Math 229: Introduction to Analytic Number Theory

The product formula for \( \xi(s) \) and \( \zeta(s) \); vertical distribution of zeros

Behavior on vertical lines. We next show that \((s^2 - s)\xi(s)\) is an entire function of order 1; more precisely:

**Lemma.** There exists a constant \( C \) such that \((s^2 - s)\xi(s) \ll \exp(C|s|\log|s|)\), but no constant \( C' \) such that \((s^2 - s)\xi(s) \ll \exp(C'|s|)\).

**Proof:** The second part is easy, because \( \xi(s) \) already grows faster than \( \exp(C'|s|) \) as \( s \to \infty \) along the positive real axis. Indeed for \( s > 1 \) we have \( \xi(s) > 1 \), while the factor \((s^2 - s)\pi^{-s/2}\Gamma(s/2)\) of \((s^2 - s)\xi(s)\) grows faster than \( \exp(C'|s|) \) by Stirling.

For the first part, since \( |s|\log|s| \) is bounded below we need only give an upper bound on \((s^2 - s)\xi(s)\) for large \( |s| \), and the functional equation \( \xi(s) = \xi(1-s) \) lets us assume that \( s = \sigma + it \) with \( \sigma \geq 1/2 \). Moreover, the sum and integral formulas for \( \xi(s) \) and \( \Gamma(s) \) yield \( |\xi(\sigma + it)| \leq \pi^{-\sigma/2}\Gamma(\sigma/2)|\zeta(\sigma)| \) whenever \( \sigma > 1 \). By Stirling we readily conclude that \((s^2 - s)\xi(s) \ll \exp(C|s|\log|s|)\) holds if \( s \) is far enough from the critical strip, say in the half-plane \( \sigma \geq 2 \). To extend this bound to \( \sigma \geq 1/2 \), it will be more than enough to prove that \( \zeta(s) \ll |t| \) for \( \sigma \geq 1/2 \) and \( |t| \geq 1 \); and this follows from our formula

\[
\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \int_{n}^{n+1} (n^{-s} - x^{-s}) \, dx
\]

for \( \zeta(s) \) as a meromorphic function on the half-plane \( \sigma > 0 \), in which we saw that the sum is \( O(|s|) \). □

Remarks: While the bound \( \zeta(s) \ll |t| \) on the critical strip (away from \( s = 1 \)) is more than sufficient for our present purpose, we can do considerably better. To begin with, we can choose some \( N \) and leave alone the terms with \( n < N \) in \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \), using the integral approximation only for the \( n \geq N \) terms:

\[
\zeta(s) = \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + \sum_{n=N}^{\infty} \int_{n}^{n+1} (n^{-s} - x^{-s}) \, dx.
\]

For large \( t, N \) this is \( \ll N^{1-\sigma} + |t|N^{-\sigma} \), uniformly at least for \( \sigma \geq 1/2 \). Taking \( N = |t| + O(1) \) we find \( \zeta(\sigma + it) \ll |t|^{1-\sigma} \) for \( \sigma \geq 1/2, |t| > 1 \). At the end of this chapter we describe further improvements. Meanwhile, note our choice of \( N = |t| + O(1) \), which may seem suboptimal. We wanted to make the bound as good as possible, that is, to minimize \( N^{1-\sigma} + |t|N^{-\sigma} \). In calculus we learned to minimize such an expression by setting its derivative equal to zero. That would give \( N \) proportional to \( |t| \), but we arbitrarily set the constant of proportionality to 1 even though another choice would make \( N^{1-\sigma} + |t|N^{-\sigma} \) somewhat smaller. In general when we bound some quantity by a sum \( O(f(N) + g(N)) \) of an increasing and a decreasing function of some parameter \( N \), we shall simply choose \( N \) so that \( f(N) = g(N) \) (or, if \( N \) is constrained to be an integer, so that...
\(f(N)\) and \(g(N)\) are nearly equal. This is much simpler and less error-prone than fumbling with derivatives, and is sure to give the minimum to within a factor of 2, which is good enough when we are dealing with \(O(\cdots)\) bounds.

