Computing with Elliptic Surfaces:  
A bit of theoretical background

AMS Short Course  
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Noam D. Elkies  
Harvard University
Overview

**Elliptic surface** [over \( P^1 \)]: birationally the curve 
\((a_1, a_2, a_3, a_4, a_6)\) over \( k(t) \), i.e.

\[Y^2 + a_1(t)XY + a_3(t)Y = X^3 + a_2(t)X^2 + a_4(t)X + a_6(t)\]

with each \( a_i(t) \) a polynomial (really homog. poly. in \((t : 1)\)) of degree \( i \cdot \nu \).

NB often useful to **NOT** mechanically go to narrow Weierstrass form \((a_1 = a_2 = a_3 = 0)\)
even in characteristic zero when such a form is always available. E.g. the universal curve with
7-torsion [Tate 1966]:

\[(a_1, a_2, a_3, a_4, a_6) = (1 + d - d^2, d^2 - d^3, d^2 - d^3, 0, 0)\]

with torsion point at \((0, 0)\); narrow Weierstrass form is **MUCH** less pleasant!
Standard picture/cartoon of an elliptic surface:

\[ E_{t_3} \quad E_{t_2} \quad E_{t_1} \]

\[ \pi \ (\text{a.k.a. } t) \]

\[ s_0 \quad s \]

\[ t_3 \quad t_2 \quad t_1 \]
To make a very beautiful but long story short:

This picture, together with intersection theory on $\mathcal{E}$, yields a Euclidean lattice $N_{\text{ess}}$ of rank $\rho - 2$, the “essential lattice” of the elliptic surface: $N_{\text{ess}}$ is the orthogonal complement in $\text{NS}(\mathcal{E})\langle -1 \rangle$ of the indefinite lattice spanned by $s_0$ and the fiber $E_t$. (Here $\rho$ is the Picard number, that is, the rank of $\text{NS}(\mathcal{E})$; and “$\langle -1 \rangle$” means: multiply inner product by $-1$.) The essential lattice is even: $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in N_{\text{ess}}$. 
The vertical divisors (components of reducible fibers) generate a sublattice $R$ of $N_{\text{ess}}$, each fiber contributing a factor $A_n$, $D_n$, or $E_n$ according to its Kodaira type, and thus computable via Tate’s algorithm. If $\nu \geq 2$ this is also the root lattice of $N_{\text{ess}}$ (the $\mathbb{Z}$-span of vectors of norm 2); when $\nu = 1$, there may be roots not in $R$: in this case $N_{\text{ess}} \cong E_8$ always, but usually $R$ is strictly smaller than $E_8$.

It follows that the MW rank is $\rho - 2 - \text{rank}(R)$. The pairing (inner product) on $(N_{\text{ess}}/R) \otimes \mathbb{Q}$ gives canonical height, and

$$|\text{disc}(\text{NS})| = |T|^{-2} \text{disc}(R) \cdot \text{Regulator}$$

(BSD/Artin–Tate).