text{Aut}(\hat{\mathcal{O}})_i \)-equivariant forgetful functor 
\[ F : D \to \text{Vect}, \]
and a (weakly) \( \text{Aut}(\hat{\mathcal{O}})_i \)-equivariant object \( d \in D \).

The composition
\[ \text{Vect} \xrightarrow{d} D \xrightarrow{F} \text{Vect} \]
is thus a (weakly) \( \text{Aut}(\hat{\mathcal{O}})_i \)-equivariant functor \( \text{Vect} \to \text{Vect} \), i.e., an object \( V \in \text{Rep}(\text{Aut}(\hat{\mathcal{O}})_i) \).

6.3.2. In addition, by functoriality, the functor \( F \) gives rise to a map of crystals
\[ D_X \to \text{QCoh}(X), \]
and the object \( d \) to a crystalline object \( d_X \in D_X \).

In particular, \( F_X(d_X) \in \text{QCoh}(X) \) has a natural structure of crystal.

6.3.3. Thus, we obtain that Theorem 6.2.9 describes the relationship between the following three pieces of data on \( \Gamma(X, F_X(d_X)) \):
- The Lie action of \( \text{Der}(X) \) that comes from viewing \( F_X(d_X) \) as associated to \( F(d) \in \text{Rep}(\text{Aut}(\hat{\mathcal{O}})_i) \).
- The action of \( \text{Der}(X) \) that comes from the crystal structure.
- The inner action arising from the \( \text{Der}(\hat{\mathcal{O}})_i \)-action on \( F(d) \).
6.3.4. Let us specialize to the case when \( X \) is a curve, \( n = 0 \) and \( D = \hat{\mathfrak{g}}_{\kappa} \)-mod.

First, we recall that \( \hat{\mathfrak{g}}_{\kappa} \)-mod carries a canonical weak action of \( \text{Aut}(\hat{\mathcal{O}}) \). We take \( F \) to be the usual forgetful functor \( \hat{\mathfrak{g}}_{\kappa} \)-mod \( \rightarrow \) Vect. It is \( \text{Aut}(\hat{\mathcal{O}}) \)-equivariant by construction.

We consider the object \( \text{Vac} \in \hat{\mathfrak{g}}_{\kappa} \)-mod, which is naturally (weakly) \( \text{Aut}(\hat{\mathcal{O}}) \)-equivariant. Note that the resulting D-module \( \text{Vac}_X \) identifies with the Kac-Moody chiral algebra \( A_{g,\kappa} \).

Thus, Theorem 6.2.9 describes the discrepancy between the crystalline and Lie actions of \( \text{Der}(X) \) on \( \Gamma(X,A_{g,\kappa}) \). In the next subsection we shall describe the discrepancy, given by the inner action in terms of the Sugawara construction.

6.4. **The Sugawara construction at the level of chiral algebras.**

6.4.1. Let us recall that the Sugawara construction on the curve defines a map of Lie-* algebras

\[ \phi^{\text{ch}} : \Theta'_X \rightarrow A_{g,\kappa} , \]

where \( \Theta'_X \) is a certain canonical central extension

\[ 0 \rightarrow \omega_X \rightarrow \Theta'_X \rightarrow \Theta_X \rightarrow 0 , \]

such that \( \phi^{\text{ch}} \) sends \( \omega_X \subset \Theta'_X \) identically to \( \omega_X \subset A_{g,\kappa} \). (Here \( \Theta_X \) is the Lie-* algebra associated to the algebroid of vector fields on \( X \).)

We are now going to formulate the relationship between the homomorphism \( \phi^{\text{ch}} \) and the inner action of \( \text{Der}(X) \) on \( \Gamma(X,A_{g,\kappa}) \).

6.4.2. Let \( T'_X \) (resp., \( \overset{\circ}{T}'_X \)) denote the crystal of topological Lie algebras associated to \( \Theta' \) by the procedure of Sect. 4.4. Let \( (T'_{tw})_X \) (resp., \( (\overset{\circ}{T}'_{tw})_X \)) be their modifications as in Sect. 5.3.4 obtained by pushing out the unit.

Note that \( (T'_{tw})_X \) identifies with the crystal of topological Lie algebras \( \text{Der}(\hat{\mathcal{O}})_X \) on \( X \), obtained from \( \text{Der}(\hat{\mathcal{O}}) \) by the procedure of Sect. 3.3.4.

I.e., it is the crystal of topological Lie algebras that corresponds to the crystal of formal groups

\[ (\text{Aut}(\hat{\mathcal{O}}))^{\wedge}_1 \]

of Sect. 6.2.3.

6.4.3. Thus, on the one hand, from (6.4), for any \( S \rightarrow X_{\text{dR}} \) and any \( M \in A_{g,\kappa}\text{-mod}(\text{QCoh}(S)) \), we obtain a canonical action of \( T'_X(S) \) on \( M \), and in particular, an action of \( T'_X(S) \).

Let us take \( S = X \). And let us take \( M = V_X \), for \( V \) being a weakly \( \text{Aut}(\hat{\mathcal{O}})_1 \)-equivariant object of \( \hat{\mathfrak{g}}_{\kappa} \)-mod. Then from Sect. 6.2.7, we obtain an action of \( T'_X(S) \simeq \text{Der}(\hat{\mathcal{O}})_X \) on \( V_X \).