**Product and logarithmic-derivative formulas.** By our general product formula for an entire function of finite order we know that \(\xi(s)\) has a product expansion:

\[
\xi(s) = e^{A + Bs} \prod_{\rho} (1 - (s/\rho)) e^{s/\rho},
\]

for some constants \(A, B\), with the product ranging over zeros \(\rho\) of \(\xi\) (that is, the nontrivial zeros of \(\zeta\)) listed with multiplicity. Moreover, \(\sum_{\rho} |\rho|^{-1 - \epsilon} < \infty\) for all \(\epsilon > 0\), but \(\sum_{\rho} |\rho|^{-1} = \infty\) because \(\xi(s)\) is not \(O(\exp C' |s|)\). The logarithmic derivative of (1) is

\[
\frac{\xi'(s)}{\xi(s)} = B - \frac{1}{s - 1} + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right);
\]

since \(\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)\) we also get a product formula for \(\zeta(s)\), and a partial-fraction expansion of its logarithmic derivative:

\[
\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s - 1} + \frac{1}{2} \log \pi - \frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} + 1 + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right).
\]

(We have shifted from \(\Gamma(s/2)\) to \(\Gamma(s/2 + 1)\) to absorb the term \(-1/s\); note that \(\zeta(s)\) does not have a pole or zero at \(s = 0\).)

**Vertical distribution of zeros.** Since the zeros \(\rho\) of \(\xi(s)\) are limited to a strip we can find much more precise information about the distribution of their sizes than the convergence of \(\sum_{\rho} |\rho|^{-1 - \epsilon}\) and the divergence of \(\sum_{\rho} |\rho|^{-1}\). Let \(N(T)\) be the number of zeros in the rectangle \(\sigma \in [0, 1], t \in [0, T]\); note that this is very nearly half of what we would call \(n(T)\) in the context of the general product formula for \((s^2 - s)\xi(s)\).

**Theorem (von Mangoldt).** As \(T \to \infty\),

\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).
\]

**Proof:** We follow chapter 15 of [Davenport 1967], keeping in mind that Davenport’s \(\xi\) and ours differ by a factor of \((s^2 - s)/2\).

We may assume that \(T\) does not equal the imaginary part of any zero of \(\zeta(s)\). Then

\[
2N(T) - 2 = \frac{1}{2\pi i} \oint_{C_R} \frac{\xi'(s)}{\xi(s)} ds = \frac{1}{2\pi i} \oint_{C_R} d(\log \xi(s)) = \frac{1}{2\pi i} \oint_{C_R} d(\text{Im} \log \xi(s)),
\]

where \(C_R\) is the boundary of the rectangle \(\sigma \in [-1, 2], t \in [-T, T]\). Since \(\xi(s) = \xi(1 - s) = \bar{\xi}(\bar{s})\), we may by symmetry evaluate the last integral by
integrating over a quarter of $C_R$ and multiplying by 4. We use the top right quarter, going from 2 to $2 + iT$ to $(1/2) + iT$. Because $\log \xi(s)$ is real at $s = 2$, we have

$$\pi(N(T) - 1) = \text{Im} \log \xi(\frac{1}{2} + iT) = \text{Im}(\log \Gamma(\frac{1}{4} + \frac{iT}{2})) - \frac{T}{2} \log \pi + \text{Im}(\log \xi(\frac{1}{2} + iT)).$$

By Stirling, the first term is within $O(T^{-1})$ of

$$\text{Im} \left( \left( \frac{iT}{2} - \frac{1}{4} \right) \log \left( \frac{iT}{2} + \frac{1}{4} \right) \right) - \frac{T}{2} = \frac{T}{2} \log \left| \frac{iT}{2} + \frac{1}{4} \right| - \frac{1}{4} \text{Im} \log \left( \frac{iT}{2} + \frac{1}{4} \right) - \frac{T}{2} = \frac{T}{2} \left( \log \frac{T}{2} - 1 \right) + O(1).$$

