Mathematical Sequences
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In mathematics, informally speaking, a sequence is an ordered list of objects (or events). Like a set, it contains members (also called elements, or terms). The number of ordered elements (possibly infinite) is called the length of the sequence. Unlike a set, order matters, and exactly the same elements can appear multiple times at different positions in the sequence. Most precisely, a sequence can be defined as a function whose domain is a countable totally ordered set, such as the natural numbers.

For example, (M, A, R, Y) is a sequence of letters with the letter 'M' first and 'Y' last. This sequence differs from (A, R, M, Y). Also, the sequence (1, 1, 2, 3, 5, 8), which contains the number 1 at two different positions, is a valid sequence. Sequences can be finite, as in this example, or infinite, such as the sequence of all even positive integers (2, 4, 6,...). Finite sequences are sometimes known as strings or words and infinite sequences as streams. The empty sequence ( ) is included in most notions of sequence, but may be excluded depending on the context.

Examples and notation

A sequence can be thought of as a list of elements with a particular order. Sequences are useful in a number of mathematical disciplines ... In particular, sequences are the basis for series, which are important in differential equations and analysis. Sequences are also of interest in their own right and can be studied as patterns or puzzles, such as in the study of prime numbers.

There are a number of ways to denote a sequence, some of which are more useful for specific types of sequences. One way to specify a sequence is to list the elements. For example, the first four odd numbers form the sequence (1,3,5,7). This notation can be used for infinite sequences as well. For instance, the infinite sequence of positive odd integers can be written (1,3,5,7,...). Listing is most useful for infinite sequences with a pattern that can be easily discerned from the first few elements. Other ways to denote a sequence are discussed after the examples.

Important examples

A tiling with squares whose sides are successive Fibonacci numbers in length.
There are many important integer sequences. The **prime numbers** are numbers that have no **divisors** but 1 and themselves. Taking these in their natural order gives the sequence  
(2,3,5,7,11,13,17,...). The study of prime numbers has important applications for **mathematics** and specifically **number theory**.

The **Fibonacci numbers** are the integer sequence whose elements are the sum of the previous two elements. The first two elements are either 0 and 1 or 1 and 1 so that the sequence is  
(0,1,1,2,3,5,8,13,21,34,...).

For a list of important examples of integer sequences see **On-line Encyclopedia of Integer Sequences**.

Other important examples of sequences include ones made up of **rational numbers**, **real numbers**, and **complex numbers**. The sequence (.9,.99,.999,.9999,...) approaches the number 1. In fact, every real number can be written as the **limit of a sequence** of rational numbers. For instance, for a sequence (3,3.1,3.14,3.141,3.1415,...) the **limit of a sequence** can be written as \( \pi \). It is this fact that allows us to write any real number as the limit of a sequence of **decimals**. The decimal for \( \pi \), however, does not have any pattern like the one for the sequence (0.9,0.99,...).

### Indexing

Other notations can be useful for sequences whose pattern cannot be easily guessed, or for sequences that do not have a pattern such as the digits of \( \pi \).

The terms of a sequence are commonly denoted by a single variable, say \( a_n \), where the **index** \( n \) indicates the \( n \)th element of the sequence.

\[
\begin{align*}
    a_1 & \leftrightarrow \text{1st element} \\
    a_2 & \leftrightarrow \text{2nd element} \\
    \vdots & \vdots \\
    a_{n-1} & \leftrightarrow (n-1)\text{th element} \\
    a_n & \leftrightarrow n\text{th element} \\
    a_{n+1} & \leftrightarrow (n+1)\text{th element} \\
    \vdots & \vdots \\
\end{align*}
\]

Indexing notation is used to refer to a sequence in the abstract. It is also a natural notation for sequences whose elements are related to the index \( n \) (the element's position) in a simple way. For instance, the sequence of the first 10 square numbers could be written as

\[
(a_1, a_2, \ldots, a_{10}), \quad a_k = k^2.
\]

This represents the sequence (1,4,9,...100). This notation is often simplified further as
\[(a_k)_{k=1}^{10}, \quad a_k = k^2.\]

Here the subscript \(\{k=1\}\) and superscript 10 together tell us that the elements of this sequence are the \(a_k\) such that \(k = 1, 2, \ldots, 10.\)

Sequences can be indexed beginning and ending from any integer. The infinity symbol \(\infty\) is often used as the superscript to indicate the sequence including all integer \(k\)-values starting with a certain one. The sequence of all positive squares is then denoted

\[(a_k)_{k=1}^{\infty}, \quad a_k = k^2.\]

In cases where the set of indexing numbers is understood, such as in analysis, the subscripts and superscripts are often left off. That is, one simply writes \(a_k\) for an arbitrary sequence. In analysis, \(k\) would be understood to run from 1 to \(\infty.\) However, sequences are often indexed starting from zero, as in

\[(a_k)_{k=0}^{\infty} = (a_0, a_1, a_2, \ldots).\]

In some cases the elements of the sequence are related naturally to a sequence of integers whose pattern can be easily inferred. In these cases the index set may be implied by a listing of the first few abstract elements. For instance, the sequence of squares of odd numbers could be denoted in any of the following ways.

- \((1, 9, 25, \ldots)\)
- \((a_1, a_3, a_5, \ldots), \quad a_k = k^2\)
- \((a_{2k-1})_{k=1}^{\infty}, \quad a_k = k^2\)
- \((a_k)_{k=1}^{\infty}, \quad a_k = (2k - 1)^2\)
- \(((2k - 1)^2)_{k=1}^{\infty}\)

Moreover, the subscripts and superscripts could have been left off in the third, fourth, and fifth notations if the indexing set was understood to be the natural numbers.

Finally, sequences can more generally be denoted by writing a set inclusion in the subscript, such as in

\[(a_k)_{k\in \mathbb{N}}\]