The power series for the inverse function of $y(1-y)^t$

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Fix t (usually an integer, but may be real or even complex); For small x let $y(x)$ be the small solution of $y(1-y)^t = x$. Let a_n be the coefficient of the power series expansion

$$y = \sum_{n=1}^{\infty} a_n x^n.$$

For $t = 2$, the a_n are Catalan numbers; larger integers t yield generalized Catalan numbers. The formula

$$a_n = \frac{1}{n}\binom{(t+1)n-2}{n-1} \tag{1}$$

is usually proved using the residue theorem. The following derivation uses only elementary calculus, and as an added bonus gives the power series expansion of y^β for all β.

Differentiate both sides of $y(1-y)^t = x$ to find

$$1 = \Big(1 - \frac{ty}{1-y}\Big)(1-y)^t y' = \frac{1-(t+1)y}{1-y}\,\frac{x}{y}\,y'.$$

Multiply by $(1-y)y^\beta$ to obtain

$$y^\beta - y^{1+\beta} = x\big(y^{\beta-1} - (t+1)y^\beta\big)y' = x\frac{d}{dx}\left(\frac{y^\beta}{\beta} - (t+1)\frac{y^{\beta+1}}{\beta+1}\right) \tag{2}$$

(for $\beta \neq 0, -1$). Now recall that the operator $x\,d/dx$ takes any power series $\sum_n a_n x^n$ to $\sum_n n a_n x^n$. Thus if for any c we expand y^c in a power series

$$y^c = \sum_{m=0}^{\infty} a_{m+c}(c) x^{m+c}$$

then our identity (2) amounts to

$$a_{m+\beta}(\beta) - a_{m+\beta}(1+\beta) = (m+\beta)\left(\frac{a_{m+\beta}(\beta)}{\beta} - (t+1)\frac{a_{m+\beta}(1+\beta)}{1+\beta}\right),$$

which simplifies to the recursion

$$a_{m+\beta}(\beta) = \frac{\beta}{1+\beta}\,\frac{t(m+\beta)+m-1}{m}\,a_{m+\beta}(1+\beta).$$

Applying this m times, we find

$$a_{m+\beta}(\beta) = \frac{\beta}{m+\beta}\binom{t(m+\beta)+m-1}{m}a_{m+\beta}(m+\beta).$$

But $a_{m+\beta}(m+\beta)$ is the leading coefficient of $y^{m+\beta}$, and thus equals 1. We conclude that

$$a_{m+\beta}(\beta) = \frac{\beta}{m+\beta}\binom{t(m+\beta)+m-1}{m}.$$

In particular, taking $\beta = 1$ and $m = n-1$ we recover the formula (1) for the x^n coefficient $a_n = a_n(1)$ of y.