Geometric Topology

Harvard University — Fall 2003 — Math 99r
Course Notes

Contents

1 Introduction: Knots and Reidemeister moves

Simplicial complexes and embeddings. Any n-complex embeds in $\mathbb{R}^{2n+1}$. The utility graph and the pentagram do not embed in the plane.

Knots and links. Fundamental group, the Hopf link, the unlink, and the carabiner trick.

Knots exist. General position, Reidemeister moves, tricoloring, existence of knots.

A knot in the Science Center. Two people A and B walk from up the main stairs from floor 1 to floor 3 – arriving outside 309 – holding hands. They start with their backs to us as seen from the yard, in position A-B. The arrive on the 3rd floor facing us, so in postion B-A. Now B goes down one wing of the SC – toward Littauer – and A goes down the other wing – to the new Hist. of Sci. dept. They go down the stairs in their wings, arrive on the first floor, and rejoin. Now they are in postion B-A with their backs to us. So the two paths they have traced join together to form a closed path = a knot. The knot is simply the closure of a braid on two strands with an odd number of twists. The number of twists comes out to be 3 – which gives the trefoil.

2 1-Dimensional Topology

Groups. Free group, presentations, Tietze moves, unsolvable problems in topology. Free products and semidirect products; the word problem for each. The groups $\langle a, b \ : \ a^2, b^2\rangle$ and $\langle a, b \ : \ a^2, aba' = b'\rangle$; the dihedral groups $\langle a, b \ : \ a^2, aba' = b', b^n\rangle$.

$\mathrm{PSL}_2(\mathbb{Z}) = \langle a^2, b^3\rangle$ and the groups $G_n = \langle a^2, b^3, (ab)^n\rangle$; $G_{2,3,4,5} = S_3, A_4, S_4, A_5$.

Cayley graphs. The fundamental group and universal cover of $X = S^1 \wedge S^1$. Deck groups and irregular coverings. The coverings of X coming from S_3 and its subgroups.

Nielsen-Schreier Theorem: any f.g. subgroup of a free group is free. For any finite graph, $\chi(X) = 1 - n$ where $\pi_1(X) \cong F_n$. If $X \to Y$ is a degree d covering of graphs, then $\chi(X) = d\chi(Y)$. Thus if $H \subset F_n$ has index d, then $H \cong F_m$ with $1 - m = d(1 - n)$.

Universal covers. The universal cover of S^1. The degree of a map $f : S^1 \to S^1$. Classification of immersions; smiles and frowns.

The universal cover of S^2, $\mathbb{RP}^2$ and $S^1 \times S^1$. The product formula: $\pi_1(A \times B) = \pi_1(A) \times \pi_1(B)$.

More covering spaces of the bouquet of 2 circles X:

$$H = \langle a \rangle, \langle a, b^2 \rangle, \langle b^n a b^{-n} \ : \ n \in \mathbb{Z} \rangle.$$

The kernel of $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}$ is infinitely generated. Generators and normal generators for the kernel of $\mathbb{Z} * \mathbb{Z} \to S_3$.

Amenability. A graph is amenable if $\inf |\partial V|/|V| = 0$; a group, if its Cayley graph is amenable. Ponzi schemes and the degree three tree; the enrichment flow shows $|\partial V| \ge |V|$.

The group $\mathbb{Z}^n$ is amenable, the free group is not. A non-amenable group has exponential growth, but not conversely.

An amenable group of exponential growth. Let $G = \langle a, b \ : \ ab = b^2 a \rangle$. It is easy to see that G has exponential growth, and in fact G is isomorphic to the group of affine transformations of $\mathbb{R}$ of the form $f(x) = 2^n x + k2^m$, $n, m, k \in \mathbb{Z}$.

