Computing with Elliptic Surfaces: A bit of theoretical background

AMS Short Course
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Noam D. Elkies
Harvard University

Overview

Elliptic surface [over $\mathbf{P}^1$]: birationally the curve $(a_1, a_2, a_3, a_4, a_6)$ over $k(t)$, i.e.

$$Y^2 + a_1(t)XY + a_3(t)Y = X^3 + a_2(t)X^2 + a_4(t)X + a_6(t)$$

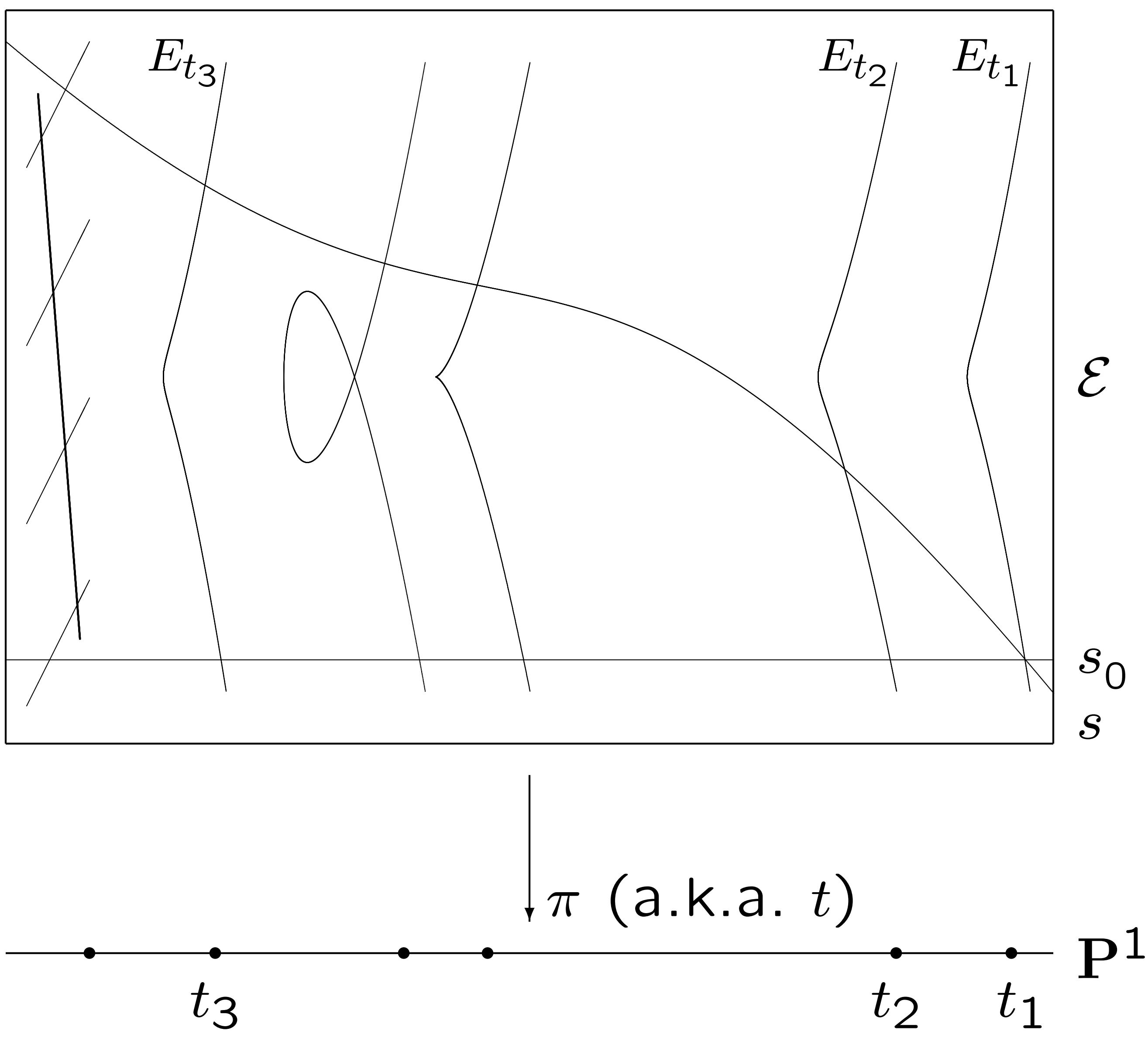
with each $a_i(t)$ a polynomial (really homog. poly. in $(t:1)$) of degree $i \cdot \nu$.

NB often useful to **NOT** mechanically go to narrow Weierstrass form ($a_1 = a_2 = a_3 = 0$) even in characteristic zero when such a form is always available. E.g. the universal curve with 7-torsion [Tate 1966]:

$$(a_1, a_2, a_3, a_4, a_6) = (1 + d - d^2, d^2 - d^3, d^2 - d^3, 0, 0)$$

with torsion point at $(0, 0)$; narrow Weierstrass form is **MUCH** less pleasant!

Standard picture/cartoon of an elliptic surface:



To make a very beautiful but long story short:

This picture, together with intersection theory on $\mathcal{E}$, yields a Euclidean lattice N_{ess} of rank $\rho - 2$, the "essential lattice" of the elliptic surface: N_{ess} is the orthogonal complement in $\mathrm{NS}(\mathcal{E})\langle -1 \rangle$ of the indefinite lattice spanned by s_0 and the fiber E_t. (Here ρ is the Picard number, that is, the rank of $\mathrm{NS}(\mathcal{E})$; and "$\langle -1 \rangle$" means: multiply inner product by -1.) The essential lattice is **even**: $\langle x, x \rangle \in 2\mathbf{Z}$ for all $x \in N_{\mathrm{ess}}$.

The vertical divisors (components of reducible fibers) generate a sublattice R of N_{ess}, each fiber contributing a factor A_n, D_n, or E_n according to its Kodaira type, and thus computable via Tate's algorithm. If $\nu \geq 2$ this is also the root lattice of N_{ess} (the $\mathbf{Z}$-span of vectors of norm 2); when $\nu = 1$, there may be roots not in R: in this case $N_{\text{ess}} \cong E_8$ always, but usually R is strictly smaller than E_8.

It follows that the MW rank is $\rho - 2 - \text{rank}(R)$. The pairing (inner product) on $(N_{\text{ess}}/R) \otimes \mathbf{Q}$ gives canonical height, and

$$|\text{disc(NS)}| = |T|^{-2}\, \text{disc}(R) \cdot \text{Regulator}$$

(BSD/Artin–Tate).