Math S305
Advanced Algebra and Trigonometry!

Sequences continued!
Fifth Class – Wednesday July 2nd

• POTD
  • A chocolate game?!  
• Final thoughts about sequences
  • ...what about Fibonacci?  
  • ...what about other recursive sequences?  
• And now... something different?!  
  • Singapore Math!
What do you make of this?!
How about this?!
Now another POTD!

**A simple yet tasty game!**

My boys are fighting over a chocolate bar... and use a game to see who gets it

Who should go first?!
Sequences – descriptions

What about something this simple?

\[ 0.9 \quad 0.99 \quad 0.999 \quad 0.9999 \quad 0.99999 \quad \ldots \]

Looks suspiciously like it’s heading towards...

Okay, so define a new term!

**Convergence!**

A sequence \( \{a_n\}_{n \in \mathbb{N}} \) converges to \( L \) if...

\[ \forall \varepsilon > 0 \quad \exists B \in \mathbb{N} \quad s.t. \forall n \geq B \quad |a_n - L| < \varepsilon \]
Closed formulas – not always!

Here’s a pretty simple sequence...

\((a_k)_{k \in \mathbb{N}}\) \(a_k = 2k - 1\)

What about \(P_n = n^{th}\) prime number

Back to \(P_k = \text{first } k \text{ digits of } \pi\)

3 3.1 3.14 3.1415 …

What is \(P_{100000000000000} = P_{100\text{ trillion}}?\)
Major classes of sequences

**GEOMETRIC SEQUENCES...**

*constant ratio between consecutive terms*

**ARITHMETIC SEQUENCES...**

*constant difference between consecutive terms*

Can we sum them?

...**Series!**
What about finite sequences? 
...can’t we sum them at least?

**GEOMETRIC SEQUENCES...**

*starting with 1, with ratio r...*

**ARITHMETIC SEQUENCES...**

*starting with 0, with difference d...*

And now vary the starting point...
Off to Infinity! Summing series

consider an arithmetic sequence...

still just starting with 0, with difference d...

take a look at the formula!

Now consider a geometric sequence...

starting with 1, with ratio r...
Off to Infinity! and more!

...does every series sum to a finite number if its sequence of terms converges to 0?

Say hello to the **Harmonic Series**!
The tipping point – hard to tell!

But if you tamper with the Harmonic Series ever so slightly...

What about a “depleted” Harmonic Series?

Toss out all the terms with a 9 in the denominator...

\[ S = \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{8} + \frac{1}{10} + ... + \frac{1}{18} + \frac{1}{20} + ... + \frac{1}{88} + \frac{1}{100} + \frac{1}{101} + ... \]
The Comparison Test et al.

The first several convergence tests are relatively self-explanatory...

If $a_n$ is a sequence such that $\sum_{n=1}^{\infty} a_n$ exists, and $0 < b_n < a_n$ for all $n$, then $\sum_{n=1}^{\infty} b_n$ exists too.

can do a bit more by considering absolute values of the terms, and also just need the comparison to eventually happen for large enough $n$...
The Ratio Test

essentially compares a sequence to a geometric series, which converges if the ratio is < 1...

If \( a_n \) is a sequence such that \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) exists and is < 1, then \( \sum_{n=1}^{\infty} a_n \) exists too (i.e. the series has a finite sum).

oops – we need to fix this every so slightly – see the hole in this test? Think negative thoughts (instead of “thinking positive thoughts”!)
And now for something different...

Suppose \( a_1 = 1 \) \( a_2 = 1 \)

and \( a_3 = a_1 + a_2 \)

...and then \( a_n = a_{n-1} + a_{n-2} \)

**RECURSIVE FORMULAS!**

The Fibonacci sequence is an example of a linear recursive sequence...

*Is it arithmetic, or geometric?*
Given that each term in the Fibonacci sequence is defined using a linear recursive sequence... *then to find the* $n^{th}$ *term don’t you have to compute all the ones before it?*

*So does that mean there’s no closed formula?*
But there are some obvious patterns, right?

The infamous Fibonacci–Pascal connection!
Time for an amazing number!

Can you find a number $X$ with a similarly recursive pattern...?

Find $X$ so that $X^n = X^{n-2} + X^{n-1}$!

If you find such an $X$ then any sequence of the form $a_k = X^k$ will follow the recursive constraint...

In fact any sequence of the form $a_k = C X^k$ (where $C$ is any constant) will work just as well!
Time for an amazing number!

So can you find a number $X$ with a similarly recursive pattern...?

Find $X$ so that $X^n = X^{n-2} + X^{n-1}$!

Aha! $\phi = \frac{1 + \sqrt{5}}{2}$

and phi’s (somewhat negative!) sidekick...

$\psi = \frac{1 - \sqrt{5}}{2}$
Now put them to work!

Big idea, yes the following two sequences obey the recursive relation:

\[(\varphi^n)_{n \in \mathbb{N}} \quad \text{and} \quad (\psi^n)_{n \in \mathbb{N}}\]

But now think about what happens to a sequence defined by their sum...

\[s_n = \varphi^n + \psi^n\]
Now put them to work!

In fact any sequence of the form

\[ s_n = A \phi^n + B \psi^n \]

with constants \( A \) and \( B \) will fulfill the recursive constraint...!

Could we find \( A \) and \( B \) so that the sequence \( s_n = A \phi^n + B \psi^n \) begins with

\[ 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad \ldots \]