Here is a sketch of a proof that G is amenable. Consider the elements of G of the form $g(i, j)$ corresponding to the transformations $2^{-i}(x + j)$, with $i \ge 1$. Then we have

$$ag(i, j) = g(i - 1, j) \quad \text{and} \quad bg(i, j) = g(i, j + 2^i).$$

This makes it easy to visualize a large part of the Cayley graph of G. For an even better picture, mod out by the action of b (look at the $\langle b \rangle$-cosets): the result is a collection of 'circles' $S_i \cong \mathbb{Z}/2^i$, with the a-edges joining the points of S_i to S_{i-1} in pairs. The result is a bifurcating tree T.

Thus we can think of the Cayley graph as a tree T, with each vertex replaced by a copy of $\mathbb{Z}$. The a-edges converge in pairs, but this is compensated for by the fact that the b-edges spread apart, interleaving the copies of $\mathbb{Z}$.

Now consider the set $V = \{g(i, j) \ : \ 1 \le i \le L, 1 \le j \le M\}$. Then using the equations above, it is easy to see that the b-edges contribute about 2^i vertices to ∂V at level i, resulting in a total of about 2^L. The a-edges obviously contribute a boundary of size M. Thus we have

$$\frac{|\partial V|}{|V|} \asymp \frac{2^L + M}{LM}.$$

By taking $0 \ll L \ll 2^L \ll M$, we can make this ratio arbitrarily small, and therefore G is amenable.

General theory of the fundamental group. Homotopy equivalence and retracts. Contracting trees in graphs (1) to show homotopy equivalence to a bouquet of circles and (2) to show covering spaces are homotopy equivalent to finite graphs.

Theta and barbells. To see these are homotopy equivalent, note they are both deformation retracts of a pair of pants.

Collapsing: a compact surface Σ with nonempty boundary is homotopy equivalent to a graph. Thus $\pi(\Sigma)$ is free.

Collapsing: the house with two rooms. Not every contractible complex collapses.

Seifert-van Kampen. A presentations for $G *_A H$ is given by the union of the presentations for G and H, together with the additional relations $\phi_G(a_i) = \phi_H(a_i)$ for each generator of A. (Note that the relators of A are irrelevant.) If H is trivial, then $G *_A H = G/N(\phi_G(A))$.

The Seifert-van Kampen theorem says

$$\pi_1(X \cup Y) = \pi_1(X) *_{\pi_1(X \cap Y)} \pi_1(Y).$$

Examples: bouquets of circles; $\pi_1(S^n)$; $\pi_1(S^1 \times S^1)$, thought of as a torus with a hole, with the hole glued back in.

Presentations and cell complexes. A compact Hausdorff space X is a finite *cell complex* if there are maps $f_i : D^{n_i} \to X$, whose images cover X, such that f_i is a homeomorphism on the interior of the n_i-disk, and f_i sends the boundary into the $(n_i - 1)$ skeleton of X, defined as the union of the cells of dimension $k \le n_i - 1$.

Theorem: given any finitely presented group G, there is a finite 2-complex K such that $\pi_1(K) \cong G$.

Example: The two complex canonically associated to $\mathbb{Z}^2 = \langle a, b \,:\, [a, b]\rangle$ *is* a torus.

Theorem (Poincaré): for any complex K, we have $\pi_1(K) = \pi_1(K^{(2)})$.

First betti number. The *cohomology group* $H^1(G, \mathbb{Z}) = G/[G, G]$. Example: $H^1(F_n, \mathbb{Z}) \cong \mathbb{Z}^n$. This shows free groups on different numbers of generators are not isomorphic.

For a topological space X, we define $b_1(X) =$ rank of free part of $H^1(G, \mathbb{Z})$. Also $b_0(X) =$ number of components of X. Then for a graph we have:

$$\chi(X) = b_0(X) - b_1(X).$$

This generalizes, and shows $\chi(X)$ is a homotopy invariant.

