Yet another proof of the series in x for the powers y^β of the solution of $y(1-y)^t = x$

Noam D. Elkies

Proposition. *Fix t, and for each $\beta \notin \{0, -1, -2, -3, \ldots\}$ let $S_{t,\beta}(x)$ be the power series*

$$S_{t,\beta}(x) := \sum_{j=0}^{\infty} \frac{\beta}{j+\beta} \binom{t(j+\beta)+j-1}{j} x^{j+\beta}.$$

Then $S_{t,\beta}(x) = y^\beta$, where $y = S_{t,1}(x)$ is the unique solution of $y(1-y)^t = x$ in $(\mathbf{Q}(t))[[x]]$.

Proof: It is enough to check that $S_{t,\beta}\big(y(1-y)^t\big) = y^\beta$. We have

$$\begin{aligned} S_{t,\beta}\big(y(1-y)^t\big) &= \sum_{j=0}^{\infty} \frac{\beta}{j+\beta} \binom{t(j+\beta)+j-1}{j} y^{j+\beta} (1-y)^{t(j+\beta)} \\ &= \sum_{j=0}^{\infty} \frac{\beta}{j+\beta} \binom{t(j+\beta)+j-1}{j} y^{j+\beta} \sum_{k=0}^{\infty} (-1)^k \binom{t(j+\beta)}{k} y^t \\ &= y^\beta + \beta \sum_{m=1}^{\infty} y^{m+\beta} \sum_{j=0}^{m} \frac{(-1)^{m-j}}{j+\beta} \binom{t(j+\beta)+j-1}{j} \binom{t(j+\beta)}{m-j}, \end{aligned}$$

where we took $m = j+k$ in the last transformation. Using the factorial formula for combinatorial coefficients we rewrite this as

$$S_{t,\beta}\big(y(1-y)^t\big) = y^\beta + \beta t \sum_{m=1}^{\infty} \frac{y^{m+\beta}}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \frac{\big(t(j+\beta)+j-1\big)!}{\big(t(j+\beta)+j-m\big)!}.$$

But the inner sum is the m-th finite difference of

$$\frac{\big(t(j+\beta)+j-1\big)!}{\big(t(j+\beta)+j-m\big)!} = \prod_{i=1}^{m-1} \big(t(j+\beta)+j-i\big),$$

which as a function of j is a polynomial of degree $m-1$. Hence its m-th finite difference vanishes. We conclude that $S_{t,\beta}\big(y(1-y)^t\big) = y^\beta$, as desired. $\diamondsuit$

Remark: This approach was suggested by R. Stanley's proof of the power series for the solution of $ye^{-y} = x$ (see the bottom of page 28 of *Enumerative Combinatorics*, Vol. 2). That proof can be recovered from ours by setting $\beta = 1$ and taking a scaled limit as $k \to \infty$.