Math 229: Introduction to Analytic Number Theory

A zero-free region for \( \zeta(s) \)

We first show, as promised, that \( \zeta(s) \) does not vanish on \( \sigma = 1 \). As usual nowadays, we give Mertens’ elegant version of the original arguments of Hadamard and (independently) de la Vallée Poussin. Recall that

\[
-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}
\]

has a simple pole at \( s = 1 \) with residue +1. If \( \zeta(s) \) were to vanish at some \( 1 + it \) then \( -\zeta'/\zeta \) would have a simple pole with residue \(-1 \) (or \(-2, -3, \ldots \)) there. The idea is that \( \sum_n \Lambda(n)/n^s \) converges for \( \sigma > 1 \), and as \( s \) approaches 1 from the right all the terms contribute towards the positive-residue pole. As \( \sigma \to 1 + it \) from the right, the corresponding terms have the same magnitude but are multiplied by \( n^{-it} \), so a pole with residue \(-1 \) would force “almost all” the phases \( n^{-it} \) to be near \(-1 \). But then near \( 1 + 2it \) the phases \( n^{-2it} \) would again approximate \((-1)^2 = +1 \), yielding a pole of positive residue, which is not possible because then \( \zeta \) would have another pole besides \( s = 1 \).

To make precise the idea that if \( n^{-it} \approx -1 \) then \( n^{-2it} \approx +1 \), we use the identity

\[
2(1 + \cos \theta)^2 = 3 + 4 \cos \theta + \cos 2 \theta,
\]

from which it follows that the right-hand side is positive. Thus if \( \theta = t \log n \) we have

\[
3 + 4 \Re(n^{-it}) + \Re(n^{-2it}) \geq 0.
\]

Multiplying by \( \Lambda(n)/n^\sigma \) and summing over \( n \) we find

\[
3 \left[ -\frac{\zeta'(\sigma)}{\zeta(\sigma)} \right] + 4 \Re \left[ -\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} \right] + \Re \left[ -\frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + 2it)} \right] \geq 0 \tag{1}
\]

for all \( \sigma > 1 \) and \( t \in \mathbb{R} \). Fix \( t \neq 0 \). As \( \sigma \to 1+ \), the first term in the LHS of this inequality is \( 3/(\sigma - 1) + O(1) \), and the remaining terms are bounded below. If \( \zeta \) had a zero of order \( r > 0 \) at \( 1 + it \), the second term would be \(-4r/(\sigma - 1) + O(1) \). Thus the inequality yields \( 4r \leq 3 \). Since \( r \) is an integer, this is impossible, and the proof is complete.

We next use (1), together with the partial-fraction formula

\[
-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + B_1 + \frac{1}{2} \frac{\Gamma'}{\Gamma}(\frac{s}{2} + 1) - \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right),
\]

to show that even the existence of a zero close to \( 1 + it \) is not possible. How close depends on \( t \); specifically, we show:

\[1\]See for instance Chapter 13 of Davenport’s book [Davenport 1967] cited earlier. This classical bound has been improved; the current record of \( 1 - \sigma \ll \log^{-2/3-\epsilon} |t| \), due to Korobov and perhaps Vinogradov, has stood for 50 years. See [Walfisz 1963] or [Montgomery 1971, Chapter 11].
Theorem. There is a constant $c > 0$ such that if $|t| > 2$ and $\zeta(\sigma + it) = 0$ then
\[ \sigma < 1 - \frac{c}{\log |t|}. \] (2)

Proof: Let $\sigma \in [1, 2]$ and $|t| \geq 2$ in the partial-fraction formula. Then the $B_1$ and $\Gamma'/\Gamma$ terms are $O(\log |t|)$, and each of the terms $1/(s - \rho), 1/\rho$ has positive real part as noted in connection with von Mangoldt’s theorem on $N(T)$. Therefore
\[ -\text{Re} \frac{\zeta'}{\zeta}(\sigma + 2it) < O(\log |t|), \]
and if some $\rho = 1 - \delta + it$ then
\[ -\text{Re} \frac{\zeta'}{\zeta}(\sigma + it) < O(\log |t|) - \frac{1}{\sigma + \delta - 1}. \]

Thus (1) yields
\[ \frac{4}{\sigma + \delta - 1} < \frac{3}{\sigma - 1} + O(\log |t|). \]

In particular, taking $\sigma = 1 + 4\delta$ yields $1/20\delta < O(\log |t|)$. Hence $\delta \gg (\log |t|)^{-1}$, and our claim (2) follows. \qed

Once we obtain the functional equation and partial-fraction decomposition for Dirichlet $L$-functions $L(s, \chi)$, the same argument will show that (2) also gives a zero-free region for $L(s, \chi)$, though with the implied constant depending on $\chi$.

Remarks

The only properties of $\Lambda(n)$ that we used in the proof of $\zeta(1 + it) \neq 0$ are that facts that $\Lambda(n) \geq 0$ for all $n$ and that $\sum_n \Lambda(n)/n^s$ has an analytic continuation with a simple pole at $s = 1$ and no other poles of real part $\geq 1$. Thus the same argument exactly will show that $\prod_{\chi \mod q} L(s, \chi)$, and thus each of the factors $L(s, \chi)$, has no zero on the line $\sigma = 1$.

The $3 + 4 \cos \theta + \cos 2\theta$ trick is worth remembering, since it has been adapted to other uses. For instance, we shall revisit and generalize it when we develop the Drinfeld-Vlăduţ upper bounds on points of a curve over a finite field and the Odlyzko-Stark lower bounds on discriminants of number fields. See also the following Exercises.

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2A lower bound $|t| \geq t_0$ would do for any $t_0 > 1$ — and the only reason we cannot go lower is that our bounds are in terms of $\log |t|$ so we do not want to allow $\log |t| = 0$.

3Note that we write $< O(\log |t|)$, not $= O(\log |t|)$, to allow the possibility of an arbitrarily large negative multiple of $\log |t|$.

4$1 + \alpha \delta$ will do for any $\alpha > 3$. This requires that $\alpha \delta \leq 1$, e.g. $\delta \leq 1/4$ for our choice of $\alpha = 4$, else $\sigma > 2$; but we’re concerned only with $\delta$ near zero, so this does not matter.
Exercises

1. Use the inequality $3 + 4 \cos \theta + \cos 2 \theta \geq 0$ to give an alternative proof that $L(1, \chi) \neq 0$ when $\chi$ is a complex Dirichlet character (a character such that $\chi \neq \overline{\chi}$).

2. Show that for each $\alpha > 2$ there exists $t \in \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \exp(-|x|^\alpha + itx) \, dx < 0.$$  

(Yes, this is related to the present topic; see [EOR 1991, p.633]. The integral is known to be positive for all $t \in \mathbb{R}$ when $\alpha \in (0, 2]$; see for instance [EOR 1991, Lemma 5].)

References

