# MATH 224, PROBLEM SET 1.

**1.** Let $G$ be a linear algebraic group and $V$ a vector space.

(a) Give a complete proof of the fact that the structure on $V$ of $G$-representation is equivalent to that of co-action of $\mathcal{O}_G$.

(b) Assume that $V$ is finite-dimensional. Show that the structure on $V$ of $G$-representation is also equivalent to the datum of homomorphism $G \to GL(V)$.

(c) Consider the algebraic variety $\mathbb{V}$ attached to $V$ (or, rather, ind-scheme so that we don't need to assume that $V$ is finite-dimensional). Show that the vector space structure on $V$ gives rise to the maps $\mathbb{V} \times \mathbb{V} \to \mathbb{V}$ and $\mathbb{A}^1 \times \mathbb{V} \to \mathbb{V}$. Show that the structure of $G$-representation on $V$ is equivalent to that of action of $G$ on $\mathbb{V}$ that commutes with the above two pieces of structure on $\mathbb{V}$.

**2.** Let $\mathbf{C}$ be a category, and let $\mathrm{TwArr}(\mathbf{C})$ denote the categories of twisted arrows in $\mathbf{C}$. Its objects are $\mathbf{c}_0 \overset{\phi}{\to} \mathbf{c}_1$, and its morphisms are commutative diagrams

$$\begin{array}{ccc} \mathbf{c}_0 & \xrightarrow{\phi} & \mathbf{c}_1 \\ \big\uparrow & & \big\downarrow \\ \mathbf{c}'_0 & \xrightarrow{\phi'} & \mathbf{c}'_1. \end{array}$$

(a) Recall that the Groethendieck construction associates to a category $\mathbf{D}$ and a functor $F$ : $\mathbf{D} \to \mathrm{Sets}$ a category $\widetilde{\mathbf{D}}$ equipped with a functor $\widetilde{\mathbf{D}} \to \mathbf{D}$, such that the category-fiber over each $\mathbf{d} \in \mathbf{D}$ is the set $F(\mathbf{d})$ viewed as a discrete category. Take $\mathbf{D} = \mathbf{C} \times \mathbf{C}^{\mathrm{op}}$ and $F(\mathbf{c}_0, \mathbf{c}_1) = \mathrm{Hom}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)$. Show that the resulting category $\widetilde{\mathbf{D}}$ identifies with $\mathrm{TwArr}(\mathbf{C})$.

(b) Let $F', F''$ be two functors from $\mathbf{C}$ to some category $\mathbf{D}$. Define the functor

$$H : \mathrm{TwArr}(\mathbf{C}) \to \mathrm{Sets}$$

by setting $H(\mathbf{c}_0 \overset{\phi}{\to} \mathbf{c}_1) = \mathrm{Hom}_{\mathbf{D}}(F'(\mathbf{c}_0), F''(\mathbf{c}_1))$. Construct an isomorphism between the set $\mathrm{Hom}_{\mathrm{Funct}(\mathbf{C},\mathbf{D})}(F', F'')$ and $\lim\limits_{\mathrm{TwArr}(\mathbf{C})} H$.

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