Thus (4) is equivalent to

$$\text{Im} \log \xi(\frac{1}{2} + iT) \ll \log T. \quad (5)$$

We shall show that for $s = \sigma + it$ with $\sigma \in [-1, 2]$, $|t| > 1$ we have

$$\frac{\zeta'}{\zeta}(s) = \sum_{|\text{Im}(s - \rho)| < 1} \frac{1}{s - \rho} + O(\log |t|), \quad (6)$$

and that the sum comprises at most $O(\log |t|)$ terms, from which our desired estimate will follow by integrating from $s = 2 + iT$ to $s = 1/2 + iT$. We start by taking $s = 2 + it$ in (3). At that point the LHS\(^1\) is uniformly bounded (use the Euler product), and the RHS is

$$\sum_{\rho} \left( \frac{1}{2 + it - \rho} + \frac{1}{\rho} \right) + O(\log |t|)$$

by Stirling. Thus the sum, and in particular its real part, is $O(\log |t|)$. But each summand has positive real part, which is at least $1/(4 + (t - \text{Im} \rho)^2)$. Our second claim, that $|t - \text{Im} \rho| < 1$ holds for at most $O(\log |t|)$ zeros $\rho$, follows immediately. It also follows that

$$\sum_{|\text{Im}(s - \rho)| \geq 1} \frac{1}{\text{Im}(s - \rho)^2} \ll \log |t|.$$ 

Now by (3) we have

$$\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(2 + it) = \sum_{\rho} \left( \frac{1}{s - \rho} - \frac{1}{2 + it - \rho} \right) + O(1).$$

\(^{1}\)“LHS” and “RHS” are the left-hand side and right-hand side of an equation.
The LHS differs from that of (6) by $O(1)$, as noted already; the RHS summed over zeros with $|\text{Im}(s - \rho)| < 1$ is within $O(\log |t|)$ of the RHS of (6); and the remaining terms are

$$(2 - \sigma) \sum_{|\text{Im}(s - \rho)| \geq 1} \frac{1}{(s - \rho)(2 + it - \rho)} \ll \sum_{|\text{Im}(s - \rho)| \geq 1} \frac{1}{\text{Im}(s - \rho)^2} \ll \log |t|.$$ 

This proves (6) and thus also (5); von Mangoldt’s theorem (4) follows. □

For much more about the vertical distribution of the nontrivial zeros $\rho$ of $\zeta(s)$ see [Titchmarsh 1951], Chapter 9.

**Further remarks**

Recall that as part of our proof that $(s^2 - s)\xi(s)$ has order 1 we showed that for each $\sigma$ there exists $\nu(\sigma)$ such that $|\zeta(\sigma + it)| \ll |t|^{\nu(\sigma)}$ as $|t| \to \infty$. Any bounded $\nu(\sigma)$ suffices for this purpose, but one naturally asks how small $\nu(\sigma)$ can become. Let $\mu(\sigma)$ be the infimum of all such $\nu(\sigma)$; that is,

$$\mu(\sigma) := \limsup_{|t| \to \infty} \frac{\log |\zeta(\sigma + it)|}{\log |t|}.$$ 

We have seen that $\mu(\sigma) = 0$ for $\sigma > 1$, that $\mu(1 - \sigma) = \mu(\sigma) + \sigma - \frac{1}{2}$ by the functional equation (so in particular $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma < 0$), and that $\mu(\sigma) \leq 1$ for $\sigma < 1$, later improving the upper bound from 1 to $1 - \sigma$. For $\sigma \in (0, 1)$ an even better bound is obtained using the “approximate functional equation” for $\zeta(s)$ (usually attributed to Siegel, but now known to have been used by Riemann himself) to show that $\mu(\sigma) \leq (1 - \sigma)/2$; this result and the fact that $\mu(\sigma) \geq 0$ for all $\sigma$, also follow from general results in complex analysis, which indicate that since $\mu(\sigma) < \infty$ for all $\sigma$ the function $\mu(\cdot)$ must be convex. For example, $\mu(1/2) \leq 1/4$, so $|\zeta(1/2 + it)| \ll |t|^\frac{1}{2} + \epsilon$.