3 2-Dimensional Topology

Background. Closed manifolds, manifolds with boundary. A combinatorial n-manifold is a simplicial complex such that the link of every vertex is a PL sphere S^{n-1}. This implies every $(n-1)$-simplex is the face of exactly two n-simplices. A manifold is *orientable* if we can orient its n-simplices so the two induced orientations on each $(n-1)$-simplex are opposite.

Why are manifolds important? Because they arise as level sets of functions, e.g. $x^n + y^n = 1$.

Surfaces and Euler characteristic. The basic surfaces: S^2, $\mathbb{RP}^2$, $S^1 \times S^1$, the Klein bottle (thought of as a cylinder with ends suitably identified). Euler characteristic.

A closed orientable surface is uniquely determined by $\chi(S) = 2 - 2g$. (A nonorientable surface is also determined by its Euler characteristic.)

Examples of closed surfaces. Orientable surfaces: $\Sigma_0 = S^2$, $\Sigma_1 = S^1 \times S^1$, $\Sigma_{n+1} = \Sigma_n \# \Sigma_1$.

Nonorientable surfaces: $N_0 = \mathbb{RP}^2$, $N_{h+1} = N_n \# \mathbb{RP}^2$.

Classification of surfaces. Theorem. Every closed surface is homeomorphic to Σ_g or N_h, some g or h.

Basic properties. We have $\chi(\Sigma_g) = 2 - 2g$, $\chi(N_h) = 1 - h$. The orientable double cover of N_h is Σ_h.

Branched coverings and degree. Riemann-Hurwitz: if $f : A \to B$ is a branched cover, then

$$\chi(A) = \deg(f)\chi(B) - \text{the number of branched points of } f.$$

Examples: (1) rational maps on the Riemann sphere; complex analysis. (2) elliptic functions. (3) hyperelliptic surfaces.

Degree of general maps between surfaces. Turning the sphere inside out.

Surfaces with boundary. Definition: $\Sigma_{g,n}$ and $N_{h,n}$ are obtained by removing n disks from Σ_g and N_h respectively.

Theorem. Any compact surface with n boundary components is homeomorphic to $\Sigma_{g,n}$ or $N_{h,n}$.

Note that $\chi(\Sigma_{g,n} = 2 - 2g - n$. Examples: $\Sigma_{0,n}$ for $n = 0, 1, 2, 3$ are S^2, D^2, $S^1 \times I$ and a pair of pants respectively.

Cor. Up to the action of $\operatorname{Aut}(\Sigma_g)$, there are $g + 1$ types of simple closed curves on Σ_g.

Cor. A maximal system of disjoint, non-separating simple closed curves on Σ_g has cardinality g. Note that $\chi(\Sigma_g) = \chi(\Sigma_{0,2g})$.

The mapping-class group of a torus. $\operatorname{Mod}(\Sigma_g) = \operatorname{Aut}^+(\Sigma_g)/\operatorname{Aut}_0(\Sigma_g)$. Thus $\operatorname{Mod}(S^2)$ is trivial.

Theorem. $\operatorname{Mod}(\Sigma_1) \cong \operatorname{SL}_2(\mathbb{Z})$.

Cor. Isotopy classes of essential simple closed curves on Σ_1 correspond naturally to extended rational numbers $p/q \in \mathbb{Q} \cup \{\infty\}$.

Pairs of pants. For $g > 1$, Σ_g can be decomposed by $3g - 3$ simple closed curves into $2g - 2$ pairs of pants. It is then described by a trivalent graph.

Morse theory. Möbius, 1863, used a version of Morse theory (1892–1977) to show every closed surface embedded in $\mathbb{R}^3$ is homeomorphic to a standard surface of genus g. His method was similar to modern Morse theory: he determined how the surface changed upon passing a critical point of the height function.

Universal covers of surfaces. Theorem. For $g \geq 2$, the universal cover of Σ_g can be identified with the hyperbolic plane. In fact, there is a regular $4g$-gon in $\mathbb{H}$ with internal angles, $2\pi/4g$, which tiles $\mathbb{H}$ and glues up to give a complete hyperbolic structure on Σ_g.