6.4.4. The sought-for relationship between \( \phi^{\text{ch}} \) and the inner action of \( \text{Der}(X) \) on objects of the form \( V_X \) is given by the following proposition:

**Proposition 6.4.5.** Assume that the structure of weakly \( \text{Aut}(\hat{\mathcal{O}})_1 \)-equivariant object on \( V \) comes from a structure of strong \( \text{Aut}(\hat{\mathcal{O}})_1 \)-equivariance. Then the resulting two actions of \( T'_X(S) \) on \( V_X \) are canonically homotopic.

The proposition will be proved in the next subsection.
6.4.6. Let us again take \( i = 0 \) and \( V = \text{Vac} \), so that \( V_X = A_{g, \kappa} \). We obtain the following concrete description of the inner action of \( \text{Der}(X) \) on \( \Gamma(X, A_{g, \kappa}) \).

Namely, let \( \xi \in \text{Der}(X) \) be vector field on \( X \), and let \( \xi' \) be its arbitrary lift to a section of \( \Theta_X' \). Let \( a \) be a section of \( \Gamma(X, A_{g, \kappa}) \), where \( A_{g, \kappa} \) is the left \( D \)-module corresponding to \( A_{g, \kappa} \).

We have
\[
\xi \cdot a = (h \boxtimes \text{Id}) \{ \phi^{ch}(\xi') \boxtimes a \},
\]
where \( h \boxtimes \text{Id} \) denotes the projection \( \Gamma(X \times X, \Delta_{dR} \ast (A_{g, \kappa}) \to \Gamma(X, A_{g, \kappa}) \).

Here we regard \( \Delta_{dR} \ast (A_{g, \kappa}) \) as an object of the category of right \( D \)-modules with respect to the first coordinate, and left \( D \)-modules, with respect to the second coordinate.

To summarize, we obtain the following relation:
(6.5)
\[
\xi \text{ Lie} \cdot a = \xi \text{ crys} \cdot a + (h \boxtimes \text{Id}) \{ \phi^{ch}(\xi') \boxtimes a \}.
\]

6.5. **Chiral vs. categorical versions of the Sugawara construction.** In order to prove Proposition 6.4.5 we will first have to understand how the Sugawara construction at the level of chiral algebras, given by homomorphism \( \phi^{ch} \), is related to the categorical Sugawara construction as formulated in Theorem 5.2.2.

6.5.1. Consider the topological associative algebra \( \hat{U}(\widehat{g}_{\kappa})_{tw} \), where the subscript “tw” refers to the fact that we identify the unit of the enveloping algebra with the element \( 1 \in \widehat{g}_{\kappa} \), and \( \hat{U} \) refers to the fact that we are taking an appropriate completion.

By Proposition 2.7.8, adapted to the non-finite type situation, the datum of upgrading the weak action of \( \text{Aut}(\hat{\mathcal{O}}) \) on \( \hat{g}_{\kappa} \)-mod to a strong action amounts to a homomorphism of Lie algebras
\[
\phi : \text{Der}(\hat{\mathcal{O}}) \to \hat{U}(\widehat{g}_{\kappa})_{tw},
\]
satisfying some natural conditions. (These are indeed conditions and not additional pieces of structure, since our objects all lie in cohomological degree 0 and \( \text{Aut}(\hat{\mathcal{O}}) \) is connected.)

6.5.2. Let \( (\hat{U}(\widehat{g}_{\kappa})_{tw})_X \) be the crystal of topological associative algebras on \( X \), corresponding to \( \hat{U}(\widehat{g}_{\kappa})_{tw} \) via Sect. 3.3.4. Let us apply Sect. 3.3.4 to the homomorphism \( \phi \). We obtain a map of crystals of topological Lie algebras
\[
\phi_X : (\mathcal{T}_{tw})_X \simeq \text{Der}(\hat{\mathcal{O}})_X \to (\hat{U}(\widehat{g}_{\kappa})_{tw})_X.
\]

By construction, the crystal \( (\hat{U}(\widehat{g}_{\kappa})_{tw})_X\text{-mod}(\text{QCoh}(X)) \) identifies with \( \widehat{g}_{\kappa}\text{-mod}_X \simeq A_{g, \kappa}\text{-mod}_X \), in a way compatible with the forgetful functor to \( \text{QCoh}(X) \).

Hence, we obtain that when we view \( A_{g, \kappa} \) as Lie-* algebra, we have a canonical map of topological Lie algebras
\[
(\mathfrak{A}_{tw})_X \to (\hat{\mathfrak{A}}_\mathcal{I})_X \to (\hat{U}(\widehat{g}_{\kappa})_{tw})_X,
\]
where \( \mathfrak{A}_X \) and \( \hat{\mathfrak{A}}_X \) are crystals of topological Lie algebras associated to \( A_{g, \kappa} \) by the procedure of Sect. 4.4, and \( (\mathfrak{A}_{tw})_X \) and \( (\hat{\mathfrak{A}}_\mathcal{I})_X \) are their respective versions obtained by pushing out the units as in Sect. 5.3.4.
6.5.3. The sought-for relationship between $\phi^\text{ch}$ and $\phi$ is formulated as follows:

**Lemma 6.5.4.** The composed map

$$(\mathcal{T}^\prime_{tw})_X \xrightarrow{\phi^\text{ch}} (A_{tw})_X \rightarrow (\hat{U}(\hat{g}_\kappa)_{tw})_X$$

identifies with $\phi_X$.

6.5.5. In view of Lemma 6.5.4, the assertion of Proposition 6.4.5 follows from Lemma 2.7.12.

**References**