The value of $\mu(\sigma)$ is not known for any $\sigma \in (0, 1)$. The Lindelöf conjecture asserts that $\mu(1/2) = 0$, from which it would follow that $\mu(\sigma) = 0$ for all $\sigma \geq 1/2$ while $\mu(\sigma) = \frac{1}{2} - \sigma$ for all $\sigma \leq 1/2$. Equivalently, the Lindelöf conjecture asserts that $\zeta(\sigma + it) \ll \epsilon |t|^\epsilon$ for all $\sigma \geq 1/2$ (excluding a neighborhood of the pole $s = 1$), and thus by the functional equation that also $\zeta(\sigma + it) \ll \epsilon |t|^{1/2 - \sigma + \epsilon}$ for all $\sigma \leq 1/2$. We shall see that this conjecture is implied by the Riemann hypothesis, and also that it holds on average in the sense that $\int_0^T |\zeta(1/2 + it)|^2 \, dt \ll T^{1 + \epsilon}$. However, the best upper bound currently proved on $\mu(1/2)$ is only a bit smaller than 1/6; when we get to exponential sums later this term we shall derive the upper bound of 1/6.

**Exercises**

1. Show that in the product formula (1) we may take $A = 0$. Prove the formula

$$\gamma = \lim_{s \to 1} \left( \zeta(s) - \frac{1}{s - 1} \right)$$
for Euler’s constant, and use it to compute

\[ B = \lim_{s \to 0} \left( \frac{\xi'(s)}{\xi(s)} + \frac{1-s}{s} \right) = \lim_{s \to 1} \left( -\frac{\xi'(s)}{\xi(s)} + \frac{s}{1-s} \right) \]

\[ = \frac{1}{2} \log 4\pi - 1 - \frac{\gamma}{2} = -0.0230957 \ldots \]

Show also (starting by pairing the \( \rho \) and \( \bar{\rho} \) terms in the infinite product) that

\[ B = -\sum_{\rho} \frac{\Re(\rho)}{|\rho|^2}, \]

and thus that \( |\Im(\rho)| > 6 \) for every nontrivial zero \( \rho \) of \( \zeta(s) \). [From [Davenport 1967], Chapter 12. It is known that in fact the smallest zeros have (real part \( 1/2 \) and) imaginary part \( \pm 14.134725 \ldots \)]

2. By the functional equation, \( \xi(s) \) can be regarded as an entire function of \( s^2 - s \); what is the order of this function? Use this to obtain the alternative infinite product

\[ \xi(s) = \frac{\xi(1/2)}{4(s-s^2)} \prod_{\rho} \left[ 1 - \left( \frac{s - 1/2}{\rho - 1/2} \right)^2 \right], \quad (7) \]

the product extending over zeros \( \rho \) of \( \xi \) whose imaginary part is positive.

[This symmetrical form eliminates the exponential factors \( e^{A+Bs} \) and \( e^{s/\rho} \) of (1).

We nevertheless use (1) rather than (7) to develop the properties of \( \xi \) and \( \zeta \) because (1) generalizes to arbitrary Dirichlet \( L \)-series, whereas (7) generalizes only to Dirichlet series associated to real characters \( \chi \) since in general the functional equation relates \( L(s,\chi) \) with \( L(1-s,\chi) \), not with \( L(1-s,\chi) \).]

3. Let \( f \) be any analytic function on the vertical strip \( a < \sigma < b \) such that

\[ M_f(\sigma) := \limsup_{|t| \to \infty} \frac{\log |f(\sigma + it)|}{\log |t|} \]

is finite for all \( \sigma \in (a, b) \). Prove that \( M_f \) is a convex function on that interval.

[Hint: Apply the maximum principle to \( \alpha f \) for suitable analytic functions \( \alpha(\sigma) \).]

It follows in particular that \( M_f \) is continuous on \( (a, b) \). While \( \zeta(s) \) is not analytic on vertical strips that contain \( s = 1 \), we can still deduce the convexity of \( \mu : \mathbb{R} \to \mathbb{R} \) from \( \mu(\sigma) = M_f(\sigma) \) for \( f(s) = \zeta(s) - (1/(s-1)) \).

Much the same argument proves the “three lines theorem”: if \( f \) is actually bounded on the strip then \( \log \sup_{|t|} |f(\sigma + it)| \) is a convex function of \( \sigma \). The name of this theorem alludes to the equivalent formulation: if \( a < \sigma_1 < \sigma_2 < \sigma_3 < b \) then the supremum of \( |f(s)| \) on the line \( s = \sigma_2 + it \) is bounded by a weighted geometric mean of its suprema on the lines \( s = \sigma_1 + it \) and \( s = \sigma_3 + it \).

Reference