Homology theory. One can define $H_1(X, \mathbb{Z})$ as the abelianization of $\pi_1(X)$. Loops that form the boundary of an (orientable) subsurface represent the same element of $H_1(X, \mathbb{Z})$.

4 Knot theory

The Wirtinger presentation. Tricolorings, *reprise.*

The Jones polynomial. The braid group B_n has the presentation

$$\langle \sigma_1, \ldots, \sigma_{n-1} \; : \; \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \rangle.$$

There is a natural homomorphism $e : B_n \to \mathbb{Z}$ satisfying $e(\sigma_i) = 1$.

On the other hand, there is also an algebra of matrices A_n generated by $\langle 1, e_1, \ldots, e_{n-1} \rangle$, equipped with a trace $\operatorname{tr} : A_n \to \mathbb{R}$, satisfying for a real parameter τ,

1. $e_i^2 = e_i^* = e_i$ (the e_i are projections);
2. $e_i e_{i \pm 1} e_i = \tau e_i$;
3. $e_i e_j = e_j e_i$ if $|i - j| > 1$;
4. $\operatorname{tr}(x e_{i+1}) = \tau \operatorname{tr}(x)$ if $x \in A_i$; and
5. $\operatorname{tr}(1) = 1$.

These rules determine the value of $\operatorname{tr}(x)$ for all $x \in A_n$.

A representation of B_n in the algebra A_n is obtained by setting $\sigma_i = (t + 1)e_i - 1$, where

$$\tau^{-1} = t + t^{-1} + 2 + (t^{-1/2} + t^{1/2})^2.$$

Fixing this representation, let $x \in B_n$ be a braid whose closure is the oriented link $L = \widehat{x}$. Then one obtains a polynomial invariant of L by setting:

$$V_{\widehat{x}}(t) = (-t^{-1/2} - t^{1/2})^{n-1} t^{e(\alpha)/2} \operatorname{tr}(\alpha).$$

It is clear that the expression above remains the same if we replace x by a conjugate braid xyx^{-1}. The factor in front of the trace insures that $V_{\widehat{x}}(t)$ is also invariant under the *second* Markov move. That is, if $x \in B_n$ and $y = x\sigma_n$, then we have $y = (1 + t)xe_n - x$, and therefore

$$\operatorname{tr}(x\sigma_n) = ((1 + t)\tau - 1) \operatorname{tr}(x) = -(1 + t)^{-1} \operatorname{tr}(x).$$

On the other hand, $e(y) = e(x)+1$, so the factor in front of the trace is multiplied by an additional factor of $-(1 + t)$ upon replacing x with y; thus $V_{\widehat{y}}(t) = V_{\widehat{x}}(t)$.

Example: the trefoil knot. The trefoil knot K is the closure of the braid $\sigma_1^3 \in B_2$. We have $\sigma_1 = (t+1)e_1 - 1$ and $e_1^2 = e_1$, and therefore

$$\sigma_1^3 = e_1((t+1)^3 - 3(t+1)^2 + 3(t+1)) - = e_1((t+1)(t^2 - t + 1)) - 1.$$

Since $\mathrm{tr}(e_1) = \tau = t(1+t)^{-2}$, we have

$$\mathrm{tr}(\sigma_1^3) == \tau((t+1)(t^2 - t + 1)) - 1.$$

Since $n = 2$ and $e(\sigma_1^3) = 3$, the factor in front of the trace is

$$(-t^{-1/2} - t^{1/2})t^{3/2} = -t(t+1).$$

Taking the product, we find

$$V_K(t) = \tau(-t(t+1)^2(t^2 - t + 1)) + t + t^2 = -t^2(t^2 - t + 1) + t + t^2 = -t^4 + t^3 + t.$$